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Some oscillation results for nonlinear second-order differential equations with damping

Cuimei Jiang, Yazhou Tian, Ying Jiang and Tongxing Li*

*Correspondence:
litongx2007@163.com
Qingdao Technological University, Feixian, Shandong 273400, P.R. China

Abstract

The aim of this paper is to investigate oscillatory properties of a class of second-order nonlinear differential equations with damping. Employing the generalized Riccati transformation and a class of functions, several oscillation criteria are presented that improve the results obtained in the literature. Two examples are presented to demonstrate the main results.

MSC: 34K11

Keywords: oscillation; second-order differential equation; damping term; generalized Riccati transformation

1 Introduction

During the past few decades, the oscillation of differential equations has attracted a great deal of interest in various fields due to its theoretical and practical applications in natural sciences and technology. For instance, the oscillation of a building or a machine, the beam vibration in a synchrotron accelerator, the complicated oscillation in a chemical reaction, and so on; see, *e.g.*, [1, 2]. For some related contributions on the oscillation of various classes of differential equations, we refer the reader to [3–27] and the references cited therein. In particular, the oscillatory behavior of second-order damped differential equations has been studied by many authors due to the fact that such equations arise in the study of noise, vibration, and harshness (NVH) of vehicles, see, *e.g.*, the paper by Fu *et al.* [4].

In this paper, we are concerned with the oscillation of a nonlinear second-order damped differential equation

$$\bigl(r(t)\psi\bigl(x(t)\bigr)f\bigl(x'(t)\bigr)\bigr)' + h(t)f\bigl(x'(t)\bigr) + q(t)g\bigl(x(t)\bigr) = H\bigl(t,x'(t),x(t)\bigr), \quad t \geq t_0, \tag{1.1}$$

where $h,q \in C([t_0,\infty),\mathbb{R})$, $\psi,f,g \in C(\mathbb{R},\mathbb{R})$, and $H \in C([t_0,\infty)\times\mathbb{R}^2,\mathbb{R})$. We suppose also that the following hypotheses are satisfied:

(H$_1$) $r \in C([t_0,\infty),(0,\infty))$;
(H$_2$) $0 < k_1 \leq \psi(x) \leq k_2$ for all $x \neq 0$;
(H$_3$) there exists a constant $m > 0$ such that $f^2(y) \leq myf(y)$ for all $y \in \mathbb{R}$;
(H$_4$) $H(t,y,x)/g(x) \leq p(t)$ for $t \in [t_0,\infty)$, $x,y \in \mathbb{R}$, $x \neq 0$, and $p \in C([t_0,\infty),\mathbb{R})$.

As usual, a solution x of (1.1) is called oscillatory if the set of its zeros is unbounded from above; otherwise, it is said to be nonoscillatory. Equation (1.1) is termed oscillatory if all its solutions are oscillatory.

In what follows, we present some background details that motivate the study of this paper. Grace and Lalli [6], Kirane and Rogovchenko [7], Li and Agarwal [8], Rogovchenko [9], Rogovchenko and Tuncay [10], and Sun [23] considered a particular case of (1.1), namely, the second-order damped equation

$$\bigl(r(t)x'(t)\bigr)' + h(t)x'(t) + q(t)g\bigl(x(t)\bigr) = 0. \tag{1.2}$$

In particular, Sun [23] used a class of functions Y defined in the sequel to establish several Kamenev-type (see [25]) oscillation criteria for (1.2). Grace [11], Grace and Lalli [17], Kirane and Rogovchenko [18], Manojlović [19], Rogovchenko and Tuncay [20], and Tunç and Avci [21] investigated the oscillation of the nonlinear damped differential equations

$$\bigl(r(t)\psi\bigl(x(t)\bigr)x'(t)\bigr)' + h(t)x'(t) + q(t)g\bigl(x(t)\bigr) = 0 \tag{1.3}$$

and

$$\bigl(r(t)\psi\bigl(x(t)\bigr)f\bigl(x'(t)\bigr)\bigr)' + h(t)f\bigl(x'(t)\bigr) + q(t)g\bigl(x(t)\bigr) = 0. \tag{1.4}$$

Very recently, Salhin *et al.* [22] established several oscillation criteria for (1.1) by using a generalized Riccati transformation, some of which we present below for convenience of the reader.

Theorem 1.1 ([22], Corollary 2.1) *Let assumptions* (H_1)-(H_4) *be fulfilled and assume*

(H_5) $g'(x)$ *exists and* $g'(x) \ge k > 0$ *for all* $x \ne 0$.

If there exist functions $\delta, \tilde{\phi} \in C([t_0,\infty),\mathbb{R})$ *such that* $(r\delta) \in C^1([t_0,\infty),\mathbb{R})$,

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{\tilde{\phi}_+^2(s)}{\tilde{\rho}(s)r(s)}\,ds = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_T^t \left((t-s)^{n-1}\widetilde{Q}(s) - \frac{\alpha m k_2}{4k}\tilde{\rho}(s)r(s)(t-s)^{n-3} \right) ds \ge \tilde{\phi}(T)$$

for any $T \ge t_0$ *and* $\alpha > 1$, *where*

$$\begin{aligned}
\widetilde{Q}(t) &= \tilde{\rho}(t)\biggl\{ q(t) - p(t) + \frac{k}{mk_2}r(t)\delta^2(t) - \bigl(r(t)\delta(t)\bigr)' \\
&\qquad - \frac{1}{k_2}h(t)\delta(t) - \frac{m}{4k}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\frac{h^2(t)}{r(t)} \biggr\}, \\
\tilde{\rho}(t) &= \exp\left(-\frac{2}{k_2}\int^t \left(\frac{k\delta(s)}{m} - \frac{h(s)}{2r(s)} \right) ds \right),
\end{aligned}$$

and $\tilde{\phi}_+(t) = \max\{\tilde{\phi}(t), 0\}$, *then* (1.1) *is oscillatory.*

