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Positive periodic solution of p-Laplacian Liénard type differential equation with singularity and deviating argument

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Abstract
In this paper, we consider the following p-Laplacian Liénard type differential equation with singularity and deviating argument:

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(t, x(t-\sigma)) = e(t).$$

By applications of coincidence degree theory and some analysis techniques, sufficient conditions for the existence of positive periodic solutions are established.

MSC: 34C25; 34K13; 34K40

Keywords: positive solution; p-Laplacian; Liénard equation; singularity; deviating argument

1 Introduction

In this paper, we consider the following p-Laplacian Liénard type differential equation with singularity and deviating argument:

$$\bigl(\varphi_p\bigl(x'(t)\bigr)\bigr)' + f\bigl(x(t)\bigr)x'(t) + g\bigl(t, x(t-\sigma)\bigr) = e(t), \tag{1.1}$$

where $\varphi_p : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_p(s) = |s|^{p-2}s$, here $p > 1$ is a constant, f is continuous function; g is a continuous function defined on $\mathbb{R}^2$ and periodic in t with $g(t, \cdot) = g(t + T, \cdot)$, g has a singularity at $x = 0$; σ is a constant and $0 \le \sigma < T$; $e : \mathbb{R} \to \mathbb{R}$ are continuous periodic functions with $e(t + T) \equiv e(t)$ and $\int_0^T e(t)\,dt = 0$.

As is well known, the existence of periodic solutions for Liénard type differential equations was extensively studied (see [1–10] and the references therein). In recent years, there also appeared some results on a Liénard type differential equation with singularity; see [11, 12]. In 1996, using coincidence degree theory, Zhang considered the existence of T-periodic solutions for the scalar Liénard equation

$$x''(t) + f\bigl(x(t)\bigr)x'(t) + g\bigl(t, x(t)\bigr) = 0,$$

when g becomes unbounded as $x \to 0^+$. The main emphasis was on the repulsive case, *i.e.* when $g(t, x) \to +\infty$, as $x \to 0^+$. Afterwards, Wang [12] studied the existence of periodic solutions of the Liénard equation with a singularity and a deviating argument,

$$x''(t) + f\big(x(t)\big)x'(t) + g\big(t, x(t-\sigma)\big) = 0,$$

where σ is a constant. When g has a strong singularity at $x = 0$ and satisfies a new small force condition at $x = \infty$, the author proved that the given equation has at least one positive T-periodic solution.

However, the Liénard type differential equation (1.1), in which there is a p-Laplacian Liénard type differential equation, has not attracted much attention in the literature. There are not so many existence results for (1.1) even as regards the p-Laplacian Liénard type differential equation with singularity and deviating argument. In this paper, we try to fill this gap and establish the existence of a positive periodic solution of (1.1) using coincidence degree theory. Our new results generalize in several aspects some recent results contained in [11, 12].

2 Preparation

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider supplementary subspaces X_1, Y_1 of X, Y, respectively, such that $X = \operatorname{Ker} L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y_1$ denote the natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$ and so the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Let K denote the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1 (Gaines and Mawhin [13]) *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;

(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

For the sake of convenience, throughout this paper we will adopt the following notation:

$$|u|_\infty = \max_{t\in[0,T]} \big|u(t)\big|, \qquad |u|_0 = \min_{t\in[0,T]} \big|u(t)\big|,$$

$$|u|_p = \left(\int_0^T |u|^p \, dt\right)^{\frac{1}{p}}, \qquad \bar{h} = \frac{1}{T}\int_0^T h(t)\, dt.$$

Lemma 2.2 ([14]) *If $\omega \in C^1(\mathbb{R}, \mathbb{R})$ and $\omega(0) = \omega(T) = 0$, then*

$$\int_0^T \big|\omega(t)\big|^p \, dt \le \left(\frac{T}{\pi_p}\right)^p \int_0^T \big|\omega'(t)\big|^p \, dt,$$

where $1 \le p < \infty$, $\pi_p = 2\int_0^{(p-1)/p} \frac{ds}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$.

