Song and Meng *Journal of Inequalities and Applications* (2017) 2017:128
DOI 10.1186/s13660-017-1401-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Some normed binomial difference sequence spaces related to the $\ell_p$ spaces

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**Abstract**

The aim of this paper is to introduce the normed binomial sequence spaces $b_p^{r,s}(\nabla)$ by combining the binomial transformation and difference operator, where $1 \leq p \leq \infty$. We prove that these spaces are linearly isomorphic to the spaces $\ell_p$ and $\ell_\infty$, respectively. Furthermore, we compute Schauder bases and the $\alpha$-, $\beta$- and $\gamma$-duals of these sequence spaces.

**Keywords:** sequence space; matrix domain; Schauder basis; $\alpha$-, $\beta$- and $\gamma$-duals

## 1 Introduction and preliminaries

Let $w$ denote the space of all sequences. By $\ell_p$, $\ell_\infty$, $c$ and $c_0$, we denote the spaces of $p$-absolutely summable, bounded, convergent and null sequences, respectively, where $1 \leq p < \infty$. Let $Z$ be a sequence space, then Kizmaz [1] introduced the following difference sequence spaces:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = \ell_\infty, c, c_0$, where $\Delta x_k = x_k - x_{k+1}$ for each $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the set of positive integers. Since then, many authors have studied further generalization of the difference sequence spaces [2–6]. Moreover, Altay and Polat [7], Başarir and Kara [8–12], Kara [13], Kara and İlkhan [14], Polat and Başar [15], and many others have studied new sequence spaces from a matrix point of view that represent difference operators.

For an infinite matrix $A = (a_{n,k})$ and $x = (x_k) \in w$, the $A$-transform of $x$ is defined by $Ax = \{(Ax)_n\}$ and is supposed to be convergent for all $n \in \mathbb{N}$, where $(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k$. For two sequence spaces $X$, $Y$ and an infinite matrix $A = (a_{n,k})$, the sequence space $X_A$ is defined by $X_A = \{x = (x_k) \in w : Ax \in X\}$, which is called the domain of matrix $A$ in the space $X$. By $(X : Y)$, we denote the class of all matrices such that $X \subseteq Y_A$.

The Euler means $E^r$ of order $r$ is defined by the matrix $E^r = (e^r_{n,k})$, where $0 < r < 1$ and

$$e^r_{n,k} = \begin{cases} \binom{n}{k}(1-r)^{n-k} r^k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

The Euler sequence spaces $e_p^r$ and $e_\infty^r$ were defined by Altay, Başar and Mursaleen [16] as follows:

$$e_p^r = \left\{ x = (x_k) \in w : \sum_n \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \right|^p < \infty \right\}$$

and

$$e_\infty^r = \left\{ x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \right| < \infty \right\}.$$

Altay and Polat [7] defined further generalization of the Euler sequence spaces $e_0^r(\nabla)$, $e_c^r(\nabla)$ and $e_\infty^r(\nabla)$ by

$$e_0^r(\nabla) = \{ x = (x_k) \in w : (\nabla x_k) \in e_0^r \},$$
$$e_c^r(\nabla) = \{ x = (x_k) \in w : (\nabla x_k) \in e_c^r \}$$

and

$$e_\infty^r(\nabla) = \{ x = (x_k) \in w : (\nabla x_k) \in e_\infty^r \},$$

where $\nabla x_k = x_k - x_{k-1}$ for each $k \in \mathbb{N}$. Here any term with negative subscript is equal to naught. Many authors have used especially the Euler matrix for defining new sequence spaces, for instance, Kara and Başarir [17], Karakaya and Polat [18] and Polat and Başar [15].

Recently Bişgin [19, 20] defined another type of generalization of the Euler sequence spaces and introduced the binomial sequence spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$. Let $r, s \in \mathbb{R}$ and $r + s \neq 0$. Then the binomial matrix $B^{r,s} = (b_{n,k}^{r,s})$ is defined by

$$b_{n,k}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. For $sr > 0$ we have

(i) $\|B^{r,s}\| < \infty$,

(ii) $\lim_{n\to\infty} b_{n,k}^{r,s} = 0$ for each $k \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \sum_k b_{n,k}^{r,s} = 1$.