Theorem 1.2 ([22], Corollary 2.2) *Let assumptions* (H_1)-(H_5) *hold. Suppose that there exists a function* $\delta \in C([t_0,\infty),\mathbb{R})$ *such that* $(r\delta) \in C^1([t_0,\infty),\mathbb{R})$ *and*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \left((t-s)^{n-1} \left(\widetilde{Q}(s) - \frac{mh^2(s)\tilde{\rho}(s)}{2kk_2 r(s)} \right) \right. \\ \left. - \frac{\alpha m k_2 (n-1)^2}{2k} \tilde{\rho}(s) r(s) (t-s)^{n-3} \right) ds = \infty$$

for some integer $n > 2$ *and* $\alpha \geq 1$, *where*

$$\tilde{\rho}(t) = \exp\left(-\frac{2}{mk_2} \int^{t} k\delta(s)\, ds \right)$$

and $\widetilde{Q}$ *is defined as in Theorem* 1.1. *Then* (1.1) *is oscillatory.*

Equations (1.2), (1.3), and (1.4) are special cases of (1.1). Note that Theorems 1.1 and 1.2 are Kamenev-type or Philos-type (see [26]) criteria for (1.1). The natural question now is: Can one apply methods reported in [23] to (1.1) and improve Theorems 1.1 and 1.2? The objective of this paper is to give an affirmative answer to this question.

Now, we introduce a class of functions Y. Let $\mathbb{E} = \{(t,s,l) : t_0 \leq l \leq s \leq t < \infty\}$. We say that a function $\Phi \in C(\mathbb{E},\mathbb{R})$ belongs to Y, denoted by $\Phi \in Y$, if

(i) $\Phi(t,t,l) = 0$, $\Phi(t,l,l) = 0$, and $\Phi(t,s,l) \neq 0$ for $l < s < t$;

(ii) Φ has the partial derivative $\partial\Phi/\partial s$ in $\mathbb{E}$ such that $\partial\Phi/\partial s$ is locally integrable with respect to s in $\mathbb{E}$ and satisfies

$$\frac{\partial \Phi(t,s,l)}{\partial s} = \phi(t,s,l)\Phi(t,s,l). \tag{1.5}$$

Next, we define the operator $A[\cdot;l,t]$ by

$$A[g;l,t] = \int_{l}^{t} \Phi^2(t,s,l) g(s)\, ds \quad \text{for } t \geq s \geq l \geq t_0 \text{ and } g \in C\bigl([t_0,\infty),\mathbb{R}\bigr). \tag{1.6}$$

It is obvious that $A[\cdot;l,t]$ is a linear operator and satisfies

$$A\bigl[g';l,t\bigr] = -2A[g\phi;l,t] \quad \text{for } g \in C^1\bigl([t_0,\infty),\mathbb{R}\bigr). \tag{1.7}$$

In what follows, all functional inequalities are assumed to hold for all t large enough, unless mentioned otherwise.

2 Oscillation criteria for increasing *g*

Theorem 2.1 *Assume conditions* (H_1)-(H_5) *hold. Equation* (1.1) *is oscillatory provided that, for each* $l \geq t_0$, *there exist three functions* $\Phi \in Y$, $\rho \in C^1([t_0,\infty),(0,\infty))$, *and* $b \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} A\left[\rho(s)Q(s) - \frac{2mk_2}{k} r(s)\rho(s)\phi^2; l,t \right] > 0, \tag{2.1}$$

where

$$Q(t) = q(t) - p(t) - b'(t) - \frac{m}{4k}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\frac{h^2(t)}{r(t)} + \frac{kb^2(t)}{mk_2 r(t)} - \frac{b(t)h(t)}{k_2 r(t)} - \frac{mk_2}{2k} r(t)a^2(t),$$
$$a(t) = \frac{\rho'(t)}{\rho(t)} + \frac{2kb(t)}{mk_2 r(t)} - \frac{h(t)}{k_2 r(t)},$$

$\phi = \phi(t,s,l)$ *and* A *are defined by* (1.5) *and* (1.6), *respectively.*

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may suppose that $x(t) > 0$ for $t \geq t_1 \geq t_0$. A similar argument holds for the case when x is eventually negative. Define a generalized Riccati transformation w by

$$w(t) = \rho(t)\left(\frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))} + b(t)\right), \quad t \geq t_1. \tag{2.2}$$

Differentiating (2.2) and using (1.1), we have

$$w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t)b'(t) + \rho(t)\left(\frac{H(t,x'(t),x(t))}{g(x(t))} - \frac{h(t)f(x'(t))}{g(x(t))} - q(t)\right)$$
$$- \rho(t)\frac{r(t)\psi(x(t))x'(t)f(x'(t))g'(x(t))}{g^2(x(t))}.$$

By virtue of (H_1) and (H_3)-(H_5),

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t)b'(t) + \rho(t)\left(p(t) - q(t) - \frac{h(t)f(x'(t))}{g(x(t))}\right)$$
$$- \frac{k}{m}\rho(t)\frac{r(t)\psi(x(t))f^2(x'(t))}{g^2(x(t))}$$
$$= \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t)\bigl(b'(t) + p(t) - q(t)\bigr) + \frac{m\rho(t)h^2(t)}{4kr(t)\psi(x(t))} - \frac{\rho(t)}{\psi(x(t))}$$
$$\times \left(\sqrt{\frac{kr(t)}{m}}\frac{\psi(x(t))f(x'(t))}{g(x(t))} + \frac{h(t)}{2}\sqrt{\frac{m}{kr(t)}}\right)^2.$$

