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Basic Fourier transform on the space of entire functions of logarithm order 2

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Abstract

We study in this paper the basic Fourier transform, q -translation and q -convolution associated to the q -difference operator in the space of entire functions with logarithmic order 2 and finite logarithmic type and their dual.

1 Introduction

The concept of the basic Fourier transform is related to the quantum group which is a q -deformation of the Lie group. The deformation parameter q is always assumed to satisfy $0 < q < 1$. The basic Fourier transform was defined firstly in [1] and studied after that from the point of view of harmonic analysis in [2–7], ...

In this work, we are interested in the basic Fourier transforms of entire functions with logarithmic order 2 and finite logarithmic type which is introduced by [8]. This notion of logarithmic order is a refinement order of entire functions of order zero which is used to study the growth of order of some basic hypergeometric series. Our investigation is inspired by the ideas developed by [9, 10] and [11]. Some of the arguments used here are similar to the one considered in [10] and [11]. However, we need to introduce new procedures to prove the results in the q -theory setting.

The paper is organized as follows. In Section 2, we give a brief introduction and recall some known results about q -shift factorial, q -derivative and q -exponential function. In Section 3, firstly we describe the space of entire functions and its dual. Also, we give a new q -Taylor expansion of an entire function. Secondly, we introduce the logarithmic order and logarithmic type. In Section 4, we study a new q -translation operator and its related q -convolution and we give several characterizations from the space of entire functions into itself that commute with the q -translation. Finally, in Section 5, we define the q -Fourier transform on the dual space of entire functions and we establish a q -Paley-Wiener theorem type.

2 Preliminaries

We assume that $z \in \mathbb{C}$ and $0 < q < 1$, unless specified otherwise. We recall some notations [12]. For an arbitrary complex number a ,

$$(a, q)_n := \begin{cases} 1 & \text{for } n = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) & \text{for } n \geq 1, \end{cases}$$
$$(a, q)_\infty := \lim_{n \rightarrow \infty} (a, q)_n,$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q, q)_n}{(q, q)_k (q, q)_{n-k}}.$$

The q -derivative of a function $f(z)$ is defined as in [13] by

$$D_q f(z) := \frac{f(z) - f(qz)}{(1 - q)z}, \quad z \neq 0,$$

and we say that f has the q -derivative at zero if the limit

$$\lim_{n \rightarrow \infty} \frac{f(zq^n) - f(0)}{zq^n}$$

exists and does not depend on z . We define the operator

$$\Lambda_q f(z) := f(q^{-1}z).$$

Obviously,

$$\Lambda_q D_q = q D_q \Lambda_q$$

and for an arbitrary positive integer n ,

$$\frac{(1 - q)^k}{(q; q)_k} (\Lambda_q D_q)^k z^n = \begin{cases} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2} - nk} z^{n-k} & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k > n. \end{cases} \quad (1)$$

Consider the q -exponentials (see [12]) defined by

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q, q)_n} = \frac{1}{(z, q)_{\infty}}, \quad |z| < 1$$

and

$$E_q(z) := \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{(q, q)_n} = (-z, q)_{\infty}. \quad (2)$$

The q -exponentials functions $e_q(z)$ and $E_q(z)$ satisfy

$$D_q e_q(z) = e_q(z) \quad \text{and} \quad \Lambda_q D_q E_q(z) = E_q(z).$$

3 Space of entire functions of finite logarithmic order

We denote by \mathcal{A} the space of entire functions on \mathbb{C} . This space is endowed with the topology of the uniform convergence on compact subsets of \mathbb{C} . Thus, \mathcal{A} is a Fréchet space. \mathcal{A}' denotes the strong dual space of \mathcal{A} . If $T \in \mathcal{A}'$, there exist a constant $C > 0$ and a compact subset K of \mathbb{C} such that

$$|\langle T, f \rangle| \leq C \sup_{z \in K} |f(z)|, \quad f \in \mathcal{A}.$$

According to the Hahn-Banach theorem, the mapping T can be extended to the space $C(K)$ of continuous functions on K , as an element of the dual space of $C(K)$. Then the Riesz representation theorem implies there exists a regular complex Borel measure γ supported in K such that

$$\langle T, f \rangle = \int_{\mathbb{C}} f(z) d\gamma(z), \quad f \in \mathcal{A}.$$