Lemma 2.3 *If $x \in C^1(\mathbb{R},\mathbb{R})$ with $x(t+T) = x(t)$, and $t_0 \in [0,T]$ such that $|x(t_0)| < d$, then*

$$\left(\int_0^T |x(t)|^p\,dt\right)^{\frac{1}{p}} \le \left(\frac{T}{\pi_p}\right)\left(\int_0^T |x'(t)|^p\,dt\right)^{\frac{1}{p}} + dT^{\frac{1}{p}}.$$

Proof Let $\omega(t) = x(t+t_0) - x(t_0)$, and then $\omega(0) = \omega(T) = 0$. By Lemma 2.2 and Minkowski's inequality, we have

$$\begin{aligned}\left(\int_0^T |x(t)|^p\,dt\right)^{\frac{1}{p}} &= \left(\int_0^T |\omega(t) + x(t_0)|^p\,dt\right)^{\frac{1}{p}}\\ &\le \left(\int_0^T |\omega(t)|^p\,dt\right)^{\frac{1}{p}} + \left(\int_0^T |x(t_0)|^p\,dt\right)^{\frac{1}{p}}\\ &\le \left(\frac{T}{\pi_p}\right)\left(\int_0^T |\omega'(t)|^p\,dt\right)^{\frac{1}{p}} + dT^{\frac{1}{p}}\\ &= \left(\frac{T}{\pi_p}\right)\left(\int_0^T |x'(t)|^p\,dt\right)^{\frac{1}{p}} + dT^{\frac{1}{p}}.\end{aligned}$$

This completes the proof of Lemma 2.3. □

In order to apply the topological degree theorem to study the existence of a positive periodic solution for (1.1), we rewrite (1.1) in the form

$$\begin{cases} x_1'(t) = \varphi_q(x_2(t)), \\ x_2'(t) = -f(x_1(t))x_1'(t) - g(t, x_1(t-\sigma)) + e(t), \end{cases} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is an T-periodic solution to (2.1), then $x_1(t)$ must be an T-periodic solution to (1.1). Thus, the problem of finding an T-periodic solution for (1.1) reduces to finding one for (2.1).

Now, set $X = Y = \{x = (x_1(t), x_2(t)) \in C^1(\mathbb{R},\mathbb{R}^2) : x(t+T) \equiv x(t)\}$ with the norm $\|x\| = \max\{|x_1|_\infty, |x_2|_\infty\}$. Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) = \{x \in C^1(\mathbb{R},\mathbb{R}^2) : x(t+T) = x(t), t \in \mathbb{R}\} \subset X \to Y$$

by

$$(Lx)(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1(t))x_1'(t) - g(t, x_1(t-\sigma)) + e(t) \end{pmatrix}. \tag{2.2}$$

Then (2.1) can be converted to the abstract equation $Lx = Nx$. From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^2, \qquad \operatorname{Im} L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q \subset \mathbb{R}^2$ be defined by

$$Px = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix}; \qquad Qy = \frac{1}{T} \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. Let K denote the inverse of $L|_{\operatorname{Ker} p \cap D(L)}$. It is easy to see that $\operatorname{Ker} L = \operatorname{Im} Q = \mathbb{R}^2$ and

$$[Ky](t) = \int_0^T G(t,s) y(s)\, ds,$$

where

$$G(t,s) = \begin{cases} \frac{s}{T}, & 0 \le s < t \le T; \\ \frac{s-t}{T}, & 0 \le t \le s \le T. \end{cases} \tag{2.3}$$

From (2.2) and (2.3), it is clear that QN and $K(I - Q)N$ are continuous, $QN(\overline{\Omega})$ is bounded and then $K(I - Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means N is L-compact on $\overline{\Omega}$.

