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Fixed point theorems for set-valued mappings on TVS-cone metric spaces

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Abstract

In the context of TVS-cone metric spaces, we prove a Bishop-Phelps and a Caristi type theorem. These results allow us to prove a fixed point theorem for (δ, L)-weak contraction according to a pseudo Hausdorff metric defined by means of a cone metric.

MSC: Primary 47H10; secondary 47H04

Keywords: TVS-cone metric space; Bishop-Phelps type theorem; Caristi type theorem; Berinde weak contraction; set-valued mapping

1 Introduction

Huang and Zhang in [1], introduced the concept of cone metric space as a generalization of metric space. The most relevant feature of their work is that these authors gave an example of a contraction on a cone metric space, which is not a contraction in a standard metric space. This fact makes it clear that the theory of metric spaces is not flexible enough for the fixed point theory, which has prompted several authors to publish numerous works on fixed point theory for operators defined on cone metric spaces. Most of these are based on cone metrics taking values in a Banach space, and even, some of them suppose this space is normal, in the sense that this space has a base of neighborhood of zero consisting of order-convex subsets. The main aim of this paper is to provide results for set-valued mappings defined on a cone metric space, whose metric takes values in a quite general topological vector space, since it is only assumed this space is σ-order complete. In [2] (see also, [3]), Agarwal and Khamsi proved a version of Caristi's theorem based on a Bishop-Phelps type result for a cone metric taking values in a Banach space. In this paper, we extend this result, which enables us to prove a more general version of Caristi's theorem for cone metric spaces. Natural consequences are deduced from this fact and, as an application, we prove the existence of a fixed point for an analogous weak contraction of set-valued mapping defined by Berinde and Berinde in [4], which, in our case, is defined according to a pseudo Hausdorff cone metric.

The paper is organized as follows. In Section 2 some preliminary definitions and facts are given, while in Section 3, Bishop-Phelps' and Caristi's theorems are proved. Finally, Section 4 is devoted to an application to set-valued weak contractions defined by means of a cone metric.

2 Preliminaries

Let E be a topological vector space with θ as zero element and usual notations for addition and scalar product. A cone is a nonempty closed subset P of E such that $P \cap (-P) = \{\theta\}$ and for each $\lambda \geq 0$, $\lambda P + P \subseteq P$. Given a cone P of E, a partial order is defined on E as $x \preceq y$, if and only if $y - x \in P$. We denote by $x \prec y$ whenever $x \preceq y$ and $x \neq y$. Moreover, the notations $x \ll y$ means that $y - x$ belongs to int(P), the interior of P. As natural, the notations $x \succeq y$, $x \succ y$, and $x \gg y$ mean $y \preceq x$, $y \prec x$, and $y \ll x$, respectively. In the following, we assume P is a cone of E such that E is a Riesz space, *i.e.* given $x, y \in E$, the greatest lower bound (infimum) of $\{x, y\}$ exists, which also implies that the least upper bound (supremum) of $\{x, y\}$ exists. Additionally, E is assumed order complete (Dedekind), which means that every decreasing bounded from below net has an infimum. Of course, from this we see that every increasing bounded from the above net has a supremum. For notations and facts as regards ordered vector spaces, we refer to [5]. In particular, since E is a Riesz space, Theorem 1.20 in [5] implies that every bounded from below subset of E has an infimum. This fact is used in Section 4 when a kind of Hausdorff pseudo metric is defined.

Remark 1 For each $a, b, c \in E$ such that $a \preceq b \ll c$, we have $a \ll c$.

A cone metric space is a pair (X, d), where X is a nonempty set and $d : X \times X \to E$ is a function satisfying the following two conditions: (i) for all $x, y \in X$, $d(x, y) = \theta$, if and only if $x = y$, and (ii) for all $x, y, z \in X$, $d(x, y) \preceq d(x, z) + d(y, z)$.

In the sequel, (X, d) stands for a cone metric space.