From (H_2) and (2.2), we conclude that

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t)\bigl(b'(t) + p(t) - q(t)\bigr) + \frac{m\rho(t)h^2(t)}{4kk_1 r(t)} - \frac{1}{k_2}\rho(t)$$
$$\times \left(\sqrt{\frac{kr(t)}{m}}\left(\frac{w(t)}{r(t)\rho(t)} - \frac{b(t)}{r(t)}\right) + \frac{h(t)}{2}\sqrt{\frac{m}{kr(t)}}\right)^2$$
$$= \rho(t)\left(b'(t) + p(t) - q(t) + \frac{m}{4k}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\frac{h^2(t)}{r(t)} - \frac{kb^2(t)}{mk_2 r(t)} + \frac{b(t)h(t)}{k_2 r(t)}\right)$$
$$+ a(t)w(t) - \frac{kw^2(t)}{mk_2 r(t)\rho(t)}.$$

Using the inequality

$$-az^2 + bz \leq -\frac{a}{2}z^2 + \frac{b^2}{2a}, \quad a > 0, b, z \in \mathbb{R},$$

we get, for $t \geq t_1$,

$$w'(t) \leq -\rho(t)Q(t) - \frac{kw^2(t)}{2mk_2 r(t)\rho(t)}. \tag{2.3}$$

Applying $A[\cdot; l, t]$ $(t \geq l \geq t_1)$ to (2.3), we obtain

$$A\big[w'(s); l, t\big] \leq A\left[-\rho(s)Q(s) - \frac{kw^2(s)}{2mk_2 r(s)\rho(s)}; l, t\right].$$

Combining (1.7) and the latter inequality, we have, for $t \geq l \geq t_1$,

$$\begin{aligned}
&A\big[\rho(s)Q(s); l, t\big] \\
&\quad \leq A\left[2w(s)\phi - \frac{kw^2(s)}{2mk_2 r(s)\rho(s)}; l, t\right] \\
&\quad = A\left[\frac{2mk_2}{k} r(s)\rho(s)\phi^2 - \left(\sqrt{\frac{k}{2mk_2 r(s)\rho(s)}} w(s) - \sqrt{\frac{2mk_2 r(s)\rho(s)}{k}} \phi\right)^2; l, t\right] \\
&\quad \leq A\left[\frac{2mk_2}{k} r(s)\rho(s)\phi^2; l, t\right].
\end{aligned}$$

Hence

$$A\left[\rho(s)Q(s) - \frac{2mk_2}{k} r(s)\rho(s)\phi^2; l, t\right] \leq 0$$

for $t \geq l \geq t_1$, which contradicts (2.1). Therefore, all solutions of (1.1) are oscillatory. The proof is complete. □

With an appropriate choice of the functions Φ, we can obtain a number of oscillation criteria for (1.1) by Theorem 2.1. For example, assume that $\Phi(t, s, l) = (R(t) - R(s))^\gamma (R(s) - R(l))^\beta$ for $\gamma, \beta > 1/2$ and $R \in C^1([t_0, \infty), \mathbb{R})$ satisfying $(R(t) - R(s))(R(s) - R(l)) \neq 0$ for $l < s < t$. By a simple calculation,

$$\phi(t, s, l) = R'(s) \frac{\beta R(t) - (\beta + \gamma)R(s) + \gamma R(l)}{(R(t) - R(s))(R(s) - R(l))}.$$

Thus, we derive the following oscillation result.

Corollary 2.1 *Let conditions* (H_1)-(H_5) *be satisfied. If there exist three functions* $\rho \in C^1([t_0, \infty), (0, \infty))$, $b \in C^1([t_0, \infty), \mathbb{R})$, $R \in C^1([t_0, \infty), \mathbb{R})$, *and two constants* $\gamma, \beta > 1/2$ *such that* $(R(t) - R(s))(R(s) - R(l)) \neq 0$ *for* $l < s < t$ *and, for all* $l \geq t_0$,

$$\begin{aligned}
&\limsup_{t\to\infty} \int_l^t \big(R(t) - R(s)\big)^{2\gamma} \big(R(s) - R(l)\big)^{2\beta} \left(\rho(s)Q(s) - \frac{2mk_2}{k} r(s)\rho(s)\right. \\
&\quad \times \left.\left(R'(s) \frac{\beta R(t) - (\beta + \gamma)R(s) + \gamma R(l)}{(R(t) - R(s))(R(s) - R(l))}\right)^2\right) ds > 0,
\end{aligned}$$

where the functions Q *and* a *are the same as in Theorem* 2.1, *then* (1.1) *is oscillatory.*

Letting $r(t) = 1$, $\rho(t) = 1$, and $\Phi(t,s,l) = (t-s)^{\gamma}(s-l)$, we have the following criterion.

Corollary 2.2 *Let conditions* (H_1)-(H_5) *hold. Equation* (1.1) *with* $r(t) = 1$ *is oscillatory provided that, for each* $l \geq t_0$, *there exist a function* $b \in C^1([t_0,\infty),\mathbb{R})$ *and a constant* $\gamma > 1/2$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^{2\gamma+1}} \int_l^t (t-s)^{2\gamma}(s-l)^2 Q(s)\,ds > \frac{2mk_2\gamma}{k(2\gamma-1)(2\gamma+1)}, \tag{2.4}$$

where the functions Q *and* a *are as in Theorem* 2.1.