3 Main results

Assume that

$$\psi(t) = \lim_{x \to +\infty} \sup \frac{g(t,x)}{x^{p-1}} \tag{3.1}$$

exists uniformly a.e. $t \in [0, T]$, *i.e.*, for any $\varepsilon > 0$ there is $g_\varepsilon \in L^2(0, T)$ such that

$$g(t,x) \le \big(\psi(t) + \varepsilon\big)x + g_\varepsilon(t), \tag{3.2}$$

for all $x > 0$ and a.e. $t \in [0, T]$. Moreover, $\psi \in C(\mathbb{R}, \mathbb{R})$ and $\psi(t + T) = \psi(t)$.

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

(H_1) (Balance condition) There exist constants $0 < D_1 < D_2$ such that if x is a positive continuous T-periodic function satisfying

$$\int_0^T g\big(t, x(t)\big)\, dt = 0,$$

then

$$D_1 \le x(\tau) \le D_2,$$

for some $\tau \in [0, T]$.

(H_2) (Degree condition) $\bar{g}(x) < 0$ for all $x \in (0, D_1)$, and $\bar{g}(x) > 0$ for all $x > D_2$.

(H_3) (Decomposition condition) $g(t,x) = g_0(x) + g_1(t,x)$, where $g_0 \in C((0,\infty);\mathbb{R})$ and $g_1 : [0,T] \times [0,\infty) \to \mathbb{R}$ is an L^2-Carathéodory function, *i.e.* it is measurable in the first variable and continuous in the second variable, and for any $b > 0$ there is $h_b \in L^2(0,T;\mathbb{R}_+)$ such that

$$\left|g_1(t,x)\right| \le h_b(t), \quad \text{a.e. } t \in [0,T], \forall 0 \le x \le b.$$

(H_4) (Strong force condition at $x = 0$) $\int_0^1 g_0(x)\,dx = -\infty$.

Theorem 3.1 *Assume that conditions* (H_1)-(H_4) *hold. Suppose the following condition is satisfied*:

(H_5) $(\frac{T}{\pi_p})^p |\psi|_\infty < 1$.

Then (1.1) *has at least one positive* T*-periodic solution.*

Proof Consider the equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1).$$

Set $\Omega_1 = \{x : Lx = \lambda Nx, \lambda \in (0,1)\}$. If $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} x_1'(t) = \lambda\varphi_q(x_2(t)), \\ x_2'(t) = -\lambda f(x_1(t))x_1'(t) - \lambda g(t, x_1(t-\sigma)) + \lambda e(t). \end{cases} \tag{3.3}$$

Substituting $x_2(t) = \frac{1}{\lambda^{p-1}}\varphi_p(x_1'(t))$ into the second equation of (3.3)

$$\left(\varphi_p\left(x_1'(t)\right)\right)' + \lambda^p f\left(x_1(t)\right)x_1'(t) + \lambda^p g\left(t, x_1(t-\sigma)\right) = \lambda^p e(t). \tag{3.4}$$

Integrating both sides of (3.4) over $[0,T]$, we have

$$\int_0^T g\left(t, x_1(t-\sigma)\right) dt = 0. \tag{3.5}$$

From (H_1), there exist positive constants D_1, D_2, and $\xi \in [0,T]$ such that

$$D_1 \le x_1(\xi) \le D_2. \tag{3.6}$$

Then we have

$$\left|x_1(t)\right| = \left|x_1(\xi) + \int_\xi^t x_1'(s)\,ds\right| \le D_2 + \int_\xi^t \left|x_1'(s)\right| ds, \quad t \in [\xi, \xi + T],$$

and

$$\left|x_1(t)\right| = \left|x_1(t-T)\right| = \left|x_1(\xi) - \int_{t-T}^\xi x_1'(s)\,ds\right| \le D_2 + \int_{t-T}^\xi \left|x_1'(s)\right| ds, \quad t \in [\xi, \xi + T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned}|x_1|_\infty &= \max_{t\in[0,T]}\bigl|x_1(t)\bigr| = \max_{t\in[\xi,\xi+T]}\bigl|x_1(t)\bigr| \\ &\le \max_{t\in[\xi,\xi+T]}\left\{D_2 + \frac{1}{2}\left(\int_\xi^t \bigl|x_1'(s)\bigr|\,ds + \int_{t-T}^{\xi}\bigl|x_1'(s)\bigr|\,ds\right)\right\} \\ &\le D_2 + \frac{1}{2}\int_0^T \bigl|x_1'(s)\bigr|\,ds. \end{aligned} \tag{3.7}$$