Proposition 1 *The operator $\Lambda_q D_q$ is continuous from \mathcal{A} into its self.*

Proof According to the Cauchy integral formula, for $|z| < q^2 r$, we can write

$$\Lambda_q D_q f(z) = \frac{1}{2i\pi} \int_{|u|=r} \frac{f(u)}{(u - q^{-1}z)(u - z)} du.$$

Then

$$|\Lambda_q D_q f|_{q^2 r} \leq \frac{1}{(1-q)(1-q^2)r} |f|_r,$$

where $|f|_r = \sup_{|z| \leq r} |f(z)|$. Thus, we conclude that the operator $\Lambda_q D_q$ is continuous from \mathcal{A} into itself. \square

In the following theorem, we establish a new q -Taylor formula different from the one considered in [14] and [15].

Theorem 2 *If $f \in \mathcal{A}$, then f admits the representation*

$$f(z) = \sum_{n=0}^{\infty} \frac{(1-q)^n q^{\binom{n}{2}} z^n}{(q; q)_n} (\Lambda_q D_q)^n f(0).$$

Proof Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then for $k = 0, 1, \dots$, we have

$$\frac{(1-q)^k}{(q; q)_k} (\Lambda_q D_q)^k f(z) = \sum_{n=k}^{\infty} a_n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2} - nk} z^{n-k}. \tag{3}$$

By taking $z = 0$, we obtain for $k = 0, 1, \dots$ that

$$a_k = \frac{(1-q)^k}{(q; q)_k} q^{\binom{k}{2}} (\Lambda_q D_q)^k f(0), \tag{4}$$

This proves the result. \square

Recent research concerning q -difference equations, moment problems (see [8, 16–20]) strongly suggests that in order to deal with basic hypergeometric functions, one should use the following concept.

We recall (see [8]) that an entire function f of order zero has logarithmic order ϱ if

$$\varrho := \limsup_{r \rightarrow +\infty} \frac{\ln \ln |f|_r}{\ln \ln r}, \tag{5}$$

and when the logarithmic order ϱ of f is finite, we define the logarithmic type τ as

$$\tau := \limsup_{r \rightarrow +\infty} \frac{\ln |f|_r}{(\ln r)^\varrho}. \tag{6}$$

It is easy to see that:

f is of logarithmic order ρ if and only if, for every $\varepsilon > 0$,

$$|f|_r = O(e^{(\ln r)^{\rho+\varepsilon}}).$$

If f is of logarithmic order ρ , then its logarithmic type is equal to τ if and only if

$$|f|_r = O(e^{(\tau+\varepsilon)(\ln r)^\rho}).$$

We denote by $E_{\tau,q}(\mathbb{C})$ the space of entire functions of logarithmic order 2 and logarithmic type τ . In particular, when $\tau = \frac{1}{2 \ln q - 1}$, $E_{\tau,q}(\mathbb{C})$ is denoted simply by $E_q(\mathbb{C})$. That is, an entire function f is in $E_{\tau,q}(\mathbb{C})$ if and only if for every $\varepsilon > 0$,

$$|f(z)| = O(e^{(\tau+\varepsilon)(\ln |z|)^2}), \quad \text{as } |z| \rightarrow \infty.$$

The space $E_{\tau,q}(\mathbb{C})$ is endowed by the topology associated to the family $\{\rho_{\tau,\varepsilon}\}$ of seminorms, where for every $\varepsilon > 0$,

$$\rho_{\tau,\varepsilon}(f) = \sup_{z \in \mathbb{C}} |f(z) e^{-(\tau+\varepsilon) \ln^2 |z|}|.$$

Thus, $E_{\tau,q}(\mathbb{C})$ is a Banach space. The dual of $E_{\tau,q}(\mathbb{C})$ is denoted, as usual, by $E'_{\tau,q}(\mathbb{C})$.