Proof Note that

$$\begin{aligned}
&\int_l^t (t-s)^{2\gamma-2}\bigl(t-(\gamma+1)s+\gamma l\bigr)^2\,ds \\
&\quad = \int_l^t (t-s)^{2\gamma-2}\bigl((t-s)-\gamma(s-l)\bigr)^2\,ds \\
&\quad = \int_l^t (t-s)^{2\gamma}\,ds - 2\gamma\int_l^t (t-s)^{2\gamma-1}(s-l)\,ds + \gamma^2\int_l^t (t-s)^{2\gamma-2}(s-l)^2\,ds \\
&\quad = \int_l^t (t-s)^{2\gamma}\,ds - \int_l^t (t-s)^{2\gamma}\,ds + \frac{2\gamma^2}{2\gamma-1}\int_l^t (t-s)^{2\gamma-1}(s-l)\,ds \\
&\quad = \frac{\gamma}{2\gamma-1}\int_l^t (t-s)^{2\gamma}\,ds \\
&\quad = \frac{\gamma}{(2\gamma-1)(2\gamma+1)}(t-l)^{2\gamma+1}.
\end{aligned} \tag{2.5}$$

It follows from (2.5) that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^{2\gamma+1}} \int_l^t (t-s)^{2\gamma}(s-l)^2\left(Q(s) - \frac{2mk_2}{k}\left(\frac{t-(\gamma+1)s+\gamma l}{(t-s)(s-l)}\right)^2\right)ds \\
&\quad = \limsup_{t\to\infty} \frac{1}{t^{2\gamma+1}} \int_l^t \left((t-s)^{2\gamma}(s-l)^2 Q(s) - \frac{2mk_2}{k}(t-s)^{2\gamma-2}\bigl(t-(\gamma+1)s+\gamma l\bigr)^2\right)ds \\
&\quad = \limsup_{t\to\infty} \frac{1}{t^{2\gamma+1}} \int_l^t (t-s)^{2\gamma}(s-l)^2 Q(s)\,ds - \frac{2mk_2\gamma}{k(2\gamma-1)(2\gamma+1)}.
\end{aligned} \tag{2.6}$$

Thus, by (2.4) and (2.6), we have

$$\limsup_{t\to\infty} \frac{1}{t^{2\gamma+1}} \int_l^t (t-s)^{2\gamma}(s-l)^2\left(Q(s) - \frac{2mk_2}{k}\left(\frac{t-(\gamma+1)s+\gamma l}{(t-s)(s-l)}\right)^2\right)ds > 0.$$

Consequently, (1.1) with $r(t) = 1$ is oscillatory by Corollary 2.1. This completes the proof. □

Similarly, the following result can be obtained with the choice of $r(t) = \rho(t) = 1$ and $\Phi(t,s,l) = (t-s)(s-l)^{\beta}$.

Corollary 2.3 *Let conditions* (H_1)-(H_5) *hold. Equation* (1.1) *with* $r(t) = 1$ *is oscillatory provided that, for each* $l \geq t_0$, *there exist a function* $b \in C^1([t_0,\infty),\mathbb{R})$ *and a constant* $\beta > 1/2$

such that

$$\limsup_{t\to\infty} \frac{1}{t^{2\beta+1}} \int_l^t (t-s)^2(s-l)^{2\beta} Q(s)\,ds > \frac{2mk_2\beta}{k(2\beta-1)(2\beta+1)},$$

where the functions Q and a are as in Theorem 2.1.

Let

$$\mathbb{D} = \{(t,s) : t_0 \le s \le t < \infty\} \quad \text{and} \quad \mathbb{D}_0 = \{(t,s) : t_0 \le s < t < \infty\}.$$

A function $H = H(t,s) \in C(\mathbb{D},[0,\infty))$ is said to belong to the function class P, if $H(t,t) = 0$ for $t \ge t_0$, $H(t,s) > 0$ for $t > s$, and H has partial derivatives $\partial H/\partial s$ and $\partial H/\partial t$ on $\mathbb{D}_0$ satisfying

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)} \quad \text{and} \quad \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)},$$

where h_1 and h_2 are locally integrable with respect to t and s, respectively, in $\mathbb{D}_0$.

Set $\Phi(t,s,l) = \sqrt{H_1(s,l)H_2(t,s)}$, $H_1, H_2 \in P$. It follows from (1.5) that

$$\phi(t,s,l) = \frac{1}{2}\left(\frac{h_1^{(1)}(s,l)}{\sqrt{H_1(s,l)}} - \frac{h_2^{(2)}(t,s)}{\sqrt{H_2(t,s)}}\right),$$

where $h_1^{(1)}$ and $h_2^{(2)}$ are defined by

$$\frac{\partial H_1}{\partial s} = h_1^{(1)}(s,l)\sqrt{H_1(s,l)} \quad \text{and} \quad \frac{\partial H_2}{\partial s} = -h_2^{(2)}(t,s)\sqrt{H_2(t,s)}. \tag{2.7}$$

After a simple computation, we have the following result when using Theorem 2.1.

Theorem 2.2 *Suppose assumptions* (H_1)-(H_5) *are satisfied. If there exist four functions* $H_1, H_2 \in P$, $\rho \in C^1([t_0,\infty),(0,\infty))$, *and* $b \in C^1([t_0,\infty),\mathbb{R})$ *such that, for each* $l \ge t_0$,

$$\limsup_{t\to\infty} \int_l^t H_1(s,l)H_2(t,s)\left(\rho(s)Q(s) - \frac{mk_2}{2k} r(s)\rho(s)\left(\frac{h_1^{(1)}(s,l)}{\sqrt{H_1(s,l)}} - \frac{h_2^{(2)}(t,s)}{\sqrt{H_2(t,s)}}\right)^2\right) ds > 0,$$

where the functions Q and a are defined as in Theorem 2.1, $h_1^{(1)}$ *and* $h_2^{(2)}$ *are defined as in* (2.7), *then* (1.1) *is oscillatory.*

3 Oscillation results for nonmonotonic *g*

Theorem 3.1 *Assume conditions* (H_1)-(H_4) *and*

(H_6) *g satisfies* $g(x)/x \ge k > 0$ *for all* $x \ne 0$ *and* $q(t) - p(t) \ge 0$ *for* $t \ge t_0$.