Thus, the binomial matrix $B^{r,s}$ is regular for $sr > 0$. Unless stated otherwise, we assume that $sr > 0$. If we take $s + r = 1$, we obtain the Euler matrix $E^r$. So the binomial matrix generalizes the Euler matrix. Bişgin [20] defined the following spaces of binomial sequences:

$$b_p^{r,s} = \left\{ x = (x_k) \in w : \sum_n \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^p < \infty \right\}$$

and

$$b_\infty^{r,s} = \left\{ x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right| < \infty \right\}.$$

The main purpose of the present paper is to study the normed difference spaces $b_p^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ of the binomial sequence whose $B^{r,s}(\nabla)$-transforms are in the spaces $\ell_p$ and $\ell_\infty$, respectively. These new sequence spaces are the generalization of the sequence spaces defined in [7] and [20]. Also, we compute the bases and $\alpha$-, $\beta$- and $\gamma$-duals of these sequence spaces.

## 2 The binomial difference sequence spaces

In this section, we introduce the spaces $b_p^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ and prove that these sequence spaces are linearly isomorphic to the spaces $\ell_p$ and $\ell_\infty$, respectively.

We first define the binomial difference sequence spaces $b_p^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ by

$$b_p^{r,s}(\nabla) = \{x = (x_k) \in w : (\nabla x_k) \in b_p^{r,s}\}$$

and

$$b_\infty^{r,s}(\nabla) = \{x = (x_k) \in w : (\nabla x_k) \in b_\infty^{r,s}\}.$$

Let us define the sequence $y = (y_n)$ as the $B^{r,s}(\nabla)$-transform of a sequence $x = (x_k)$, that is,

$$y_n = [B^{r,s}(\nabla x_k)]_n = \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (\nabla x_k) \tag{2.1}$$

for each $n \in \mathbb{N}$. Then the binomial difference sequence spaces $b_p^{r,s}(\nabla)$ or $b_\infty^{r,s}(\nabla)$ can be redefined by all sequences whose $B^{r,s}(\nabla)$-transforms are in the space $\ell_p$ or $\ell_\infty$.

**Theorem 2.1** *The sequence spaces $b_p^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ are complete linear metric spaces with the norm defined by*

$$f_{b_p^{r,s}(\nabla)}(x) = \|y\|_p = \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

*and*

$$f_{b_\infty^{r,s}(\nabla)}(x) = \|y\|_\infty = \sup_{n\in\mathbb{N}} |y_n|,$$

*where $1 \le p < \infty$ and the sequence $y = (y_n)$ is defined by the $B^{r,s}(\nabla)$-transform of $x$.*

*Proof* The proof of the linearity is a routine verification. It is obvious that $f_{b_p^{r,s}}(\alpha x) = |\alpha| f_{b_p^{r,s}}(x)$ and $f_{b_p^{r,s}}(x) = 0$ if and only if $x = \theta$ for all $x \in b_p^{r,s}(\nabla)$, where $\theta$ is the zero element in $b_p^{r,s}$ and $\alpha \in \mathbb{R}$. We consider $x, z \in b_p^{r,s}(\nabla)$, then we have

$$\begin{aligned} f_{b_p^{r,s}(\nabla)}(x+z) &= \left(\sum_n \left|\left(B^{r,s}[\nabla(x_k+z_k)]\right)_n\right|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_n \left|[B^{r,s}(\nabla x_k)]_n\right|^p\right)^{\frac{1}{p}} + \left(\sum_n \left|[B^{r,s}(\nabla z_k)]_n\right|^p\right)^{\frac{1}{p}} = f_{b_p^{r,s}(\nabla)}(x) + f_{b_p^{r,s}(\nabla)}(z). \end{aligned}$$

Hence $f_{b_p^{r,s}(\nabla)}$ is a norm on the space $b_p^{r,s}(\nabla)$.