Lemma 3 ([8]) *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order 0. Its logarithmic order ρ and logarithmic type τ satisfy*

$$\rho = 1 + \limsup_{n \rightarrow \infty} \frac{\ln n}{\ln \ln \left(\frac{1}{\sqrt[n]{|a_n|}} \right)} \tag{7}$$

and

$$\tau = \frac{(\rho - 1)^{\rho-1}}{\rho^\rho} \limsup_{n \rightarrow \infty} \frac{n}{\left(\ln \left(\frac{1}{\sqrt[n]{|a_n|}} \right) \right)^{\rho-1}}. \tag{8}$$

Proposition 4 *The logarithmic order of the q -exponential function $E_q(z)$ is equal to 2 and its logarithmic type is $\frac{1}{-2 \ln(q)}$.*

Proof The result follows from Lemma 3. □

Proposition 5 Let χ be a functional on $E_{\tau,q}(\mathbb{C})$, then $\chi \in E'_{\tau,q}(\mathbb{C})$ if and only if there exists a Borel measure γ on \mathbb{C} and $\varepsilon > 0$ such that for every $f \in E_{\tau,q}(\mathbb{C})$,

$$\langle \chi, f \rangle = \int_{\mathbb{C}} f(z) e^{-(\tau+\varepsilon)(\ln|z|)^2} d\gamma(z). \tag{9}$$

Proof Suppose that χ admits the representation (9) for a certain complex regular Borel measure γ on \mathbb{C} and $\varepsilon > 0$. Then we have

$$|\langle \chi, f \rangle| = C |f(z)| \sup_z e^{-(\tau+\varepsilon)(\ln|z|)^2}.$$

Then $\chi \in E'_{\tau,q}(\mathbb{C})$. Conversely, let $\chi \in E'_{\tau,q}(\mathbb{C})$, then there exists C such that

$$|\langle \chi, f \rangle| = C |f(z)| \sup_z e^{-(\tau+\varepsilon)(\ln|z|)^2}.$$

We denote by C_0 the space of continuous functions in \mathbb{C} vanishing at infinity. If $f \in E_{\tau,q}(\mathbb{C})$, then $f(z) e^{-(\tau+\varepsilon)(\ln|z|)^2} \in C_0$. Indeed for $0 < \eta < \varepsilon$, we have

$$|f(z)| e^{-(\tau+\varepsilon)(\ln|z|)^2} \leq \sup_z (|f(z)| e^{-(\tau+\eta)(\ln|z|)^2}) e^{-(\varepsilon-\eta)(\ln|z|)^2} \rightarrow 0,$$

as $|z| \rightarrow \infty$.

We consider the mapping I from $E_{\tau,q}(\mathbb{C})$ into C_0 defined by $I(f)(z) = f(z) e^{-(\tau+\varepsilon)(\ln|z|)^2}$ and the functional λ from $I(E_{\tau,q}(\mathbb{C}))$ into \mathbb{C} given by $\lambda(f) = \langle \chi, f \rangle$.

It is clear that λ is continuous when on $I(E_{\tau,q}(\mathbb{C}))$ we consider the topology induced in it by the usual topology of C_0 . By using Hanh-Banach and Riesz representation theorems in a standard way, we can conclude that χ admits a representation like (9) for a certain complex regular Borel measure γ on \mathbb{C} and $\varepsilon > 0$. □

4 q -translation and q -convolution

In this section, we define a new q -translation operator related to q -difference operator $\Lambda_q D_q$, on the space of entire functions of logarithm order 2.

Definition 6 The q -translation T_ξ is defined on monomials z^n by

$$T_\xi z^n := \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-(n-k)k} \frac{(1-q)^k}{(q; q)_k} \xi^k z^{n-k}. \tag{10}$$

Proposition 7 For $\xi \in \mathbb{C}$, the q -translation operator T_ξ can be extended on the entire function $f(z) = \sum_{n=0}^\infty a_n z^n$ of logarithmic order 2 and logarithmic type lesser than $\frac{1}{-2 \ln q}$ in the following manner:

$$T_\xi f(z) = \sum_{n=0}^\infty a_n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(1-q)^k}{(q; q)_k} q^{-(n-k)k} \xi^k z^{n-k}. \tag{11}$$

Proof It suffices to prove the convergence of the following series:

$$A = \sum_{n=0}^\infty |a_n| \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(1-q)^k}{(q; q)_k} q^{-(n-k)k} |\xi|^k |z|^{n-k}.$$