Equation (1.1) *is oscillatory provided that, for each* $l \ge t_0$, *there exist three functions* $\Phi \in Y$, $\rho \in C^1([t_0,\infty),(0,\infty))$, *and* $b \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} A\left[\rho(s)\overline{Q}(s) - 2mk_2 r(s)\rho(s)\phi^2; l, t\right] > 0, \tag{3.1}$$

where

$$\begin{aligned}\overline{Q}(t) &= k\big(q(t)-p(t)\big) - b'(t) - \frac{m}{4}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\frac{h^2(t)}{r(t)} \\ &\quad + \frac{b^2(t)}{mk_2r(t)} - \frac{b(t)h(t)}{k_2r(t)} - \frac{mk_2}{2}r(t)\overline{a}^2(t), \\ \overline{a}(t) &= \frac{\rho'(t)}{\rho(t)} + \frac{2b(t)}{mk_2r(t)} - \frac{h(t)}{k_2r(t)},\end{aligned}$$

$\phi = \phi(t,s,l)$ *and* A *are defined by* (1.5) *and* (1.6), *respectively.*

Proof As in the proof of Theorem 2.1, suppose x is a nonoscillatory solution of (1.1) which satisfies $x(t) > 0$ for $t \geq t_1 \geq t_0$ since the case $x < 0$ can be treated similarly. We introduce a generalized Riccati transformation by

$$w(t) = \rho(t)\left(\frac{r(t)\psi(x(t))f(x'(t))}{x(t)} + b(t)\right), \quad t \geq t_1. \tag{3.2}$$

Differentiating (3.2) and using (1.1), we have, for $t \geq t_1$,

$$\begin{aligned}w'(t) &= \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)b'(t) + \rho(t)\left(\frac{H(t,x'(t),x(t))}{x(t)} - \frac{h(t)f(x'(t))}{x(t)} - \frac{q(t)g(x(t))}{x(t)}\right) \\ &\quad - \rho(t)\frac{r(t)\psi(x(t))x'(t)f(x'(t))}{x^2(t)}.\end{aligned}$$

From (H_1)-(H_4) and (H_6), we obtain

$$\begin{aligned}w'(t) &\leq \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)b'(t) + \rho(t)\left(\big(p(t)-q(t)\big)\frac{g(x(t))}{x(t)} - h(t)\frac{f(x'(t))}{x(t)}\right) \\ &\quad - \frac{\rho(t)r(t)\psi(x(t))}{m}\frac{f^2(x'(t))}{x^2(t)} \\ &\leq \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)b'(t) + k\rho(t)\big(p(t)-q(t)\big) - \rho(t)h(t)\frac{f(x'(t))}{x(t)} \\ &\quad - \frac{\rho(t)r(t)\psi(x(t))}{m}\frac{f^2(x'(t))}{x^2(t)} \\ &= \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)\Big(b'(t) + k\big(p(t)-q(t)\big)\Big) + \frac{m\rho(t)h^2(t)}{4r(t)\psi(x(t))} - \frac{\rho(t)}{\psi(x(t))} \\ &\quad \times \left(\sqrt{\frac{r(t)}{m}}\left(\frac{w(t)}{r(t)\rho(t)} - \frac{b(t)}{r(t)}\right) + \frac{h(t)}{2}\sqrt{\frac{m}{r(t)}}\right)^2 \\ &\leq \rho(t)\left(b'(t) + k\big(p(t)-q(t)\big) + \frac{m}{4}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\frac{h^2(t)}{r(t)} - \frac{b^2(t)}{mk_2r(t)} + \frac{b(t)h(t)}{k_2r(t)}\right) \\ &\quad + \overline{a}(t)w(t) - \frac{w^2(t)}{mk_2r(t)\rho(t)} \\ &\leq -\rho(t)\overline{Q}(t) - \frac{w^2(t)}{2mk_2r(t)\rho(t)}.\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1 and one can get a contradiction to (3.1). This completes the proof. □

In what follows, we derive some corollaries from Theorem 3.1 by choosing different $\Phi(t,s,l)$. If we choose $r(t)=1$, $\rho(t)=1$, and $\Phi(t,s,l)=(t-s)^{\gamma}(s-l)$, then the following oscillation result can be obtained.

Corollary 3.1 *Let conditions* (H_1)-(H_4) *and* (H_6) *hold. Equation* (1.1) *with* $r(t)=1$ *is oscillatory provided that, for each* $l\geq t_0$, *there exist a function* $b\in C^1([t_0,\infty),\mathbb{R})$ *and a constant* $\gamma>1/2$ *such that*

$$\limsup_{t\to\infty}\frac{1}{t^{2\gamma+1}}\int_l^t (t-s)^{2\gamma}(s-l)^2\overline{Q}(s)\,ds>\frac{2mk_2\gamma}{(2\gamma-1)(2\gamma+1)},$$

where the functions $\overline{Q}$ *and* $\overline{a}$ *are as in Theorem* 3.1.

Proof The proof of this corollary is similar to that of Corollary 2.2, and hence it is omitted. □

Similarly, letting $r(t)=1$, $\rho(t)=1$, and $\Phi(t,s,l)=(t-s)(s-l)^{\beta}$, we have the following result.

Corollary 3.2 *Let conditions* (H_1)-(H_4) *and* (H_6) *be satisfied. Equation* (1.1) *with* $r(t)=1$ *is oscillatory provided that, for each* $l\geq t_0$, *there exist a function* $b\in C^1([t_0,\infty),\mathbb{R})$ *and a constant* $\beta>1/2$ *such that*

$$\limsup_{t\to\infty}\frac{1}{t^{2\beta+1}}\int_l^t (t-s)^{2}(s-l)^{2\beta}\overline{Q}(s)\,ds>\frac{2mk_2\beta}{(2\beta-1)(2\beta+1)},$$

where the functions $\overline{Q}$ *and* $\overline{a}$ *are as in Theorem* 3.1.

As discussion in Section 2, we choose $\Phi(t,s,l)=\sqrt{H_1(s,l)H_2(t,s)}$, then we get the following result.

