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Minimization of eigenvalues for some differential equations with integrable potentials

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Abstract

In this paper we use the limiting approach to solve the minimization problem of the Dirichlet eigenvalues of Sturm-Liouville equations when the L^1 norm of integrable potentials is given. The construction of an approximating problem in this paper can simplify the analysis in the limiting process.

MSC: 34L15; 34L40

Keywords: eigenvalue; Sturm-Liouville equations; minimization problem; integrable potential; L^1 ball

1 Introduction

Extremal problems for eigenvalues are important in applied sciences like optimal control theory, population dynamics [1–3] and propagation speeds of traveling waves [4, 5]. These are also interesting mathematical problems [6–10], because the solutions are applied in many different branches of mathematics. The aim of this paper is to solve the minimization problem of the Dirichlet eigenvalues of Sturm-Liouville equations when the L^1 norm of integrable potentials is given.

For $1 \le p \le \infty$, let $\mathcal{L}^p := L^p([0,1],\mathbb{R})$ denote the Lebesgue space of real functions with the L^p norm $\|\cdot\|_p = \|\cdot\|_{L^p[0,1]}$. Given a potential $q \in \mathcal{L}^p$, we consider Sturm-Liouville equations

$$\ddot{y} + \big(\lambda + q(t)\big)y = 0, \quad t \in [0,1], \tag{1.1}$$

with the Dirichlet boundary condition

$$y(0) = y(1) = 0. \tag{1.2}$$

It is known that problem (1.1)-(1.2) has countably many eigenvalues (see [11, 12]). They are denoted by λ_j, $j = 1, 2, \ldots$, and ordered in such a way that

$$-\infty < \lambda_1(q) < \lambda_2(q) < \cdots < \lambda_n(q) < \cdots, \quad \text{and} \quad \lim_{n\to\infty} \lambda_n(q) = +\infty.$$

For $r \in (0,\infty)$, denote

$$B_p[r] = \big\{q \in \mathcal{L}^p : \|q\|_p \le r\big\}.$$

In this paper, we study the following minimization problem:

$$\mathbf{L}(r) := \inf_{q\in B_1[r]} \lambda_1(q). \tag{1.3}$$

Note that the minimization problems in [1, 3, 7] are taking over order intervals of potentials/weights which are compact in the weak topologies, and therefore always have minimizers. For example, Krein studied in [7] the minimization problem of weighted Dirichlet eigenvalues of

$$\ddot{y} + \lambda w(t)y = 0, \quad t \in [0,1].$$

Given constants $0 < r \le H < \infty$, denote

$$S := \left\{ w \in \mathcal{L}^\infty : 0 \le w \le H, \int_0^1 w(t)\,\mathrm{d}t = r \right\}.$$

The problem is to find

$$\tilde{\mathbf{L}} := \inf_{w\in S} \lambda_1(w). \tag{1.4}$$

Using compactness of the class S and continuity of the eigenvalues in the weak topologies, problem (1.4) can be realized by some optimal weight w. However, our problem (1.3) is taking over L^1 balls, which are not compact even in the weak topology w_1. In order to overcome this difficulty in topology, we first solve the following approximating minimization problem of eigenvalues.

Theorem 1.1 *Let*

$$\mathbf{L}(r,H) := \inf_{q\in B_1[r]\cap B_\infty[H]} \lambda_1(q), \tag{1.5}$$

where $r, H \in (0,\infty)$. *We have that*

(i) *If* $r \ge H$, *then*

$$\mathbf{L}(r,H) = \pi^2 - H. \tag{1.6}$$

Moreover, $\mathbf{L}(r,H)$ *is attained for*

$$q_1 = H. \tag{1.7}$$

(ii) *If* $r < H$, *then* $\mathbf{L}(r,H) \in (-\infty, \pi^2 - r)$ *is the unique solution of* $\mathbf{Z}_{r,H}(x) = 0$. *Here, the function* $\mathbf{Z}_{r,H} : (-\infty, \pi^2 - r) \to \mathbb{R}$ *is defined as*