Remark 2 Note that for all $x, y \in X$, $d(x, y) \succeq \theta$ and $d(x, y) = d(y, x)$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. We say $\{x_n\}_{n\in\mathbb{N}}$ converges to x, if and only if, for every $\epsilon \gg \theta$, there exists $N \in \mathbb{N}$ such that, for any $n \geq \mathbb{N}$, we have $d(x_n, x) \ll \epsilon$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ it said to be a Cauchy sequence, if and only if, for every $\epsilon \gg \theta$, there exists $N \in \mathbb{N}$ such that, for any $m, n \geq \mathbb{N}$, we have $d(x_m, x_n) \ll \epsilon$. The cone metric space (X, d) is said to be complete, if and only if every Cauchy sequence in X converges to some point $x \in X$. A subset F of X is said to be closed, if, for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in F converging to $x \in E$, we have $x \in F$.

Remark 3 If X is complete and $F \subseteq X$ is closed, then F is complete.

Let $\varphi : X \to E$ be a function. We say φ is lower semicontinuous, if and only if, for any $\alpha \in E$, the set $\{x \in X : \varphi(x) \preceq \alpha\}$ is closed. For this function, a Brønsted type order $\preceq_\varphi$ is defined on X as follows:

$$x \preceq_\varphi y \quad \text{if and only if} \quad d(x, y) \preceq \varphi(x) - \varphi(y).$$

It is easy to see that $\preceq_\varphi$ is in effect an order relation on X.

In the sequel, $\mathcal{LS}(X)$ stands for the space of all lower semicontinuous and bounded below functions from X to E.

Remark 4 The function φ defining $\preceq_\varphi$ is non-increasing.

3 Bishop-Phelps and Caristi type theorems

The following theorem is an extension of the well-known results by Bishop-Phelps lemma [6].

Theorem 5 *Suppose X is d-complete. Then, for each $\varphi \in \mathcal{LS}(X)$ and $x_0 \in X$, there exists a maximal element $x^* \in X$ such that $x_0 \preceq_\varphi x^*$.*

Proof For each $x \in X$, let $S(x) = \{y \in X : x \preceq_\varphi y\}$, $x_0 \in X$ and C be a chain in $S(x_0)$. Since $S(x) = \{y \in X : \varphi(y) + d(x,y) \preceq \varphi(x)\}$, the lower semicontinuity of $\varphi + d(x,\cdot)$ implies $S(x)$ is a closed set. Let $e \gg \theta$ and, inductively, define an increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ as

$$x_n \in S(x_{n-1}) \cap C \quad \text{with } \varphi(x_n) \prec (1/2n)e + L_n \ll (1/n)e + L_n,$$

where x_0 is given, $A_n = \{\varphi(y) : y \in S(x_{n-1}) \cap C\}$ and $L_n = \inf(A_n)$. Due to φ being non-increasing and bounded below, A_n is a chain in P and consequently $\{x_n\}_{n\in\mathbb{N}}$ is well defined. Moreover, for each $n, p \in \mathbb{N}$, $x_{n+p} \in A_n$, and hence

$$d(x_n, x_{n+p}) \preceq \varphi(x_n) - \varphi(x_{n+p}) \ll (1/n)e.$$

Thus, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and accordingly there exists $x^* \in X$ such that this sequence converges to x^*. Since for each $n \in \mathbb{N}$, $S(x_n)$ is a closed set, we have $x^* \in S(x_n)$ and thus $x_0 \preceq x_n \preceq x^*$. Suppose $y \in X$ satisfies $x^* \preceq_\varphi y$. We have, for each $n \in \mathbb{N}$, $d(x_n, y) \preceq \varphi(x_n) - \varphi(y) \prec (1/n)e$, and hence $\lim_{n\to\infty} d(x_n, y) = 0$. This fact implies that $x^* = y$ and therefore $x^* \in X$ is a maximal element satisfying $x_0 \preceq_\varphi x^*$. This concludes the proof. □

A set $B \subseteq X$ is said to be bounded, whenever $\{d(x,y) : x, y \in X\}$ is bounded in E. In the sequel, we denote by 2^X the family of all nonempty subsets of X and by $B(X)$ the subfamily of 2^X consisting of all closed, nonempty and bounded subsets of X. For a set-valued mapping $T : X \to 2^X$ and $x \in X$, we usually denote Tx instead of $T(x)$.

Theorem 5 enables us to state below a generalized version of Caristi's theorem.