Theorem 3.2 *Suppose* (H_1)-(H_4) *and* (H_6) *are satisfied. If there exist four functions* $H_1,H_2\in P$, $\rho\in C^1([t_0,\infty),(0,\infty))$, *and* $b\in C^1([t_0,\infty),\mathbb{R})$ *such that, for each* $l\geq t_0$,

$$\limsup_{t\to\infty}\int_l^t H_1(s,l)H_2(t,s)\left(\rho(s)\overline{Q}(s)-\frac{mk_2}{2}r(s)\rho(s)\left(\frac{h_1^{(1)}(s,l)}{\sqrt{H_1(s,l)}}-\frac{h_2^{(2)}(t,s)}{\sqrt{H_2(t,s)}}\right)^2\right)ds>0,$$

where the functions $\overline{Q}$ *and* $\overline{a}$ *are defined as in Theorem* 3.1, $h_1^{(1)}$ *and* $h_2^{(2)}$ *are defined by* (2.7), *then* (1.1) *is oscillatory.*

4 Interval oscillation criteria

Our purpose in this section is to establish some interval oscillation criteria for (1.1). First of all, we consider the case where g is an increasing function.

Theorem 4.1 *Let conditions* (H_1)-(H_5) *hold. Equation* (1.1) *is oscillatory provided that, for each* $T\geq t_0$, *there exist three functions* $\Phi\in Y$, $\rho\in C^1([t_0,\infty),(0,\infty))$, $b\in C^1([t_0,\infty),\mathbb{R})$, *and two constants* $d>c\geq T$ *such that*

$$A\left[\rho(s)Q(s)-\frac{2mk_2}{k}r(s)\rho(s)\phi^2;c,d\right]>0,$$

where ϕ, Q, a, *and* A *are defined as in Theorem* 2.1.

Proof The proof is similar to that of Theorem 2.1, where t and l are replaced by d and c, respectively. Then it can be seen that every solution of (1.1) has at least one zero in (c,d), *i.e.*, every solution of (1.1) has arbitrarily large zeros on $[t_0,\infty)$. The proof is complete. □

From Theorem 4.1, we have the following corollaries by choosing $\Phi(d,s,c) = (d-s)^{\gamma}(s-c)^{\beta}$ and $\Phi(d,s,c) = \sqrt{H_1(s,c)H_2(d,s)}$, respectively.

Corollary 4.1 *Let conditions* (H_1)-(H_5) *be satisfied. Assume there exist two functions* $\rho \in C^1([t_0,\infty),(0,\infty))$, $b \in C^1([t_0,\infty),\mathbb{R})$, *two constants* $\gamma,\beta > 1/2$, *and two constants* $d > c \geq T$ *such that, for all* $T \geq t_0$,

$$\int_c^d (d-s)^{2\gamma}(s-c)^{2\beta}\left(\rho(s)Q(s) - \frac{2mk_2}{k}r(s)\rho(s)\left(\frac{\beta d-(\beta+\gamma)s+\gamma c}{(d-s)(s-c)}\right)^2\right)ds > 0,$$

where the functions Q *and* a *are as in Theorem* 2.1. *Then* (1.1) *is oscillatory.*

Corollary 4.2 *Let conditions* (H_1)-(H_5) *hold. Equation* (1.1) *is oscillatory provided that, for each* $T \geq t_0$, *there exist four functions* $H_1, H_2 \in P$, $\rho \in C^1([t_0,\infty),(0,\infty))$, $b \in C^1([t_0,\infty),\mathbb{R})$, *and two constants* $d > c \geq T$ *such that*

$$\int_c^d H_1(s,c)H_2(d,s)\left(\rho(s)Q(s) - \frac{mk_2}{2k}r(s)\rho(s)\left(\frac{h_1^{(1)}(s,c)}{\sqrt{H_1(s,c)}} - \frac{h_2^{(2)}(d,s)}{\sqrt{H_2(d,s)}}\right)^2\right)ds > 0,$$

where the functions Q *and* a *are as in Theorem* 2.1, $h_1^{(1)}$ *and* $h_2^{(2)}$ *are defined as in* (2.7).

Similarly, we can obtain the following oscillation results when g is a nonmonotonic function.

Theorem 4.2 *Assume that conditions* (H_1)-(H_4) *and* (H_6) *are satisfied. Equation* (1.1) *is oscillatory provided that, for each* $T \geq t_0$, *there exist three functions* $\Phi \in Y$, $\rho \in C^1([t_0,\infty),(0,\infty))$, $b \in C^1([t_0,\infty),\mathbb{R})$, *and two constants* $d > c \geq T$ *such that*

$$A\left[\rho(s)\overline{Q}(s) - 2mk_2 r(s)\rho(s)\phi^2; c, d\right] > 0,$$

where ϕ, $\overline{Q}$, $\overline{a}$, *and* A *are the same as in Theorem* 3.1.

Corollary 4.3 *Let* (H_1)-(H_4) *and* (H_6) *be satisfied. Assume there exist two functions* $\rho \in C^1([t_0,\infty),(0,\infty))$, $b \in C^1([t_0,\infty),\mathbb{R})$, *two constants* $d > c \geq T$, *and two constants* $\gamma,\beta > 1/2$ *such that, for all* $T \geq t_0$,

$$\int_c^d (d-s)^{2\gamma}(s-c)^{2\beta}\left(\rho(s)\overline{Q}(s) - 2mk_2 r(s)\rho(s)\left(\frac{\beta d-(\beta+\gamma)s+\gamma c}{(d-s)(s-c)}\right)^2\right)ds > 0,$$

where the functions $\overline{Q}$ *and* $\overline{a}$ *are as in Theorem* 3.1. *Then* (1.1) *is oscillatory.*

Corollary 4.4 *Let* (H_1)-(H_4) *and* (H_6) *hold. Equation* (1.1) *is oscillatory provided that, for each* $T \geq t_0$, *there exist four functions* $H_1, H_2 \in P$, $\rho \in C^1([t_0,\infty),(0,\infty))$, $b \in C^1([t_0,\infty),\mathbb{R})$, *and two constants* $d > c \geq T$ *such that*

$$\int_c^d H_1(s,c)H_2(d,s)\left(\rho(s)\overline{Q}(s) - \frac{mk_2}{2}r(s)\rho(s)\left(\frac{h_1^{(1)}(s,c)}{\sqrt{H_1(s,c)}} - \frac{h_2^{(2)}(d,s)}{\sqrt{H_2(d,s)}}\right)^2\right)ds > 0,$$

where the functions $\overline{Q}$ *and* $\overline{a}$ *are as in Theorem* 3.1, $h_1^{(1)}$ *and* $h_2^{(2)}$ *are defined by* (2.7).