Multiplying both sides of (3.4) by $x_1(t)$ and integrating over the interval $[0, T]$, we get

$$\begin{aligned}&\int_0^T \bigl(\varphi_p\bigl(x_1'(t)\bigr)\bigr)' x_1(t)\,dt + \lambda^p \int_0^T f\bigl(x_1(t)\bigr)x_1'(t)x_1(t)\,dt + \lambda^p \int_0^T g\bigl(t, x_1(t-\sigma)\bigr)x_1(t)\,dt \\ &\quad = \lambda^p \int_0^T e(t)x_1(t)\,dt. \end{aligned} \tag{3.8}$$

Substituting $\int_0^T (\varphi_p(x_1'(t)))' x_1(t)\,dt = -\int_0^T |x_1'(t)|^p\,dt$, $\int_0^T f(x_1(t))x_1'(t)x_1(t)\,dt = 0$ into (3.8), we have

$$\int_0^T \bigl|x_1'(t)\bigr|^p d = \lambda^p \int_0^T g\bigl(t, x_1(t-\sigma)\bigr)x_1(t)\,dt - \lambda^p \int_0^T e(t)x_1(t)\,dt. \tag{3.9}$$

For any $\varepsilon > 0$, there exists a function $g_\varepsilon \in L^2(0, T)$ such that (3.2) holds. Since $x_1(t) > 0$, $t \in [0, T]$, it follows from (3.4) that

$$g\bigl(t, x_1(t-\sigma)\bigr)x_1(t) \le \bigl(\psi(t) + \varepsilon\bigr)x_1^{p-1}(t-\sigma)x_1(t) + g_\varepsilon(t)x_1(t). \tag{3.10}$$

We infer from (3.9) and (3.10)

$$\begin{aligned}&\int_0^T \bigl|x_1'(t)\bigr|^p\,dt \\ &\quad\le \lambda^p \int_0^T \bigl(\psi(t) + \varepsilon\bigr)x_1^{p-1}(t-\sigma)x_1(t)\,dt + \lambda^p \int_0^T \bigl(g_\varepsilon(t) + e(t)\bigr)x_1(t)\,dt \\ &\quad\le \int_0^T \bigl(\bigl|\psi(t)\bigr| + \varepsilon\bigr)\bigl|x_1^{p-1}(t-\sigma)\bigr|\bigl|x_1(t)\bigr|\,dt + \int_0^T \bigl(\bigl|g_\varepsilon(t)\bigr| + \bigl|e(t)\bigr|\bigr)\bigl|x_1(t)\bigr|\,dt \\ &\quad\le \bigl(|\psi|_\infty + \varepsilon\bigr)\left(\int_0^T \bigl|x_1(t-\sigma)\bigr|^p\,dt\right)^{\frac{p-1}{p}}\left(\int_0^T \bigl|x_1(t)\bigr|^p\,dt\right)^{\frac{1}{p}} \\ &\qquad + |x_1|_\infty\left(\int_0^T \bigl|g_\varepsilon(t)\bigr|\,dt + \int_0^T \bigl|e(t)\bigr|\,dt\right) \\ &\quad\le \bigl(|\psi|_\infty + \varepsilon\bigr)\left(\int_0^T \bigl|x_1(t)\bigr|^p\,dt\right) + |x_1|_\infty\left(\int_0^T \bigl|g_\varepsilon(t)\bigr|\,dt + \int_0^T \bigl|e(t)\bigr|\,dt\right). \end{aligned} \tag{3.11}$$

From Lemma 2.3 and (3.7), we have

$$\left(\int_0^T \bigl|x_1(t)\bigr|^p\right)^{\frac{1}{p}} \le \left(\frac{T}{\pi_p}\right)\left(\int_0^T \bigl|x_1'(t)\bigr|^p\,dt\right)^{\frac{1}{p}} + D_2 T^{\frac{1}{p}}. \tag{3.12}$$