Let $(x_m)$ be a Cauchy sequence in $b_p^{r,s}(\nabla)$, where $x_m = (x_{m_k})_{k=1}^{\infty} \in b_p^{r,s}(\nabla)$ for each $m \in \mathbb{N}$. For every $\varepsilon > 0$, there is a positive integer $m_0$ such that $f_{b_p^{r,s}(\nabla)}(x_m - x_l) < \varepsilon$ for $m, l \geq m_0$. Then we get

$$\left|\left(B^{r,s}\left[\nabla(x_{m_k} - x_{l_k})\right]\right)_n\right| \leq \left(\sum_n \left|\left(B^{r,s}\left[\nabla(x_{m_k} - x_{l_k})\right]\right)_n\right|^p\right)^{\frac{1}{p}} < \varepsilon$$

for $m, l \geq m_0$ and each $k \in \mathbb{N}$. So $(B^{r,s}(\nabla x_{m_k}))_{m=1}^{\infty}$ is a Cauchy sequence in the set of real numbers $\mathbb{R}$. Since $\mathbb{R}$ is complete, we have $\lim_{m\to\infty} B^{r,s}(\nabla x_{m_k}) = B^{r,s}(\nabla x_k)$ for each $k \in \mathbb{N}$. We compute

$$\sum_{n=0}^{i} \left|\left(B^{r,s}\left[\nabla(x_{m_k} - x_{l_k})\right]\right)_n\right| \leq f_{b_p^{r,s}(\nabla)}(x_m - x_l) < \varepsilon \tag{2.2}$$

for $m > m_0$. We take $i$ and $l \to \infty$, then the inequality (2.2) implies that

$$f_{b_p^{r,s}(\nabla)}(x_m - x) \to 0.$$

We have

$$f_{b_p^{r,s}(\nabla)}(x) \leq f_{b_p^{r,s}(\nabla)}(x_m - x) + f_{b_p^{r,s}(\nabla)}(x_m) < \infty,$$

that is, $x \in b_p^{r,s}(\nabla)$. Thus, the space $b_p^{r,s}(\nabla)$ is complete. For the space $b_\infty^{r,s}(\nabla)$, the proof can be completed in a similar way. So, we omit the detail. □

**Theorem 2.2** *The sequence spaces $b_p^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ are linearly isomorphic to the spaces $\ell_p$ and $\ell_\infty$, respectively, where $1 \leq p < \infty$.*

*Proof* Similarly, we only prove the theorem for the space $b_p^{r,s}(\nabla)$. To prove $b_p^{r,s}(\nabla) \cong \ell_p$, we must show the existence of a linear bijection between the spaces $b_p^{r,s}(\nabla)$ and $\ell_p$.

Consider $T : b_p^{r,s}(\nabla) \to \ell_p$ by $T(x) = B^{r,s}(\nabla x_k)$. The linearity of $T$ is obvious and $x = \theta$ whenever $T(x) = \theta$. Therefore, $T$ is injective.

Let $y = (y_n) \in \ell_p$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=0}^{k} (s+r)^i \sum_{j=i}^{k} \binom{j}{i} r^{-j}(-s)^{j-i} y_i \tag{2.3}$$

for each $k \in \mathbb{N}$. Then we have

$$\begin{aligned}
f_{b_p^{r,s}(\nabla)}(x) &= \left\|\left[B^{r,s}(\nabla x_k)\right]_n\right\|_p \\
&= \left(\sum_{n=1}^{\infty} \left|\frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (\nabla x_k)\right|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}} = \|y\|_p < \infty,
\end{aligned}$$

which implies that $x \in b_p^{r,s}(\nabla)$ and $T(x) = y$. Consequently, $T$ is surjective and is norm preserving. Thus, $b_p^{r,s}(\nabla) \cong \ell_p$. □

## 3 The Schauder basis and $\alpha$-, $\beta$- and $\gamma$-duals

For a normed space $(X, \|\cdot\|)$, a sequence $\{x_k : x_k \in X\}_{k\in\mathbb{N}}$ is called a *Schauder basis* [21] if for every $x \in X$, there is a unique scalar sequence $(\lambda_k)$ such that $\|x - \sum_{k=0}^{n} \lambda_k x_k\| \to 0$ as $n \to \infty$. Next, we shall give a Schauder basis for the sequence space $b_p^{r,s}(\nabla)$.