Theorem 6 *Suppose X is d-complete, $T : X \to 2^X$ is a set-valued mapping and $\varphi \in \mathcal{LS}(X)$. The following two propositions hold.*

(6.1) *If for each $x \in X$, there exists $y \in Tx$ such that $d(x,y) \preceq \varphi(x) - \varphi(y)$, then there exists $x^* \in X$ such that $x^* \in Tx^*$.*

(6.2) *If for each $x \in X$ and $y \in Tx$, $d(x,y) \preceq \varphi(x) - \varphi(y)$, then there exists $x^* \in X$ such that $\{x^*\} = Tx^*$.*

Proof From Theorem 5, $\preceq_\varphi$ has a maximal element $x^* \in X$. Suppose there exists $y \in Tx^*$ such that $d(x^*, y) \preceq \varphi(x^*) - \varphi(y)$. That is, $x^* \preceq_\varphi y$. The maximality of x^* implies $y = x^*$ and hence (6.1) holds.

Since Tx^* is nonempty, (6.1) implies $\{x^*\} \subseteq Tx^*$. By applying assumption in (6.2) again and the maximality of x^*, we have $Tx^* \subseteq \{x^*\}$, which proves (6.2), and the proof is complete. □

For single-valued mappings the following corollary holds.

Corollary 7 *Suppose X is d-complete. Let $f: X \to X$ be a mapping and $\varphi \in \mathcal{LS}(X)$ such that for each $x \in X$, $d(x, f(x)) \preceq \varphi(x) - \varphi(f(x))$. Then there exists $x^* \in X$ such that $x^* = f(x^*)$.*

A cone metric version of the nonconvex minimization theorem according to Takahashi [7] is stated as follows.

Theorem 8 *Let $\varphi \in \mathcal{LS}(X)$ such that for any $x_0 \in X$ satisfying $\inf_{x\in X} \varphi(x) \prec \varphi(x_0)$, the following condition holds: there exists $x \in X \setminus \{x_0\}$ such that $d(x_0, x) \preceq \varphi(x_0) - \varphi(x)$. Then there exists $x^* \in X$ such that $\inf_{y\in X} \varphi(y) = \varphi(x^*)$.*

Proof Suppose for every $z \in X$, $\inf_{y\in X} \varphi(y) \prec \varphi(z)$, and let $x_0 \in X$. From Theorem 5, $\preceq_\varphi$ has a maximal element $x^* \in X$ such that $x_0 \preceq_\varphi x^*$. Since φ is non-increasing, $\varphi(x^*) \preceq \varphi(x_0)$ and the assumption implies that there exists $x \in X \setminus \{x^*\}$ such that $x^* \preceq_\varphi x$. From the maximality of x^* we have $x = x^*$, which is a contradiction. Therefore, there exists $z \in X$ such that $\inf_{x\in X} \varphi(x) = \varphi(z)$, which completes the proof. □

4 Contractions

We define $H: B(X) \times B(X) \to E$ as

$$H(A,B) = \sup\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big\}, \tag{1}$$

where for each $x \in X$ and a nonempty subset A of X, $d(x,A) = \inf_{y\in A} d(x,y)$. Since E is an order complete Riesz space, Theorem 1.20 in [5] ensures that (1) is well defined.

Remark 9 When d is a standard metric on X, H is the Hausdorff metric on $B(X)$. However, in general, $(B(X), H)$ is not a cone metric space.

An linear operator $L: E \to E$ is said to be positive, if for any $x \in P$ we have $Lx \in P$. Let $\mathcal{K}_+(E)$ be the set of all positive, injective and continuous linear operators δ from E into itself such that there exists $0 \le t < 1$ satisfying $0 \preceq \delta x \preceq tx$, for all $x \in P$. Notice that for each $\delta \in \mathcal{K}_+(E)$ and $x \in E$, $|\delta x| \preceq \delta|x|$.