5 Examples

The following examples illustrate some applications of the results presented in this paper.

Example 5.1 For $t \geq t_0 = 1$ and $\gamma > 0$, consider the equation

$$\left(\frac{2+x^2(t)}{1+x^2(t)}\frac{x'(t)}{1+(x'(t))^2}\right)' + \frac{\sqrt{\gamma}}{t}\frac{x'(t)}{1+(x'(t))^2} + \frac{\gamma}{t^2}\bigl(x^3(t)+x(t)\bigr) = \frac{\gamma}{8t^2}x^3(t)\sin x'(t). \quad (5.1)$$

Let $r(t) = 1$, $h(t) = \sqrt{\gamma}/t$, and $q(t) = \gamma/t^2$. It is not difficult to see that

$$\begin{aligned}
&1 \leq \psi(x) = \frac{2+x^2}{1+x^2} \leq 2, \quad k_1 = 1, k_2 = 2,\\
&f(y) = \frac{y}{1+y^2}, \qquad f^2(y) \leq myf(y), \quad m = 1,\\
&g(x) = x^3 + x, \qquad g'(x) = 3x^2 + 1 \geq 1, \quad k = 1,\\
&\frac{H(t,y,x)}{g(x)} = \frac{\gamma}{8t^2}\frac{x^2}{1+x^2}\sin y \leq \frac{\gamma}{8t^2} = p(t).
\end{aligned}$$

Therefore, (5.1) satisfies conditions (H_1)-(H_5). Next, we consider the following two cases separately.

(1) If we choose $\rho(t) = 1$, $\Phi(t,s,l) = (t-s)(s-l)^\beta$, and $b(t) = 0$, then $Q(t) = \gamma/(2t^2)$ and by virtue of

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{2\beta+1}}\int_l^t (s-l)^{2\beta}\,ds\\
&\quad = \frac{1}{2\beta+1}\limsup_{t\to\infty}\frac{1}{t^{2\beta+1}}(t-l)^{2\beta+1} = \frac{1}{2\beta+1},\\
&\limsup_{t\to\infty}\frac{-2t}{t^{2\beta+1}}\int_l^t \frac{(s-l)^{2\beta}}{s}\,ds\\
&\quad = -2\limsup_{t\to\infty}\frac{1}{t^{2\beta}}\left(\frac{1}{2\beta}\bigl(t^{2\beta}-l^{2\beta}\bigr) - \frac{2l\beta}{2\beta-1}\bigl(t^{2\beta-1}-t^{2\beta-1}\bigr) + \cdots + (-l)^{2\beta}(\ln t - \ln l)\right)\\
&\quad = -2\frac{1}{2\beta} = -\frac{1}{\beta},
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{t^2}{t^{2\beta+1}}\int_l^t \frac{(s-l)^{2\beta}}{s^2}\,ds\\
&\quad = \limsup_{t\to\infty}\frac{1}{t^{2\beta-1}}\left(\frac{1}{2\beta-1}\bigl(t^{2\beta-1}-l^{2\beta-1}\bigr) - \frac{2l\beta}{2\beta-2}\bigl(t^{2\beta-2}-t^{2\beta-2}\bigr)\right.
\end{aligned}$$

$$+\cdots+(-l)^{2\beta}\left(\frac{1}{l}-\frac{1}{t}\right)\Bigg)$$
$$=\frac{1}{2\beta-1},$$

we conclude that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{2\beta+1}}\int_l^t (t-s)^2(s-l)^{2\beta}\frac{\gamma}{2s^2}\,ds\\
&\quad=\frac{\gamma}{2}\limsup_{t\to\infty}\frac{1}{t^{2\beta+1}}\int_l^t\frac{(t-s)^2(s-l)^{2\beta}}{s^2}\,ds\\
&\quad=\frac{\gamma}{2}\limsup_{t\to\infty}\frac{1}{t^{2\beta+1}}\left(\int_l^t(s-l)^{2\beta}\,ds-2t\int_l^t\frac{(s-l)^{2\beta}}{s}\,ds+t^2\int_l^t\frac{(s-l)^{2\beta}}{s^2}\,ds\right)\\
&\quad=\frac{\gamma}{2\beta(2\beta-1)(2\beta+1)}.
\end{aligned}$$

Hence, by Corollary 2.3, (5.1) is oscillatory if $\gamma > 8\beta^2$ for some $\beta\in(1/2,\infty)$.

On the other hand, if we take $\alpha\geq 1$, $n=3$, and $\delta(t)=0$, then $\widetilde{Q}(t)=3\gamma/(4t^2)$ and

$$\tilde{\rho}(t)=\exp\left(-\frac{2}{mk_2}\int^t k\delta(s)\,ds\right)=1.$$

Furthermore, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^t\left((t-s)^{n-1}\left(\widetilde{Q}(s)-\frac{mh^2(s)\tilde{\rho}(s)}{2kk_2r(s)}\right)-\frac{\alpha mk_2(n-1)^2}{2k}\tilde{\rho}(s)r(s)(t-s)^{n-3}\right)ds\\
&\quad=\limsup_{t\to\infty}\frac{1}{t^2}\int_1^t\left(\frac{\gamma}{2}\frac{(t-s)^2}{s^2}-4\alpha\right)ds=\frac{\gamma}{2}\neq\infty.
\end{aligned}$$

Thus, Theorem 1.2 cannot be applied to (5.1).