For every $n \in \mathbb{N}$ and $k = 0, \dots, n$, we can write

$$-(n-k)k = \frac{k(k-1)}{2} + \frac{1}{2}k^2 + \left(\frac{1}{2} - n\right)k.$$

On the other hand, the function

$$\varphi(x) = \frac{1}{2}x^2 + \left(\frac{1}{2} - n\right)x$$

attains its minimum on $n - \frac{1}{2}$. For such a value $n - \frac{1}{2}$, the minimum value over $(0, \infty)$ is

$$-\frac{1}{2}\left(n - \frac{1}{2}\right)^2.$$

Hence,

$$q^{-(n-k)k} \leq q^{\binom{k}{2}} q^{-\frac{1}{2}(n-\frac{1}{2})^2}. \tag{12}$$

Furthermore,

$$\frac{(1-q)^k}{(q; q)_k} \leq 1.$$

Thus,

$$A \leq \sum_{n=0}^{\infty} |a_n| q^{-\frac{1}{2}(n-\frac{1}{2})^2} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} |\xi|^k |z|^{n-k}. \tag{13}$$

By q -binomial theorem, we get

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} |\xi|^k |z|^{n-k} = |z|^n \left(-\frac{\xi}{|z|}; q\right)_n. \tag{14}$$

Hence,

$$|A| \leq \sum_{n=0}^{\infty} |a_n| q^{-\frac{1}{2}(n-\frac{1}{2})^2} |z|^n \left(-\frac{|\xi|}{|z|}; q\right)_n. \tag{15}$$

The logarithmic type τ of the function f is lesser than $\frac{1}{-2\ln q}$, then there exists $0 < \beta < 1$ such that $\tau = \beta \frac{1}{-2\ln q}$. By applying Cauchy estimates, we find, for every $R > 0$ and $n \in \mathbb{N}$,

$$|a_n| \leq CR^{-n} e^{\beta \frac{(\ln R)^2}{2\ln q - 1}}. \tag{16}$$

In particular, for $R = q^{-n}$, we get

$$|a_n| \leq Cq^{(1-\frac{1}{2}\beta)n^2}.$$

Then

$$|A| \leq C \sum_{n=0}^{\infty} q^{\frac{1}{2}(1-\beta)n^2} q^{n+\frac{1}{4}} |z^n| \left(-\frac{|\xi|}{|z|}; q \right)_n < \infty.$$

This shows the result. □

Proposition 8 For $z, \xi \in \mathbb{C}$, the q -exponential function satisfies the product formula

$$T_{\xi} E_q(z) = E_q(z) E_q(\xi).$$

Proof Let f be an entire function of logarithmic order 2 and logarithmic type lesser than $\frac{1}{-2 \ln q}$. The series defined in (11) is absolutely convergent. Then, after interchanging the double sum, we can write

$$T_{\xi} f(z) = \sum_{k=0}^{\infty} \frac{(1-q)^k}{(q; q)_k} q^{-(n-k)k} q^{\binom{k}{2}} \xi^k \sum_{n=k}^{\infty} a_n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2} - nk} z^{n-k}.$$

By (3), the operator T_{ξ} can be represented by the form

$$T_{\xi} f(z) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q; q)_n} q^{\binom{n}{2}} \xi^n (\Lambda_q D_q)^n (f)(z). \tag{17}$$

The result follows from the fact that the q -exponential function $E_q(z)$ is an eigenfunction of the operator $\Lambda_q D_q$ corresponding to the eigenvalue 1. □

Proposition 9 For $f \in E_{\tau, q}(\mathbb{C})$, with τ lesser than $\frac{1}{-2 \ln q}$ and $z, \xi \in \mathbb{C}$, we have

$$\begin{aligned} T_z f(\xi) &= T_{\xi} f(z), & T_0 f &= f, \\ T_z \circ T_{\xi} f &= T_{\xi} \circ T_z f, & (\Lambda_q D_q) T_z f &= T_z (\Lambda_q D_q) f. \end{aligned}$$

Now, we define the q -convolution of $\chi \in E'_{\tau, q}(\mathbb{C})$ and $f \in E_{\tau, q}(\mathbb{C})$ as

$$\chi \star f(\xi) = \langle \chi; T_{\xi} f \rangle. \tag{18}$$

In the following theorem, we obtain several characterizations of the continuous linear mappings L from \mathcal{A} into itself that commute with the q -translation operators.