Following Berinde and Berinde in [4], a set-valued mapping $T: X \to B(X)$ is called a (δ, L)-weak contraction, if there exist a positive linear operator $L: E \to E$ and $\delta \in \mathcal{K}_+(E)$ such that

$$H(Tx, Ty) \preceq \delta d(x,y) + Ld(y, Tx), \quad \text{for all } x, y \in X. \tag{2}$$

By the symmetry of the distance, condition (2) implicitly includes the following dual inequality:

$$H(Tx, Ty) \preceq \delta d(x,y) + Ld(x, Ty), \quad \text{for all } x, y \in X. \tag{3}$$

Hence, in order to check that a set-valued mapping $T: X \to B(X)$ is a (δ, L)-weak contraction, it is necessary to check both inequalities (2) and (3).

Let $T: X \to B(X)$ be a set-valued mapping. We say T is H-continuous at $x \in A$, if, for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in A converging to x, $\{H(Tx_n, Tx)\}_{n\in\mathbb{N}}$ converges to θ in E. The mapping

T is said to be a contraction, if there exists $k \in \mathcal{K}_+(E)$ such that for any $x, y \in X$, $H(Tx, Ty) \preceq kd(x, y)$. Notice that T is a contraction, if and only if there exists $0 \leq t < 1$ such that for any $x, y \in X$, $H(Tx, Ty) \preceq td(x, y)$. When E is a Banach space, t can be chosen as the spectral ratio $\rho(k)$ of k and hence in this case, k is a contraction, if and only if $\rho(k) < 1$. Of course, any contraction is a weak contraction. A selector of T is any function $f : X \to X$ such that $f(x) \in Tx$, for all $x \in X$. We say T satisfies condition (S) if, for any $\epsilon > 0$, there exists a selector f_ϵ of T such that for each $x \in X$, $d(x, f_\epsilon(x)) \preceq (1 + \epsilon)d(x, Tx)$.

Remark 10 For $x \in X$ and $A, B \in B(X)$, we define $s(x, B)$ and $s(A, B)$ as follows:

$$s(x, B) = \bigcup_{b \in B} \{\epsilon \succ \theta : d(x, b) \preceq \epsilon\}$$

and

$$s(A, B) = \bigcap_{a \in A} s(a, B) \cap \bigcap_{b \in B} s(b, A).$$

Some references such as [8–12] define a k-contraction, for $0 \leq k < 1$, as a set-valued mapping $T : X \to B(X)$ satisfying

$$kd(x, y) \in s(Tx, Ty), \quad \text{for all } x, y \in X. \tag{4}$$

This definition is more restrictive than our definition of contraction by making $L = 0$ in (2). Indeed, even though the functional H is not properly a cone metric, it is easy to see that a set-valued mapping satisfying condition (4), it also satisfies our definition of contraction. Furthermore, condition $\theta \in s(a, A)$ implies $a \in A$ for all $a \in X$ and $A \subseteq X$, even though A is not closed. However, it is not possible to conclude that $a \in A$, if $d(a, A) = 0$, even though A is closed. Consequently, condition (4) is stronger than our definition of a contraction. See the example below.

Example 11 Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $X = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ and define $d : X \times X \to E$ as $d((a, b), (c, d)) = (|a - c|, |b - d|)$. Hence, (X, d) is a cone metric space. Let $T : X \to 2^X$ be a set-valued mapping such that $T(0, 0) = \{(0, 0), (1, 1)\}$ and

$$T(x, y) = \begin{cases} \{(0, 0), (1, 1)\} & \text{if } (x, y) = (0, 0), \\ \{(1, 0), (0, 1)\} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

It is easy to see that for each $(a, b), (c, d) \in X$, $H(T(a, b), T(c, d)) = (0, 0)$, and consequently, according to our definition, T is a k-contraction, for all $k \in \mathcal{K}_+(E)$. However,

$$s\bigl(T(0, 0), T(1, 1)\bigr) = (1, 1) = d\bigl((0, 0), (1, 1)\bigr),$$

and therefore T does not satisfy (4) for $k < 1$.

Given a set-valued mapping $T : X \to B(X)$, we denote by φ_T the mapping from X to E defined as $\varphi_T(x) = d(x, Tx)$.