Substituting (3.7), (3.12) into (3.11), we get

$$\begin{aligned}
&\int_0^T \left|x_1'(t)\right|^p dt \\
&\quad\le \left(|\psi|_\infty+\varepsilon\right)\left(\left(\frac{T}{\pi_p}\right)\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{1}{p}} + D_2 T^{\frac{1}{p}}\right)^p \\
&\qquad + \left(D_2 + \frac{1}{2}\int_0^T \left|x_1'(t)\right| dt\right)\left(\int_0^T \left|g_\varepsilon(t)\right| dt + \int_0^T \left|e(t)\right| dt\right) \\
&\quad\le \left(|\psi|_\infty+\varepsilon\right)\left(\left(\frac{T}{\pi_p}\right)^p \int_0^T \left|x_1'(t)\right|^p dt\right. \\
&\qquad \left. + p\left(\frac{T}{\pi_p}\right)^{p-1}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{p-1}{p}} D_2 T^{\frac{1}{p}} + \cdots + D_2^p T\right) \\
&\qquad + \left(D_2 + \frac{1}{2}T^{\frac{1}{q}}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{1}{p}}\right)\left(T^{\frac{1}{2}}\left(|g_\varepsilon|_2 + |e|_2\right)\right) \\
&\quad= \left(|\psi|_\infty+\varepsilon\right)\left(\frac{T}{\pi_p}\right)^p \int_0^T \left|x_1'(t)\right|^p dt \\
&\qquad + \left(|\psi|_\infty+\varepsilon\right)p\left(\frac{T}{\pi_p}\right)^{p-1}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{p-1}{p}} D_2 T^{\frac{1}{p}} + \cdots \\
&\qquad + \frac{1}{2}T^{\frac{1}{q}+\frac{1}{2}}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{1}{p}}\left(|g_\varepsilon|_2 + |e|_2\right) \\
&\qquad + \left(|\psi|_\infty+\varepsilon\right)D_2^p T + T^{\frac{1}{2}} D_2\left(|g_\varepsilon|_2 + |e|_2\right),
\end{aligned} \tag{3.13}$$

where $|g_\varepsilon|_2 = (\int_0^T |g_\varepsilon(t)|^2\,dt)^{\frac{1}{2}}$. Since ε is sufficiently small, from (H_5) we know that $(\frac{T}{\pi_p})^p|\psi|_\infty < 1$. So, it is easy to see that there exists a positive constant M_1' such that

$$\int_0^T \left|x_1'(t)\right|^p dt \le M_1'.$$

From (3.7), we have

$$\begin{aligned}
|x_1|_\infty &\le D_2 + \frac{1}{2}\int_0^T \left|x_1'(t)\right| dt \\
&\le D_2 + \frac{T^{\frac{1}{q}}}{2}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{1}{p}} \\
&\le D_2 + \frac{T^{\frac{1}{q}}}{2}\left(M_1'\right)^{\frac{1}{p}} := M_1.
\end{aligned} \tag{3.14}$$

Write

$$I_+ = \left\{t \in [0,T] : g\left(t, x_1(t-\sigma)\right) \ge 0\right\}; \qquad I_- = \left\{t \in [0,T] : g\left(t, x_1(t-\sigma)\right) \le 0\right\}.$$

Then we get from (3.2) and (3.6)

$$\begin{aligned}\int_0^T \left|g\left(t,x_1(t-\sigma)\right)\right| dt &= \int_{I_+} g\left(t,x_1(t-\sigma)\right) dt - \int_{I_-} g\left(t,x_1(t-\sigma)\right) dt\\ &= 2\int_{I_+} g\left(t,x_1(t-\sigma)\right) dt\\ &\le 2\int_{I_+} \left(\left(\psi(t)+\varepsilon\right)x_1^{p-1}(t-\sigma)+g_\varepsilon(t)\right) dt\\ &\le 2\left(|\psi|_\infty+\varepsilon\right)\int_0^T \left|x_1(t)\right|^{p-1} dt + 2\int_0^T \left|g_\varepsilon(t)\right| dt\\ &\le 2\left(|\psi|_\infty+\varepsilon\right)TM_1^{p-1}+2\sqrt{T}|g_\varepsilon|_2. \end{aligned} \tag{3.15}$$