Theorem 10 Let L be a continuous linear mapping from \mathcal{A} into itself. The following assertions are equivalent:

- (i) L commutes with the q -translation operators $T_z, z \in \mathbb{C}$, that is,

$$T_z L = L T_z, \quad z \in \mathbb{C}.$$

- (ii) L commutes with the q -difference operator $\Lambda_q D_q$.
- (iii) There exists a unique $\vartheta \in \mathcal{A}'$ such that $Lf = \vartheta * f, f \in \mathcal{A}$.

(iv) *There exists a complex Borel regular measure having compact support on \mathbb{C} , for which, for all $f \in \mathcal{A}$, we have*

$$L(f)(z) = \int_{\mathbb{C}} \sigma_z f(w) d\gamma(w).$$

(v) *There exists an entire function Ψ of logarithmic order 2 such that*

$$Lf(z) = \Psi(\Lambda_q D_q) f(z),$$

where

$$\Psi(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Proof (i) \Rightarrow (ii) Let $g \in \mathcal{A}$, we have

$$\frac{T_z g(\xi) - g(\xi)}{z} = \sum_{n=1}^{\infty} \frac{(1-q)^n}{(q; q)_n} q^{\binom{n}{2}} z^{n-1} (\Lambda_q D_q)^n (g)(\xi). \tag{19}$$

Hence,

$$\lim_{z \rightarrow 0} \frac{T_z g(\xi) - g(\xi)}{z} = \Lambda_q D_q g(\xi).$$

The operator L is continuous, then

$$\Lambda_q D_q Lg(\xi) = \lim_{z \rightarrow 0} \frac{T_z Lg(\xi) - Lg(\xi)}{z} = L \lim_{z \rightarrow 0} \frac{T_z g(\xi) - g(\xi)}{z} = L \Lambda_q D_q g(\xi).$$

Hence, (i) \Rightarrow (ii).

(ii) \Rightarrow (i) The operator L commutes with $\Lambda_q D_q$, then for every $n \in \mathbb{N}$,

$$L(\Lambda_q D_q)^n = (\Lambda_q D_q)^n L.$$

Hence,

$$\begin{aligned} LT_z g(\xi) &= \sum_{n=1}^{\infty} \frac{(1-q)^n}{(q; q)_n} q^{\binom{n}{2}} z^{n-1} L(\Lambda_q D_q)^n (g)(\xi) \\ &= \sum_{n=1}^{\infty} \frac{(1-q)^n}{(q; q)_n} q^{\binom{n}{2}} z^{n-1} (\Lambda_q D_q)^n L(g)(\xi) \\ &= T_z Lg(z). \end{aligned}$$

(i) \Rightarrow (iii) Assume that (i) holds. We define the functional ϑ on \mathcal{A} as follows:

$$\langle \vartheta; f \rangle = Lf(0).$$

It is clear that $\vartheta \in \mathcal{A}'$ and

$$\vartheta \star f(\xi) = \langle \vartheta; T_{\xi} f \rangle = LT_{\xi} f(0) = Lf(\xi).$$

(iii) \Rightarrow (iv) It follows immediately from Hahn-Banach and Riesz representation theorems.

(iv) \Rightarrow (v) Suppose that for all f , we have

$$Lf(\xi) = \int_{\mathbb{C}} T_{\xi} f(z) d\gamma(z),$$

where γ is a complex Borel regular measure with compact support. According to [14], we obtain

$$L(f)(z) = \sum_{n=0}^{\infty} (\Lambda_q D_q)^n f(z) \int_{\mathbb{C}} \frac{(1-q)^n}{(q; q)_n} q^{\binom{n}{2}} \xi^n d\gamma(\xi).$$

Hence,

$$Lf(z) = \Psi(\Lambda_q D_q) f(z),$$

where

$$\Psi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{with } a_n = \int_{\mathbb{C}} \frac{(1-q)^n}{(q; q)_n} q^{\binom{n}{2}} \xi^n d\gamma(\xi).$$

Since γ has compact support on \mathbb{C} , for certain a and C , we have

$$|a_n| \leq \frac{(1-q)^n}{(q; q)_n} q^{\binom{n}{2}} a^n.$$

Then we get

$$|\Psi(z)| \leq E_q(a|z|),$$

and Ψ is of logarithmic order 2.