Proposition 12 *Let $T : X \to B(X)$ be a H-continuous set-valued mapping. Then $\varphi_T \in \mathcal{LS}(X)$.*

Proof Let $u, v \in X$ and $y \in Tv$. Hence,

$$\begin{aligned} d(u, Tu) &\preceq d(u,v) + d(v,y) + d(y, Tu) \\ &\preceq d(u,v) + d(v,y) + H(Tv, Tu). \end{aligned}$$

Consequently, $\varphi_T(u) \preceq \varphi_T(v) + d(u,v) + H(Tu, Tv)$ and from this the lower semicontinuity of φ_T is obtained. □

Corollary 13 *Let $T : X \to B(X)$ be a contraction. Then $\varphi_T \in \mathcal{LS}(X)$.*

Theorem 14 *Let $L : E \to E$ be a positive linear operator, $\delta \in \mathcal{K}_+(E)$, and $T : X \to B(X)$ be a (δ, L)-weak contraction satisfying condition* (S). *Suppose E is d-complete and $\varphi_T \in \mathcal{LS}(X)$. Then there exists $x^* \in X$ such that $x^* \in Tx^*$.*

Proof We have

$$H(Tx, Ty) \preceq \delta d(x,y) + Ld(y, Tx), \quad \text{for all } x, y \in X.$$

Hence, for each $y \in Tx$, we have $H(Tx, Ty) \preceq \delta d(x,y)$. Define $\varphi : X \to E$ as

$$\varphi(x) = \left(\frac{1}{1+\epsilon} - \delta\right)^{-1} \varphi_T(x),$$

where $\epsilon > 0$ is chosen in such a way that $\frac{1}{1+\epsilon} > \delta$. From assumption $\varphi \in \mathcal{LS}(X)$ and since T satisfies condition (S), there exists a selector f_ϵ or T such that for each $x \in X$, $d(x, f_\epsilon(x)) \preceq (1+\epsilon)d(x, Tx)$. Hence, $d(f_\epsilon(x), Tf_\epsilon(x)) \preceq H(Tx, Tf_\epsilon(x)) \preceq \delta d(x, f_\epsilon(x))$ and thus

$$\left(\frac{1}{1+\epsilon} - \delta\right) d\bigl(x, f_\epsilon(x)\bigr) \preceq d(x, Tx) - d\bigl(f_\epsilon(x), Tf_\epsilon(x)\bigr).$$

Consequently, for each $x \in X$, $d(x, f_\epsilon(x)) \preceq \varphi(x) - \varphi(f_\epsilon(x))$, and it follows from Corollary 7 that there exists $x^* \in X$ such that $x^* \in Tx^*$, which concludes the proof. □

Corollary 15 *Suppose E is d-complete and let $T : X \to B(X)$ be a contraction satisfying condition* (S). *Then there exists $x^* \in X$ such that $x^* \in Tx^*$.*

Proof It follows from Corollary 13 and Theorem 14. □

Corollary 16 *Suppose E is d-complete and let $f : X \to X$ be a single-valued contraction. Then there exists $x^* \in X$ such that $x^* = f(x^*)$.*

Remark 17 Since the condition $d(x, Tx) = 0$ does not imply, even if Tx is closed, that $x \in Tx$, it is not possible, in the scenario of cone metric spaces, to prove existence of fixed point for weak contractions, as was done by Berinde and Berinde in [4] for set-valued mapping defined on standard metric spaces. Consequently, Corollary 7 was crucial in the proof of Theorem 14.

Some emblematic and particular cases of standard weak contractions are the Chatterjea [13] and Kannan [14] contractions. Natural extensions of these concepts are obtained for set-valued mappings defined on cone metric spaces. Corollary 18 below shows that, under the usual conditions, for these we have the existence of fixed points.

Corollary 18 *Suppose E is d-complete and let $T : X \to B(X)$ be a set-valued mapping satisfying condition* (S) *and such that $\varphi_T \in \mathcal{LS}(X)$, and at least one of the following two conditions holds*:

(18.1) $H(Tx, Ty) \preceq \alpha[d(x, Tx) + d(y, Ty)]$ (*Kannan condition*) *and*
(18.2) $H(Tx, Ty) \preceq \alpha[d(x, Ty) + d(y, Tx)]$ (*Chatterjea condition*),

where $\alpha : E \to E$ is a linear operator satisfying $2\alpha \in \mathcal{K}_+(E)$.
Then there exists $x \in X$ such that $x \in T(x)$.

Competing interests
The author declares that he has no competing interests.

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