By the second equations of (3.3) and (3.15), we obtain

$$\begin{aligned}&\int_0^T \left|x_2'(t)\right| dt\\ &\quad\le \lambda\int_0^T \left|f\left(x_1(t)\right)\right|\left|x_1'(t)\right| dt + \lambda\int_0^T \left|g\left(t,x_1(t-\sigma)\right)\right| dt + \lambda\int_0^T \left|e(t)\right| dt\\ &\quad\le \lambda|f|_{M_1}T^{\frac{1}{q}}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{1}{p}} + \lambda\left(2\left(|\psi|_\infty+\varepsilon\right)TM_1^{p-1}+2\sqrt{T}|g_\varepsilon|_2\right)+\lambda\sqrt{T}|e|_2\\ &\quad\le \lambda|f|_{M_1}T^{\frac{1}{q}}\left(M_1'\right)^{\frac{1}{p}} + \lambda\left(2\left(|\psi|_\infty+\varepsilon\right)TM_1^{p-1}+2\sqrt{T}|g_\varepsilon|_2\right)+\lambda\sqrt{T}|e|_2\\ &\quad:= \lambda M_2', \end{aligned} \tag{3.16}$$

where $|f|_{M_1} = \max_{0<x_1\le M_1} |f(x_1(t))|$. By the first equation of (3.3), we have

$$\int_0^T \left|x_2(s)\right|^{q-2} x_2(s)\, ds = 0,$$

which implies that there is a constant $t_2 \in [0,T]$ such that $x_2(t_2) = 0$, so

$$|x_2|_\infty \le \frac{1}{2}\int_0^{t_2} \left|x_2'(s)\right| ds \le \frac{1}{2}\int_0^T \left|x_2'(s)\right| ds \le \frac{\lambda}{2}M_2' := \lambda M_2. \tag{3.17}$$

On the other hand, it follows from (3.4) that

$$\left(\varphi_p\left(x_1'(t+\sigma)\right)\right)' + \lambda^p\left(f\left(x_1(t+\sigma)\right)x_1'(t+\sigma)+g\left(t+\sigma,x_1(t)\right)\right) = \lambda^p e(t+\sigma). \tag{3.18}$$

Namely,

$$\begin{aligned}&\left(\varphi_p\left(x_1'(t+\sigma)\right)\right)' + \lambda^p f\left(x_1(t+\sigma)\right)x_1'(t+\sigma)\\ &\qquad + \lambda^p g_0\left(x_1(t)\right) + g_1\left(t+\sigma,x_1(t)\right) = \lambda^p e(t+\sigma). \end{aligned} \tag{3.19}$$

Multiplying both sides of (3.19) by $x_1'(t)$, we get

$$\begin{aligned}&\left(\varphi_p\left(x_1'(t+\sigma)\right)\right)' x_1'(t) + \lambda^p f\left(x_1(t+\sigma)\right)x_1'(t+\sigma)x_1'(t)\\ &\qquad + \lambda^p g_0\left(x_1(t)\right)x_1'(t) + \lambda^p g_1\left(t+\sigma,x_1(t)\right)x_1'(t) = \lambda^p e(t+\sigma)x_1'(t). \end{aligned} \tag{3.20}$$

Let $\tau \in [0,T]$, for any $\tau \le t \le T$, we integrate (3.20) on $[\tau, t]$ and get

$$\begin{aligned}\lambda^p \int_{x_1(\tau)}^{x_1(t)} g_0(u)\,du &= \lambda^p \int_\tau^t g_0\bigl(x_1(s)\bigr)x_1'(s)\,ds \\ &= -\int_\tau^t \bigl(\varphi_p\bigl(x_1'(s+\sigma)\bigr)\bigr)'x_1'(s)\,ds - \lambda^p \int_\tau^t f\bigl(x_1(s+\sigma)\bigr)x_1'(s+\sigma)x_1'(s)\,ds \\ &\quad - \lambda^p \int_\tau^t g_1\bigl(s+\sigma, x_1(s)\bigr)x_1'(s)\,ds + \lambda^p \int_\tau^t e(s+\sigma)x_1'(s)\,ds. \end{aligned} \tag{3.21}$$