(iv) \Rightarrow (i) Suppose that for every $f \in \mathcal{A}$, we obtain

$$L(f)(z) = \sum_{n=0}^{\infty} (\Lambda_q D_q)^n f(z) \int_{\mathbb{C}} \frac{(1-q)^n}{(q; q)_n} q^{\binom{n}{2}} \xi^n d\gamma(\xi).$$

Hence, if $f \in \mathcal{A}$ since $T_z \Lambda_q D_q f = \Lambda_q D_q T_z f$,

$$\begin{aligned} T_z L(f)(z) &= \sum_{n=0}^{\infty} T_z (\Lambda_q D_q)^n f(z) \\ &= \sum_{n=0}^{\infty} (\Lambda_q D_q)^n T_z f(z) \\ &= LT_z f. \end{aligned}$$

□

5 The q -Fourier transform on the space \mathcal{A}'

We introduce the q -Fourier transform on \mathcal{A}' by

$$\mathcal{F}_q(T)(\xi) = \langle T(z), E_q(-i\xi z) \rangle, \quad \xi, z \in \mathbb{C}. \tag{20}$$

From (2) the q -Fourier transform takes the form

$$\mathcal{F}_q(T)(\xi) = \sum_{n=0}^{\infty} (-i)^n \frac{(1-q)^n q^{\binom{n}{2}}}{(q; q)_n} \langle T, z^n \rangle \xi^n, \quad \xi \in \mathbb{C}. \tag{21}$$

Theorem 11 (q -Paley-Wiener theorem) *The q -Fourier transform \mathcal{F}_q is a topological isomorphism from \mathcal{A}' onto $E_q(\mathbb{C})$.*

Proof Assume firstly that $T \in \mathcal{A}'$. Then there exists a complex regular Borel measure γ on \mathbb{C} and a $a > 0$ such that the support of it is contained in the open ball $B(0, a)$ of center 0 and radius a and

$$\langle T, f \rangle = \int_{\mathbb{C}} f(z) d\gamma(z), \quad f \in \mathcal{A}$$

so that for every $n \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$|\langle T; z^n \rangle| \leq C a^n. \tag{22}$$

By the expansion (21) of the q -Fourier of T , we can write

$$\mathcal{F}_q(T)(\lambda) = \int_{\mathbb{C}} E_q(-i\lambda z) d\gamma(z) = \sum_{n=0}^{\infty} a_n,$$

where

$$a_n = (-i)^n \frac{(1-q)^n q^{\binom{n}{2}}}{(q; q)_n} \langle T, z^n \rangle \lambda^n, \quad \lambda \in \mathbb{C}.$$

Then

$$\frac{1}{n} \ln |a_n| = \ln(1-q) + \ln |\lambda| - \ln(q; q)_n + \ln |\langle T; z^n \rangle| + \frac{n-1}{2} \ln(q)$$

and

$$\begin{aligned} \ln \ln \left(\frac{1}{\sqrt[n]{|a_n|}} \right) &= \ln(n-1) + \ln \left(-\frac{\ln(q)}{2} \right) + \ln \left[1 - \frac{2 \ln((q; q)_n)}{n(n-1) \ln(q)} \right. \\ &\quad \left. + 2 \frac{\ln(1-q) + \ln |\lambda|}{(n-1) \ln(q)} + 2 \frac{\ln |\langle T; z^n \rangle|}{n(n-1) \ln(q)} \right]. \end{aligned}$$

The inequality (22) shows that

$$\lim_{n \rightarrow \infty} \frac{\ln |\langle T; z^n \rangle|}{n(n-1) \ln(q)} = 0.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\ln \ln \left(\frac{1}{\sqrt[n]{|a_n|}} \right)}{\ln n} = 1.$$

Furthermore, the logarithmic order ρ of $\mathcal{F}_q(T)$ is equal to

$$\rho = 1 + \limsup_{n \rightarrow \infty} \frac{\ln n}{\ln \ln \left(\frac{1}{\sqrt[n]{|a_n|}} \right)} = 2.$$

Similarly, the logarithmic type of τ of $\mathcal{F}_q(T)$ is given by

$$\tau = \limsup_{n \rightarrow \infty} \frac{n}{\ln \left(\frac{1}{\sqrt[n]{|a_n|}} \right)} = \frac{1}{-2 \ln q}.$$