We define the sequence $g^{(k)}(r,s) = \{g_i^{(k)}(r,s)\}_{i\in\mathbb{N}}$ by

$$g_i^{(k)}(r,s) = \begin{cases} 0 & \text{if } 0 \le i < k, \\ (s+r)^k \sum_{j=k}^{i} \binom{j}{k} r^{-j}(-s)^{j-k} & \text{if } i \ge k, \end{cases}$$

for each $k \in \mathbb{N}$.

**Theorem 3.1** *The sequence $(g^{(k)}(r,s))_{k\in\mathbb{N}}$ is a Schauder basis for the binomial sequence spaces $b_p^{r,s}(\nabla)$ and every $x = (x_i) \in b_p^{r,s}(\nabla)$ has a unique representation by*

$$x = \sum_k \lambda_k(r,s) g^{(k)}(r,s), \tag{3.1}$$

*where $1 \le p < \infty$ and $\lambda_k(r,s) = [B^{r,s}(\nabla x_i)]_k$ for each $k \in \mathbb{N}$.*

*Proof* Obviously, $B^{r,s}(\nabla g_i^{(k)}(r,s)) = e_k \in \ell_p$, where $e_k$ is the sequence with 1 in the $k$th place and zeros elsewhere for each $k \in \mathbb{N}$. This implies that $g^{(k)}(r,s) \in b_p^{r,s}(\nabla)$ for each $k \in \mathbb{N}$.

For $x \in b_p^{r,s}(\nabla)$ and $m \in \mathbb{N}$, we put

$$x^{(m)} = \sum_{k=0}^{m} \lambda_k(r,s) g^{(k)}(r,s).$$

By the linearity of $B^{r,s}(\nabla)$, we have

$$B^{r,s}(\nabla x_i^{(m)}) = \sum_{k=0}^{m} \lambda_k(r,s) B^{r,s}(\nabla g_i^{(k)}(r,s)) = \sum_{k=0}^{m} \lambda_k(r,s) e_k$$

and

$$\left[B^{r,s}\left(\nabla\left(x_i - x_i^{(m)}\right)\right)\right]_k = \begin{cases} 0 & \text{if } 0 \le k \le m, \\ [B^{r,s}(\nabla x_i)]_k & \text{if } k > m, \end{cases}$$

for each $k \in \mathbb{N}$.

For any given $\varepsilon > 0$, there is a positive integer $m_0$ such that

$$\sum_{k=m_0+1}^{\infty} \left|\left[B^{r,s}(\nabla x_i)\right]_k\right|^p < \left(\frac{\varepsilon}{2}\right)^p$$

for all $k \geq m_0$. Then we have

$$\begin{aligned} f_{b_p^{r,s}(\nabla)}\big(x - x^{(m)}\big) &= \left( \sum_{k=m+1}^{\infty} \big| \big[B^{r,s}(\nabla x_i)\big]_k \big|^p \right)^{\frac{1}{p}} \\ &\leq \left( \sum_{k=m_0+1}^{\infty} \big| \big[B^{r,s}(\nabla x_i)\big]_k \big|^p \right)^{\frac{1}{p}} \\ &< \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

which implies that $x \in b_p^{r,s}(\nabla)$ is represented as (3.1).