$$\mathbf{Z}_{r,H}(x) = \begin{cases} 1 + \frac{\sqrt{-x-H}}{\sqrt{-x}} \tanh \sqrt{-x}\frac{H-r}{2H} \tanh \sqrt{-x-H}\frac{r}{2H} & \text{for } x \in (-\infty, -H), \\ 1 - \frac{\sqrt{x+H}}{\sqrt{-x}} \tanh \sqrt{-x}\frac{H-r}{2H} \tan \sqrt{x+H}\frac{r}{2H} & \text{for } x \in [-H, 0), \\ 1 - \frac{H-r}{2\sqrt{H}} \tan \frac{r}{2\sqrt{H}} & \text{for } x = 0, \\ 1 - \frac{\sqrt{x+H}}{\sqrt{x}} \tan \sqrt{x}\frac{H-r}{2H} \tan \sqrt{x+H}\frac{r}{2H} & \text{for } x \in (0, \pi^2 - r). \end{cases} \tag{1.8}$$

Moreover, $\mathbf{L}(r,H)$ *is attained for*

$$q_1(t) = \begin{cases} 0 & \text{for } t \in [0, \frac{H-r}{2H}], \\ H & \text{for } t \in (\frac{H-r}{2H}, \frac{H+r}{2H}), \\ 0 & \text{for } t \in [\frac{H+r}{2H}, 1]. \end{cases} \tag{1.9}$$

Then, using the continuous dependence of eigenvalues on potentials with respect to the weak topologies (see [13–16]), we obtain a complete solution for minimization problem (1.3).

Theorem 1.2 *The following holds*:

$$\mathbf{L}(r) = \lim_{H\to+\infty} \mathbf{L}(r,H) = \mathbf{Z}^{-1}(r). \tag{1.10}$$

Here, the function $\mathbf{Z} : (-\infty, \pi^2] \to [0,\infty)$ *is defined as*

$$\mathbf{Z}(x) = \begin{cases} 2\sqrt{-x}\coth(\sqrt{-x}/2) & \text{for } x \in (-\infty, 0), \\ 4 & \text{for } x = 0, \\ 2\sqrt{x}\cot(\sqrt{x}/2) & \text{for } x \in (0, \pi^2]. \end{cases} \tag{1.11}$$

This paper is organized as follows. In Section 2, we give some preliminary results on eigenvalues. In Section 3, we first consider the approximating minimization for eigenvalues and obtain Theorem 1.1. Then, by the limiting analysis, we give the proof of Theorem 1.2.

We end the introduction with the following remark. In [17, 18], the authors first considered the approximating minimization for eigenvalues on the corresponding L^p balls, $1 < p < \infty$. Then minimization problem (1.3) can be solved by complicated limiting analysis of $p \downarrow 1$. In this paper, we study a different approximating problem, which also has a sense from mathematical point of view. Such a construction can simplify the analysis in the limiting process.

2 Auxiliary lemmas

In the Lebesgue spaces $\mathcal{L}^p$, $p \in [1,\infty]$, besides the L^p norms $\|\cdot\|_p$, one has the following weak topologies [19]. For $p \in [1,\infty)$, we use w_p to indicate the topology of weak convergence in $\mathcal{L}^p$, and for $p = \infty$, by considering $\mathcal{L}^\infty$ as the dual space of $(\mathcal{L}^1, \|\cdot\|_1)$, we have the topology w_∞ of weak* convergence. In a unified way, $q_m \to q_0$ in $(\mathcal{L}^p, w_p)$ iff

$$\int_I f(t)q_m(t)\,\mathrm{d}t \to \int_I f(t)q_0(t)\,\mathrm{d}t \quad \forall f \in \mathcal{L}^{p^*}.$$

Here, $p^* := p/(p-1) \in [1,\infty]$ is the conjugate exponent of p.

To solve problem (1.5), let us quote from [14, 16] some important properties on eigenvalues.

Lemma 2.1 *As nonlinear functionals,* $\lambda_n(q)$ *are continuous in* $q \in (\mathcal{L}^p, w_p)$, $p \in [1,\infty]$.

By the continuity result above, we show that extremal problem (1.5) can be attained in $B_1[r] \cap B_\infty[H]$.

Lemma 2.2 *There exists* $q_1 \in B_1[r] \cap B_\infty[H]$ *such that* $\mathbf{L}(r,H) = \lambda_1(q_1)$.