Hence, we have proved $\mathcal{F}_q(T) \in E_q(\mathbb{C})$. Moreover, \mathcal{F}_q is a continuous mapping from \mathcal{A} into $E_q(\mathbb{C})$. Indeed, since \mathcal{A} is a bornological space, it is sufficient to see that $\mathcal{F}_q(T) \in E_q(\mathbb{C})$. The q -Fourier transform \mathcal{F}_q is a one-to-one mapping from \mathcal{A}' into $E_q(\mathbb{C})$. Indeed, assume that $T \in \mathcal{A}'$ is such that $\mathcal{F}_q(T) = 0, z \in \mathbb{C}$. Then from (20), we infer that

$$\langle T(w), w^n \rangle = 0, \quad \forall n \in \mathbb{N}.$$

Hence, (21) implies also that

$$\langle T(w), f \rangle = 0, \quad \forall f \in \mathcal{A}.$$

Thus, the transform \mathcal{F}_q is one-to-one. Suppose now that g is a function in $E_q(\mathbb{C})$. There exists $C > 0$ such that for every $r > 0$,

$$|g|_r \leq C e^{\frac{\ln^2 |r|}{2 \ln q - 1}}.$$

From Theorem 2, we have

$$g(z) = \sum_{n=0}^{\infty} \frac{(1-q)^n q^{\binom{n}{2}} z^n}{(q; q)_n} (\Lambda_q D_q)^n g(0), \quad z \in \mathbb{C}.$$

Then, for every $n \in \mathbb{N}$,

$$\frac{(1-q)^n q^{\binom{n}{2}}}{(q; q)_n} (\Lambda_q D_q)^n g(0) = \int_{C_R} \frac{g(w)}{w^{n+1}} d\gamma(w),$$

where C_R represents, for every $R > 0$, the circular path $C_R : w = R e^{i\theta}, \theta \in [0, 2)$. Hence, for every $n \in \mathbb{N}$, we have

$$\left| (\Lambda_q D_q)^n g(0) \right| \leq C \frac{(q; q)_{\infty}}{(1-q)^n R^n q^{\binom{n}{2}}} e^{\frac{\ln^2 |R|}{2 \ln q - 1}} \rho(g).$$

In particular, if we take $R = q^{-n}$, we obtain

$$|(\Lambda_q D_q)^n g(0)| \leq C' a^n, \quad a = \frac{q^{1/2}}{1-q}.$$

On the other hand, from (2), we have

$$\frac{1}{2i\pi} \int_{C_{2a}} E_q(-iz\xi) \xi^{-1-n} d\xi = \frac{(1-q)^n}{(q; q)_n} q^{\binom{n}{2}} (-iz)^n.$$

For $|z| > a$, we put

$$f(z) = \sum_{m=0}^{\infty} (-i)^{-m} z^{-1-m} (\Lambda_q D_q)^m (g)(0).$$

Then the series converges absolutely in C_{2a} . Hence, for every $z \in \mathbb{C}$, we have

$$\begin{aligned} \frac{1}{2i\pi} \int_{C_{2a}} E_q(-iz\xi) f(\xi) d\xi &= \sum_{m=0}^{\infty} (-iz)^n (\Lambda_q D_q)^n (g)(0) \frac{1}{2i\pi} \int_{C_{2a}} E_q(-iz\xi) \xi^{-1-n} d\xi \\ &= \sum_{m=0}^{\infty} \frac{(1-q)^n q^{\binom{n}{2}}}{(q; q)_n} z^n (\Lambda_q D_q)^n (g)(0) \\ &= g(z). \end{aligned}$$

We now define the functional T on \mathcal{A} by

$$\langle \chi; \varphi \rangle = \frac{1}{2i\pi} \int_{C_{2a}} f(z) \varphi(z) dz.$$

It is obvious that $\chi \in \mathcal{A}$. Moreover, $\mathcal{F}_q(T) = g$. Also, from (22), we deduce that for every bounded set Ω of \mathcal{A} , there exists $C > 0$ such that

$$\langle \mathcal{F}^{-1}(g); \varphi \rangle \leq C \sup_{z \in \Omega} |g(z)|. \tag{23}$$

We conclude that \mathcal{F}^{-1} is continuous from $E_q(\mathbb{C})$ onto \mathcal{A} . □

Competing interests

The author declares that they have no competing interests.

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