To prove the uniqueness of this representation, we assume that

$$x = \sum_k \mu_k(r,s) g^{(k)}(r,s).$$

Then we have

$$\big[B^{r,s}(\nabla x_i)\big]_k = \sum_k \mu_k(r,s) \big[B^{r,s}\big(\nabla g_i^{(k)}(r,s)\big)\big]_k = \sum_k \mu_k(r,s)(e_k)_k = \mu_k(r,s),$$

which is a contradiction with the assumption that $\lambda_k(r,s) = [B^{r,s}(\nabla x_i)]_k$ for each $k \in \mathbb{N}$. This shows the uniqueness of this representation. □

**Corollary 3.2** *The sequence space $b_p^{r,s}(\nabla)$ is separable, where $1 \leq p < \infty$.*

For the duality theory, the study of sequence spaces is more useful when we investigate them equipped with linear topologies. Köthe and Toeplitz [22] first computed duals whose elements can be represented as sequences and defined the $\alpha$-dual (or Köthe-Toeplitz dual).

For the sequence spaces $X$ and $Y$, define the multiplier space $M(X,Y)$ by

$$M(X,Y) = \big\{ u = (u_k) \in w : ux = (u_k x_k) \in Y \text{ for all } x = (x_k) \in X \big\}.$$

Then the $\alpha$-, $\beta$- and $\gamma$-duals of a sequence space $X$ are defined by

$$X^\alpha = M(X,\ell_1), \qquad X^\beta = M(X,c) \quad \text{and} \quad X^\gamma = M(X,\ell_\infty),$$

respectively.

We give the following properties:

$$\sup_{n\in\mathbb{N}} \sum_k |a_{n,k}|^q < \infty, \tag{3.2}$$

$$\sup_{k\in\mathbb{N}} \sum_n |a_{n,k}| < \infty, \tag{3.3}$$

$$\sup_{n,k\in\mathbb{N}} |a_{n,k}| < \infty, \tag{3.4}$$

$$\lim_{n\to\infty} a_{n,k} = a_k \quad \text{for each } k \in \mathbb{N}, \tag{3.5}$$

$$\sup_{K\in\Gamma}\sum_k\left|\sum_{n\in K}a_{n,k}\right|^q<\infty, \tag{3.6}$$

$$\lim_{n\to\infty}\sum_k|a_{n,k}|=\sum_k\left|\lim_{n\to\infty}a_{n,k}\right|, \tag{3.7}$$

where $\Gamma$ is the collection of all finite subsets of $\mathbb{N}$, $\frac{1}{p}+\frac{1}{q}=1$ and $1<p\leq\infty$.

**Lemma 3.3** ([23]) *Let $A=(a_{n,k})$ be an infinite matrix. Then the following statements hold*:

(i) *$A\in(\ell_1:\ell_1)$ if and only if* (3.3) *holds.*
(ii) *$A\in(\ell_1:c)$ if and only if* (3.4) *and* (3.5) *hold.*
(iii) *$A\in(\ell_1:\ell_\infty)$ if and only if* (3.4) *holds.*
(iv) *$A\in(\ell_p:\ell_1)$ if and only if* (3.6) *holds with $\frac{1}{p}+\frac{1}{q}=1$ and $1<p\leq\infty$.*
(v) *$A\in(\ell_p:c)$ if and only if* (3.2) *and* (3.5) *hold with $\frac{1}{p}+\frac{1}{q}=1$ and $1<p<\infty$.*
(vi) *$A\in(\ell_p:\ell_\infty)$ if and only if* (3.2) *holds with $\frac{1}{p}+\frac{1}{q}=1$ and $1<p<\infty$.*
(vii) *$A\in(\ell_\infty:c)$ if and only if* (3.5) *and* (3.7) *hold with $\frac{1}{p}+\frac{1}{q}=1$ and $1<p<\infty$.*
(viii) *$A\in(\ell_\infty:\ell_\infty)$ if and only if* (3.2) *holds with $q=1$.*

**Theorem 3.4** *We define the set $U_1^{r,s}$ and $U_2^{r,s}$ by*

$$U_1^{r,s}=\left\{u=(u_k)\in w:\sup_{i\in\mathbb{N}}\sum_k\left|(s+r)^i\sum_{j=i}^{k}\binom{j}{i}r^{-j}(-s)^{j-i}u_k\right|<\infty\right\}$$