Proof Notice that $B_\infty[H]$ is compact and $B_1[r] \cap \mathcal{L}^\infty$ is closed in $(\mathcal{L}^\infty, w_\infty)$. Hence $B_1[r] \cap B_\infty[H]$ is compact in $(\mathcal{L}^\infty, w_\infty)$. Consequently, the existence of minimizers of (1.5) can be deduced from Lemma 2.1 in a direct way. □

Eigenvalues possess the following monotonicity property [12].

Lemma 2.3

$$q_1, q_2 \in \mathcal{L}^1, \quad q_1 \le q_2 \quad \Longrightarrow \quad \lambda_n(q_1) \ge \lambda_n(q_2). \tag{2.1}$$

Moreover, if, in addition, $q_1(t) < q_2(t)$ *holds on a subset of* $[0,1]$ *of positive measure, the conclusion inequality in* (2.1) *is strict.*

Next, we use the theory of Schwarz symmetrization as a tool. For a given nonnegative function f defined on the interval $[0,1]$, we denote by f^+ (resp., f^-) the symmetrically increasing (resp., decreasing) rearrangement of f. We recall that the function f^+ is uniquely defined by the following conditions:

(i) f^+ and f are equimeasurable on $[0,1]$. That is, for all $s \ge 0$,

$$m\big(\{t : f^+(t) \ge s\}\big) = m\big(\{t : f(t) \ge s\}\big).$$

(ii) f^+ is symmetric about $t = 1/2$.

(iii) f^+ is decreasing in the interval $[0,1/2]$.

Similarly, f^- is (uniquely) defined by (i), (ii) and (iii)': f^- is increasing in the interval $[0,1/2]$. For more information on rearrangements, see [20] and [21].

We can compare the first eigenvalue of q with the first eigenvalue of its rearrangement.

Lemma 2.4 *For any* $q \in \mathcal{L}^p$, *we have* $\lambda_1(q) \ge \lambda_1(|q|^-)$.

Proof Let $y_1(t)$ be a positive eigenfunction corresponding to $\lambda_1(q)$. By [22, Theorem 378] and [6, Section 7], we have

$$\begin{aligned}
\lambda_1(q) &= \frac{\int_0^1 y_1'^2\,\mathrm{d}t - \int_0^1 q y_1^2\,\mathrm{d}t}{\int_0^1 (y_1^-)^2\,\mathrm{d}t} \\
&\ge \frac{\int_0^1 y_1'^2\,\mathrm{d}t - \int_0^1 |q| y_1^2\,\mathrm{d}t}{\int_0^1 (y_1^-)^2\,\mathrm{d}t} \\
&\ge \frac{\int_0^1 y_1'^2\,\mathrm{d}t - \int_0^1 |q|^- (y_1^-)^2\,\mathrm{d}t}{\int_0^1 (y_1^-)^2\,\mathrm{d}t} \\
&\ge \frac{\int_0^1 ((y_1^-)')^2\,\mathrm{d}t - \int_0^1 |q|^- (y_1^-)^2\,\mathrm{d}t}{\int_0^1 (y_1^-)^2\,\mathrm{d}t} \\
&\ge \lambda_1\big(|q|^-\big).
\end{aligned}$$

□

3 Main results

Now we are ready to prove the main results of this paper. First, we solve the minimization problem for eigenvalues when potentials $q \in B_1[r] \cap B_\infty[H]$.

Proof of Theorem 1.1 (i) If $r \geq H$, then $B_1[r] \cap B_\infty[H] = B_\infty[H]$. From the monotonicity property (2.1) of eigenvalues, we have that the minimizer $q_1 = H$ and then $\mathbf{L}(r,H) = \pi^2 - H$ by computing directly.

(ii) When $r < H$. By Lemma 2.2, Lemma 2.3 and Lemma 2.4, there exists a minimizer $0 \leq q_1 \in B_1[r] \cap B_\infty[H]$ such that $\mathbf{L}(r,H) = \lambda_1(q_1)$ and $q_1(t) = q_1^-(t)$ for a.e. t. Let $y_1(t)$ be a positive eigenfunction corresponding to the first eigenvalue $\lambda_1(q_1)$. We have from the proof of Lemma 2.4 that $y_1^-(t)$ is also an eigenfunction corresponding to the first eigenvalue $\lambda_1(q_1)$, which implies that $y_1 = y_1^-$ because $\int_0^1 y_1 \,\mathrm{d}t = \int_0^1 y_1^- \,\mathrm{d}t$.