(2) If we choose $\rho(t)=t^{\frac{\sqrt{\gamma}}{2}}$, $\Phi(t,s,l)=(t-s)(s-l)^\beta$, and $b(t)=0$, then $Q(t)=3\gamma/(4t^2)$ and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{2\beta+1}}\int_l^t (t-s)^2(s-l)^{2\beta}\frac{3\gamma}{4s^2}\,ds\\
&\quad=\frac{3\gamma}{4}\limsup_{t\to\infty}\frac{1}{t^{2\beta+1}}\int_l^t\frac{(t-s)^2(s-l)^{2\beta}}{s^2}\,ds\\
&\quad=\frac{3\gamma}{4\beta(2\beta-1)(2\beta+1)}.
\end{aligned}$$

Consequently, by Corollary 2.3, (5.1) is oscillatory if $\gamma>16\beta^2/3$ for some $\beta\in(1/2,\infty)$. By the above discussion, we observe that (5.1) is oscillatory if $\gamma>64/27\approx 2.38$ (by letting $\beta=2/3$).

On the other hand, if we take $\alpha=2$, $n=3$, and $\delta(t)=0$, then $\widetilde{Q}(t)=3\gamma t^{\frac{\sqrt{\gamma}}{2}-2}/4$, $\tilde{\rho}(t)=t^{\frac{\sqrt{\gamma}}{2}}$, and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_T^t\left((t-s)^{n-1}\widetilde{Q}(s)-\frac{\alpha mk_2}{4k}\tilde{\rho}(s)r(s)(t-s)^{n-3}\right)ds\\
&\quad=\limsup_{t\to\infty}\frac{1}{t^2}\int_T^t\left(\frac{3\gamma}{4}(t-s)^2s^{\frac{\sqrt{\gamma}}{2}-2}-s^{\frac{\sqrt{\gamma}}{2}}\right)ds.
\end{aligned}$$

By a direct computation, we obtain the following results. When $0 < \gamma < 4$,

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left(\frac{3\gamma}{4}(t-s)^2 s^{\frac{\sqrt{\gamma}}{2}-2} - s^{\frac{\sqrt{\gamma}}{2}}\right) ds = \frac{3\gamma}{4}\frac{1}{1-\frac{\sqrt{\gamma}}{2}} T^{\frac{\sqrt{\gamma}}{2}-1} = \tilde{\phi}(T),$$

whereas

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{\tilde{\phi}_+^2(s)}{\tilde{\rho}(s)r(s)} ds = \frac{9\gamma^2}{16}\frac{1}{(1-\frac{\sqrt{\gamma}}{2})^2} \limsup_{t\to\infty} \int_1^t s^{\frac{\sqrt{\gamma}}{2}-2} ds \neq \infty.$$

Therefore, Theorem 1.1 cannot be applied to (5.1) in this case where $0 < \gamma < 4$. When $\gamma \geq 4$,

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left(\frac{3\gamma}{4}(t-s)^2 s^{\frac{\sqrt{\gamma}}{2}-2} - s^{\frac{\sqrt{\gamma}}{2}}\right) ds = \infty \geq \tilde{\phi}(T).$$

Define $\tilde{\phi}$ by $\tilde{\phi}(t) = t^{\frac{\sqrt{\gamma}}{4}}$. We have

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{\tilde{\phi}_+^2(s)}{\tilde{\rho}(s)r(s)} ds = \limsup_{t\to\infty} \int_1^t 1\, ds = \infty.$$

From Theorem 1.1, we conclude that (5.1) is oscillatory for $\gamma \geq 4$.

Example 5.2 For $t \geq 1$, consider the equation

$$\begin{aligned} &\left(\frac{1+x^2(t)}{2+x^2(t)}x'(t)\right)' + \beta\sin^2 t\big(x'(t)\big) + \left(\frac{3}{4}\beta^2\sin^4 t + 3\right)\big(\sin x(t) + 2x(t)\big) \\ &\quad = 2\big(\sin x(t) + 2x(t)\big)\sin t\cos x'(t), \end{aligned} \tag{5.2}$$

where $r(t) = 1$, $h(t) = \beta\sin^2 t$, $q(t) = 3\beta^2\sin^4 t/4 + 3$, and $\beta > 8/3$. Note that

$$\begin{aligned} &\frac{1}{2} \leq \psi(x) = \frac{1+x^2}{2+x^2} \leq 1, \quad k_1 = \frac{1}{2}, k_2 = 1, \\ &f(y) = y, \qquad f^2(y) \leq myf(y), \quad m = 1, \\ &g(x) = \sin x + 2x, \qquad \frac{g(x)}{x} = \frac{\sin x}{x} + 2 \geq 1, \quad k = 1, \\ &\frac{H(t,y,x)}{g(x)} = 2\sin t\cos y \leq 2 = p(t). \end{aligned}$$

Let $c = 0$ and $d = \pi$. Choosing $\rho(t) = 1$, $b(t) = 1$, and $\Phi(d,s,c) = \sin s$, we have $\overline{Q}(s) = \beta\sin^2 s$, $\Phi^2(d,s,c)\phi^2(d,s,c) = \cos^2 s$, and

$$\begin{aligned} &A\big[\rho(s)\overline{Q}(s) - 2mk_2 r(s)\rho(s)\phi^2; c, d\big] \\ &\quad = \int_c^d \big(\Phi^2(d,s,c)\overline{Q}(s) - 2mk_2\Phi^2(d,s,c)\phi^2(d,s,c)\big) ds \\ &\quad = \int_0^\pi \beta\sin^4 s\, ds - 2\int_0^\pi \cos^2 s\, ds = \left(\frac{3\beta}{8} - 1\right)\pi. \end{aligned}$$

Using Theorem 4.2, we conclude that (5.2) is oscillatory if $\beta > 8/3$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

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