*and*

$$U_2^{r,s}=\left\{u=(u_k)\in w:\sup_{K\in\Gamma}\sum_i\left|\sum_{k\in K}(s+r)^i\sum_{j=i}^{k}\binom{j}{i}r^{-j}(-s)^{j-i}u_k\right|^q<\infty\right\}.$$

*Then $[b_1^{r,s}(\nabla)]^\alpha=U_1^{r,s}$ and $[b_p^{r,s}(\nabla)]^\alpha=U_2^{r,s}$, where $1<p\leq\infty$.*

*Proof* Let $u=(u_k)\in w$ and $x=(x_k)$ be defined by (2.3), then we have

$$u_kx_k=\sum_{i=0}^{k}(s+r)^i\sum_{j=i}^{k}\binom{j}{i}r^{-j}(-s)^{j-i}u_ky_i=\left(G^{r,s}y\right)_k$$

for each $k\in\mathbb{N}$, where $G^{r,s}=(g_{k,i}^{r,s})$ is defined by

$$g_{k,i}^{r,s}=\begin{cases}(s+r)^i\sum_{j=i}^{k}\binom{j}{i}r^{-j}(-s)^{j-i}u_k & \text{if } 0\leq i\leq k,\\ 0 & \text{if } i>k.\end{cases}$$

Therefore, we deduce that $ux=(u_kx_k)\in\ell_1$ whenever $x\in b_1^{r,s}(\nabla)$ or $b_p^{r,s}(\nabla)$ if and only if $G^{r,s}y\in\ell_1$ whenever $y\in\ell_1$ or $\ell_p$, which implies that $u=(u_k)\in[b_1^{r,s}(\nabla)]^\alpha$ or $[b_p^{r,s}(\nabla)]^\alpha$ if and only if $G^{r,s}\in(\ell_1:\ell_1)$ and $G^{r,s}\in(\ell_p:\ell_1)$ by parts (i) and (iv) of Lemma 3.3, we obtain $u=(u_k)\in[b_1^{r,s}(\nabla)]^\alpha$ if and only if

$$\sup_{i\in\mathbb{N}}\sum_k\left|(s+r)^i\sum_{j=i}^{k}\binom{j}{i}r^{-j}(-s)^{j-i}u_k\right|<\infty$$

and $u=(u_k)\in[b_p^{r,s}(\nabla)]^\alpha$ if and only if

$$\sup_{K\in\Gamma}\sum_i\left|\sum_{k\in K}(s+r)^i\sum_{j=i}^k\binom{j}{i}r^{-j}(-s)^{j-i}u_k\right|^q<\infty.$$

Thus, we have $[b_1^{r,s}(\nabla)]^\alpha=U_1^{r,s}$ and $[b_p^{r,s}(\nabla)]^\alpha=U_2^{r,s}$, where $1<p\le\infty$. □

Now, we define the sets $U_3^{r,s}$, $U_4^{r,s}$, $U_5^{r,s}$, $U_6^{r,s}$ and $U_7^{r,s}$ by

$$U_3^{r,s}=\left\{u=(u_k)\in w:\lim_{n\to\infty}(s+r)^k\sum_{i=k}^n\sum_{j=k}^i\binom{j}{k}r^{-j}(-s)^{j-k}u_i \text{ exists for each } k\in\mathbb{N}\right\},$$

$$U_4^{r,s}=\left\{u=(u_k)\in w:\sup_{n,k\in\mathbb{N}}\left|(s+r)^k\sum_{i=k}^n\sum_{j=k}^i\binom{j}{k}r^{-j}(-s)^{j-k}u_i\right|<\infty\right\},$$