Then, we have that for each $q \in B_1[r] \cap B_\infty[H]$,

$$\frac{\int_0^1 y_1'^2 \,\mathrm{d}t - \int_0^1 q_1 y_1^2 \,\mathrm{d}t}{\int_0^1 y_1^2 \,\mathrm{d}t} = \lambda_1(q_1) = \mathbf{L}(r,H) \leq \lambda_1(q) \leq \frac{\int_0^1 y_1'^2 \,\mathrm{d}t - \int_0^1 q y_1^2 \,\mathrm{d}t}{\int_0^1 y_1^2 \,\mathrm{d}t}.$$

Hence,

$$\int_0^1 q_1 y_1^2 \,\mathrm{d}t \geq \int_0^1 q y_1^2 \,\mathrm{d}t, \quad \forall q \in B_1[r] \cap B_\infty[H]. \tag{3.1}$$

Let $\eta := \frac{H-r}{2H} \in (0,1/2)$ and $\xi := y_1(\eta) > 0$. Since y_1 is symmetric about $t = 1/2$ and increasing in $[0,1/2]$, we have that

$$\begin{aligned} &0 < y_1(t) \leq \xi \quad \text{for } t \in (0,\eta) \cup (1-\eta,1), \\ &y_1(t) \geq \xi \quad \text{for } t \in (\eta, 1-\eta). \end{aligned}$$

Define

$$q_\eta := \begin{cases} 0 & \text{for } t \in [0,\eta] \cup [1-\eta,1], \\ H & \text{for } t \in (\eta, 1-\eta). \end{cases}$$

We claim that

$$\int_0^1 q_\eta y_1^2 \,\mathrm{d}t \geq \int_0^1 q_1 y_1^2 \,\mathrm{d}t. \tag{3.2}$$

In fact,

$$\begin{aligned} &\int_0^1 q_\eta y_1^2 \,\mathrm{d}t - \int_0^1 q_1 y_1^2 \,\mathrm{d}t \\ &\quad = \int_\eta^{1-\eta} (H - q_1) y_1^2 \,\mathrm{d}t - \left(\int_0^\eta + \int_{1-\eta}^1\right) q_1 y_1^2 \,\mathrm{d}t \\ &\quad \geq \xi^2 \int_\eta^{1-\eta} (H - q_1) \,\mathrm{d}t - \xi^2 \left(\int_0^\eta + \int_{1-\eta}^1\right) q_1 \,\mathrm{d}t \end{aligned}$$

$$= \xi^2 \left(r - \int_0^1 q_1 \, dt \right)$$

$$\geq 0.$$

Notice that $q_\eta \in B_1[r] \cap B_\infty[H]$. Comparing (3.1) and (3.2), we know that the equality holds in (3.2) and $y_1(t)$ is an eigenfunction corresponding to $\lambda_1(q_\eta)$. Since (3.2) is an equality, we have from the proof that $\lambda_1(q_1) = \lambda_1(q_\eta)$ and $q_1(t) = q_\eta(t)$ for a.e. t.

Let $\mu_1 := \lambda_1(q_1)$. Now, (1.1) becomes

$$\begin{cases} \ddot{y}_1 + \mu_1 y_1 = 0 & \text{for } t \in (0, \eta) \cup (1-\eta, 1), \\ \ddot{y}_1 + (\mu_1 + H) y_1 = 0 & \text{for } t \in (\eta, 1-\eta), \\ y_1(0) = y_1(1) = 0, \\ y_1(\eta) = y_1(1-\eta) = \xi. \end{cases} \tag{3.3}$$

We can find that the solution y_1 of (3.3) is given by

$$y_1(t) = \begin{cases} a \sin \sqrt{\mu_1}(t-\eta) + \xi \cos \sqrt{\mu_1}(t-\eta) & \text{for } t \in [0, \eta], \\ \frac{\xi}{\cos \sqrt{\mu_1 + H} \frac{r}{2H}} \cos \sqrt{\mu_1 + H}(t - 1/2) & \text{for } t \in (\eta, 1-\eta), \\ -a \sin \sqrt{\mu_1}(t - 1 + \eta) + \xi \cos \sqrt{\mu_1}(t - 1 + \eta) & \text{for } t \in [1-\eta, 1], \end{cases} \tag{3.4}$$

where $a = \xi \frac{\sqrt{\mu_1 + H}}{\sqrt{\mu_1}} \tan \sqrt{\mu_1 + H} \frac{r}{2H}$. Since $y_1(0) = y_1(1) = 0$ and $\mu_1 \leq \pi^2 - r$, we get from (3.4) that μ_1 is the unique solution of $\mathbf{Z}_{r,H}(x) = 0$, where $\mathbf{Z}_{r,H}(x)$ is as in (1.8). □