By (3.14), (3.15), (3.16), (3.17), and (3.18), we have

$$\begin{aligned}&\left|\int_\tau^t \bigl(\varphi_p\bigl(x_1'(t+\sigma)\bigr)\bigr)'x_1'(s)\,ds\right| \\ &\quad\le \int_\tau^t \bigl|\bigl(\varphi_p\bigl(x_1'(t+\sigma)\bigr)\bigr)'\bigr|\bigl|x_1'(s)\bigr|\,ds \\ &\quad\le \bigl|x_1'\bigr|_\infty \int_0^T \bigl|\bigl(\varphi_p\bigl(x_1'(t+\sigma)\bigr)\bigr)'\bigr|\,dt \\ &\quad\le \lambda^p \bigl|x_1'\bigr|_\infty \left(\int_0^T \bigl|f\bigl(x_1(t)\bigr)\bigr|\bigl|x_1'(t)\bigr|\,dt + \int_0^T \bigl|g\bigl(t, x_1(t-\sigma)\bigr)\bigr|\,dt + \int_0^T \bigl|e(t)\bigr|\,dt\right) \\ &\quad\le \lambda^p M_2^{p-1}\bigl(|f|_{M_1} M_1'^{\frac{1}{p}} T^{\frac{1}{q}} + 2\bigl(|\psi|_\infty + \varepsilon\bigr)TM_1^{p-1} + 2T^{\frac{1}{2}}\bigl|g_\varepsilon^+\bigr|_2 + T^{\frac{1}{2}}|e|_2\bigr).\end{aligned}$$

We have

$$\begin{aligned}\left|\int_\tau^t f\bigl(x_1(s+\sigma)\bigr)x_1'(s+\sigma)x_1'(s)\,ds\right| &\le |f|_{M_1}\left(\int_0^T \bigl|x_1'(s)\bigr|\,ds\right)^2 \\ &\le |f|_{M_1} T^{\frac{2}{q}}\left(\int_0^T \bigl|x_1'(t)\bigr|^p\,dt\right)^{\frac{2}{p}} \\ &\le |f|_{M_1} T^{\frac{2}{q}}\bigl(M_1'\bigr)^{\frac{2}{p}},\end{aligned}$$

$$\left|\int_\tau^t g\bigl(s+\sigma, x_1(s)\bigr)x_1'(s)\,ds\right| \le \bigl|x_1'\bigr| \int_0^T \bigl|g\bigl(t, x(t-\sigma)\bigr)\bigr|\,dt \le M_2^{p-1}\sqrt{T}|g_{M_1}|_2,$$

where $g_{M_1} = \max_{0\le x\le M_1} |g_1(t,x)| \in L^2(0,T)$ is as in (H_3). We have

$$\left|\int_\tau^t e(t+\sigma)x_1'(t)\,dt\right| \le M_2^{p-1} T^{\frac{1}{2}}|e|_2.$$

From these inequalities we can derive from (3.21) that

$$\left|\int_{x_1(\tau)}^{x_1(t)} g_0(u)\,du\right| \le M_3', \tag{3.22}$$

for some constant M_3' which is independent on λ, x, and t. In view of the strong force condition (H_4), we know that there exists a constant $M_3 > 0$ such that

$$x_1(t) \ge M_3, \quad \forall t \in [\tau, T]. \tag{3.23}$$

The case $t \in [0,\tau]$ can be treated similarly.