$$\begin{aligned}U_5^{r,s}=\Bigg\{u=(u_k)\in w:\lim_{n\to\infty}\sum_k\left|(s+r)^k\sum_{i=k}^n\sum_{j=k}^i\binom{j}{k}r^{-j}(-s)^{j-k}u_i\right|\\ =\sum_k\left|\lim_{n\to\infty}(s+r)^k\sum_{i=k}^n\sum_{j=k}^i\binom{j}{k}r^{-j}(-s)^{j-k}u_i\right|\Bigg\},\end{aligned}$$

$$U_6^{r,s}=\left\{u=(u_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k=0}^n\left|(s+r)^k\sum_{i=k}^n\sum_{j=k}^i\binom{j}{k}r^{-j}(-s)^{j-k}u_i\right|^q<\infty\right\},\quad 1<q<\infty,$$

and

$$U_7^{r,s}=\left\{u=(u_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k=0}^n\left|(s+r)^k\sum_{i=k}^n\sum_{j=k}^i\binom{j}{k}r^{-j}(-s)^{j-k}u_i\right|<\infty\right\}.$$

**Theorem 3.5** *We have the following relations*:
(i) $[b_1^{r,s}(\nabla)]^\beta=U_3^{r,s}\cap U_4^{r,s}$,
(ii) $[b_p^{r,s}(\nabla)]^\beta=U_3^{r,s}\cap U_6^{r,s}$, *where* $1<p<\infty$,
(iii) $[b_\infty^{r,s}(\nabla)]^\beta=U_3^{r,s}\cap U_5^{r,s}$,
(iv) $[b_1^{r,s}(\nabla)]^\gamma=U_4^{r,s}$,
(v) $[b_p^{r,s}(\nabla)]^\gamma=U_6^{r,s}$, *where* $1<p<\infty$,
(vi) $[b_\infty^{r,s}(\nabla)]^\gamma=U_7^{r,s}$.

*Proof* Let $u=(u_k)\in w$ and $x=(x_k)$ be defined by (2.3), then we consider the following equation:

$$\begin{aligned}\sum_{k=0}^n u_kx_k&=\sum_{k=0}^n u_k\left[\sum_{i=0}^k(s+r)^i\sum_{j=i}^k\binom{j}{i}r^{-j}(-s)^{j-i}y_i\right]\\&=\sum_{k=0}^n\left[(s+r)^k\sum_{i=k}^n\sum_{j=k}^i\binom{j}{k}r^{-j}(-s)^{j-k}u_i\right]y_k=\left(U^{r,s}y\right)_n,\end{aligned}$$

where $U^{r,s} = (u^{r,s}_{n,k})$ is defined by

$$u_{n,k} = \begin{cases} (s+r)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} r^{-j}(-s)^{j-k} u_i & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

Therefore, we deduce that $ux = (u_k x_k) \in c$ whenever $x \in b_1^{r,s}(\nabla)$ if and only if $U^{r,s} y \in c$ whenever $y \in \ell_1$, which implies that $u = (u_k) \in [b_1^{r,s}(\nabla)]^\beta$ if and only if $U^{r,s} \in (\ell_1 : c)$. By Lemma 3.3(ii), we obtain $[b_1^{r,s}(\nabla)]^\beta = U_3^{r,s} \cap U_4^{r,s}$. Using Lemma 3.3(i) and (iii)-(viii) instead of (ii), the proof can be completed in a similar way. So, we omit the details. □

## 4 Conclusion

By considering the definitions of the binomial matrix $B^{r,s} = (b^{r,s}_{n,k})$ and the difference operator, we introduce the sequence spaces $b_p^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$. These spaces are the natural continuations of [1, 7, 20]. Our results are the generalizations of the matrix domain of the Euler matrix of order $r$. In order to give fully inform the reader on related topics with applications and a possible line of further investigation, the e-book [24] is added to the list of references.

**Acknowledgements**
We wish to thank the referee for his/her constructive comments and suggestions.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
JM came up with the main ideas and drafted the manuscript. MS revised the paper. All authors read and approved the final manuscript.

## Publisher's Note



Received: 4 April 2017 Accepted: 18 May 2017 Published online: 02 June 2017

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