Finally, we use the limiting approach to obtain a complete solution for the minimization problem of eigenvalues when the L^1 norm of integrable potentials is given.

Proof of Theorem 1.2 By the definitions of $\mathbf{Z}_{r,H}$ in (1.8) and $\mathbf{Z}$ in (1.11), we have that

$$\lim_{H \to +\infty} \mathbf{Z}_{r,H}(x) = 1 - \frac{r}{\mathbf{Z}(x)},$$

which implies that

$$\lim_{H \to +\infty} \mathbf{L}(r, H) = \mathbf{Z}^{-1}(r).$$

Since $B_1[r] \supset B_1[r] \cap B_\infty[H]$, we have that $\mathbf{L}(r) \leq \mathbf{L}(r, H)$ and then

$$\mathbf{L}(r) \leq \lim_{H \to +\infty} \mathbf{L}(r, H). \tag{3.5}$$

On the other hand, since $\lambda_1(q)$ is continuous in $q \in (\mathcal{L}^1, w_1)$, $\lambda_1(q)$ is continuous in $q \in (\mathcal{L}^1, \|\cdot\|_1)$. Notice that $B_1[r] \subset \overline{\bigcup_{H>0}(B_1[r] \cap B_\infty[H])}$, where the closure is taken in the $\mathcal{L}^1$ space $(\mathcal{L}^1, \|\cdot\|_1)$. We have that for arbitrary $\epsilon > 0$ and $q \in B_1[r]$, there exist H_0 and $q_{H_0} \in B_1[r] \cap B_\infty[H_0]$ such that

$$\left| \lambda_1(q_{H_0}) - \lambda_1(q) \right| < \epsilon.$$

Hence,

$$\lambda_1(q) + \epsilon > \lambda_1(q_{H_0}) \geq \mathbf{L}(r, H_0) \geq \lim_{H\to+\infty} \mathbf{L}(r, H).$$

Because $q \in B_1[r]$ is arbitrary, it holds that

$$\mathbf{L}(r) + \epsilon \geq \lim_{H\to+\infty} \mathbf{L}(r, H).$$

Therefore, we have that

$$\mathbf{L}(r) \geq \lim_{H\to+\infty} \mathbf{L}(r, H) \tag{3.6}$$

since $\epsilon > 0$ is arbitrary.

By (3.5) and (3.6), the proof is complete. □

Remark 3.1 Fix $r > 0$. Assume that $q_H \in B_1[r] \cap B_\infty[H]$ is as in (1.7) when $r \geq H$ and as in (1.9) when $r < H$. Then $\mathbf{L}(r, H) = \lambda_1(q_H)$. Moreover, we have that

$$q_H \to r\delta_{1/2} \quad \text{in } \left(\left(C[0,1]\right)^*, w^*\right)$$

as $H \to +\infty$. In fact, for any $f \in C[0,1]$, it holds that

$$\begin{aligned} \int_0^1 f q_H \,\mathrm{d}t &= H \int_{\frac{H-r}{2H}}^{\frac{H+r}{2H}} f \,\mathrm{d}t \\ &= rf(\tilde{t}) \quad \left(\text{where } \tilde{t} \in \left[\frac{H-r}{2H}, \frac{H+r}{2H}\right]\right) \\ &\to rf(1/2) \\ &= \int_0^1 f r\delta_{1/2} \,\mathrm{d}t \end{aligned}$$

as $H \to +\infty$.

Notice that $r\delta_{1/2}$ is not in the $\mathcal{L}^1$ space. So, we get the so-called measure differential equations. For some basic theory for eigenvalues of the second-order linear measure differential equations, see [23].

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author completed the paper himself. The author read and approved the final manuscript.

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