From (3.14), (3.17), and (3.23), we let

$$\Omega = \{x = (x_1, x_2)^\top : E_1 \le |x_1|_\infty \le E_2, |x_2|_\infty \le E_3, \forall t \in [0, T]\},$$

where $0 < E_1 < \min(M_3, D_1)$, $E_2 > \max(M_1, D_2)$, $E_3 > M_2$. $\Omega_2 = \{x : x \in \partial\Omega \cap \operatorname{Ker} L\}$ then $\forall x \in \partial\Omega \cap \operatorname{Ker} L$

$$QNx = \frac{1}{T}\int_0^T \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1(t))x_1'(t) - g(t, x_1(t-\sigma)) + e(t) \end{pmatrix} dt.$$

If $QNx = 0$, then $x_2(t) = 0$, $x_1 = E_2$ or $-E_2$. But if $x_1(t) = E_2$, we know

$$0 = \int_0^T \{g(t, E_2) - e(t)\} \, dt.$$

From assumption (H_2), we have $x_1(t) \le D_2 \le E_2$, which yields a contradiction. Similarly if $x_1 = -E_2$. We also have $QNx \ne 0$, *i.e.*, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$, $x \notin \operatorname{Im} L$, so conditions (1) and (2) of Lemma 2.1 are both satisfied. Define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ as follows:

$$J(x_1, x_2)^\top = (x_2, -x_1)^\top.$$

Let $H(\mu, x) = -\mu x + (1-\mu)JQNx$, $(\mu, x) \in [0,1] \times \Omega$, then $\forall (\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$,

$$H(\mu, x) = \begin{pmatrix} -\mu x_1 - \frac{1-\mu}{T}\int_0^T [g(t, x_1) - e(t)] \, dt \\ -\mu x_2 - (1-\mu)|x_2|^{p-2}x_2 \end{pmatrix}.$$

We have $\int_0^T e(t) \, dt = 0$. So, we can get

$$H(\mu, x) = \begin{pmatrix} -\mu x_1 - \frac{1-\mu}{T}\int_0^T g(t, x_1) \, dt \\ -\mu x_2 - (1-\mu)|x_2|^{p-2}x_2 \end{pmatrix}, \quad \forall (\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L).$$

From (H_2), it is obvious that $x^\top H(\mu, x) < 0$, $\forall (\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$. Hence

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\{H(0, x), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{H(1, x), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \ne 0. \end{aligned}$$

So condition (3) of Lemma 2.1 is satisfied. By applying Lemma 2.1, we conclude that the equation $Lx = Nx$ has a solution $x = (x_1, x_2)^\top$ on $\bar{\Omega} \cap D(L)$, *i.e.*, (2.1) has an T-periodic solution $x_1(t)$. □

Finally, we present an example to illustrate our result.

Example 3.1 Consider the p-Laplacian Liénard type differential equation with singularity and deviating argument:

$$\big(\varphi_p\big(x'(t)\big)\big)' + f\big(x(t)\big)x'(t) + \frac{1}{5}(\cos 2t + 2)x^3(t-\sigma) - \frac{1}{x^\kappa(t-\sigma)} = \sin 2t, \tag{3.24}$$

where $\kappa \geq 1$ and $p = 4$, f is a continuous function, σ is a constant, and $0 \leq \sigma < T$.

It is clear that $T = \pi$, $g(t,x) = \frac{1}{5}(\cos 2t + 2)x^3(t-\sigma) - \frac{1}{x^\kappa(t-\sigma)}$, $\psi(t) = \frac{1}{5}(\cos 2t + 2)$. It is obvious that (H_1)-(H_4) hold. Now we consider the assumption (H_5). Since $|\psi|_\infty \leq \frac{3}{5}$, we have

$$\left(\frac{T}{\pi_p}\right)^p |\psi|_\infty = \left(\frac{T}{\frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}}\right)^p |\psi|_\infty \leq \left(\frac{\pi}{\frac{2\pi(4-1)^{1/4}}{4\sin\pi/4}}\right)^4 \times \frac{3}{5} = \frac{4}{5} < 1.$$

So by Theorem 3.1, we know (3.24) has at least one positive π-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YX and ZBC worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

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