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Classification and criteria of the limit cases for singular second-order linear equations on time scales

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Abstract

This paper is concerned with classification and criteria of the limit cases for singular second-order linear equations on time scales. By the different cases of the limiting set, the equations are divided into two cases: the limit-point and limit-circle cases just like the continuous and discrete cases. Several sufficient conditions for the limit-point cases are established. It is shown that the limit cases are invariant under a bounded perturbation. These results unify the existing ones of second-order singular differential and difference equations.

Keywords: singular second-order linear equation; time scales; limit-point case; limit-circle case

1 Introduction

In this paper, we consider classification and criteria of the limit cases for the following singular second-order linear equation:

$$-\left(p(t)y^{\Delta}(t)\right)^{\Delta} + q(t)y^{\sigma}(t) = \lambda w(t)y^{\sigma}(t), \quad t \in \left[\rho(0), \infty\right) \cap \mathbb{T}, \tag{1.1}$$

where p^{Δ}, q, and w are real and piecewise continuous functions on $[\rho(0), \infty) \cap \mathbb{T}$, $p(t) \neq 0$ and $w(t) > 0$ for all $t \in [\rho(0), \infty) \cap \mathbb{T}$; $\lambda \in \mathbb{C}$ is the spectral parameter; $\mathbb{T}$ is a time scale with $\rho(0) \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$; $\sigma(t)$ and $\rho(t)$ are the forward and backward jump operators in $\mathbb{T}$; y^{Δ} is the Δ-derivative of y; and $y^{\sigma}(t) := y(\sigma(t))$.

The spectral problems of symmetric linear differential operators and difference operators can both be divided into two cases. Those defined over finite closed intervals with well-behaved coefficients are called regular. Otherwise, they are called singular. In 1910, Weyl [1] gave a dichotomy of the limit-point and limit-circle cases for the following singular second-order linear differential equation:

$$-y''(t) + q(t)y(t) = \lambda y(t), \quad t \in [0, \infty), \tag{1.2}$$

where q is a real and continuous function on $[0, \infty)$, $\lambda \in \mathbb{C}$ is the spectral parameter. Later, Titchmarsh, Coddington, Levinson *etc.* developed his results and established the Weyl-Titchmarsh theory [2, 3]. Their work has been greatly developed and generalized to higher-order differential equations and Hamiltonian systems, and a classification and some criteria of limit cases were formulated [4–9]. Singular spectral problems of self-adjoint scalar second-order difference equations over infinite intervals were firstly studied by Atkinson [10]. His work was followed by Hinton, Jirari *etc.* [11, 12]. In 2001, some sufficient and necessary conditions and several criteria of the limit-point and limit-circle cases were obtained for the following formally self-adjoint second-order linear difference equations with real coefficients [13]:

$$-\nabla\big(p(n)\Delta y(n)\big) + q(n)y(n) = \lambda w(n)y(n), \quad n \in \{n\}_{n=0}^{\infty}, \tag{1.3}$$

where ∇ and Δ are the backward and forward difference operators respectively, namely $\nabla y(n) := y(n) - y(n-1)$ and $\Delta y(n) := y(n+1) - y(n)$; $p(n)$, $q(n)$, and $w(n)$ are real numbers with $w(n) > 0$ for $n \in [0,\infty)$ and $p(n) \neq 0$ for $n \in [-1,\infty)$; λ is a complex spectral parameter. In 2006, Shi [14] established the Weyl-Titchmarsh theory of discrete linear Hamiltonian systems. Later, several sufficient conditions and sufficient and necessary conditions for the limit-point and limit-circle cases were established for the singular second-order linear difference equation with complex coefficients (see [15]).

In the past twenty years, a lot of effort has been put into the study of regular spectral problems on time scales (see [16–23]). But singular spectral problems have started to be considered only quite recently. In 2007, we employed Weyl's method to divide the following singular second-order linear equations on time scales into two cases: limit-point and limit-circle cases [24]:

$$-y^{\Delta\Delta}(t) + q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in \big[\rho(0),\infty\big) \cap \mathbb{T},$$

where q is real and continuous on $[\rho(0),\infty) \cap \mathbb{T}$, $\lambda \in \mathbb{C}$ is the spectral parameter. By using the similar method, Huseynov [25] studied the classification of limit cases for the following singular second-order linear equations on time scales:

$$-\big(p(t)y^{\Delta}(t)\big)^{\nabla} + q(t)y(t) = \lambda y(t), \quad t \in (a,\infty) \cap \mathbb{T},$$

as well as of the form

$$-\big(p(t)y^{\Delta}(t)\big)^{\Delta} + q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in [a,\infty) \cap \mathbb{T}, \tag{1.4}$$

where p^{∇} (or p^{Δ}) and q are real and piecewise continuous functions in $(a,\infty) \cap \mathbb{T}$ (or $[a,\infty) \cap \mathbb{T}$), $p(t) \neq 0$ for all t, and $\lambda \in \mathbb{C}$ is the spectral parameter. Obviously, let $w(t) \equiv 1$ and $\rho(0) = a$, then (1.1) is the same as (1.4). In 2010, by using the properties of the Weyl matrix disks, Sun [26] established the Weyl-Titchmarsh theory of Hamiltonian systems on time scales. It has been found that the second-order singular differential and difference equations can be divided into limit-point and limit-circle cases. We wonder whether the classification of the limit cases holds on time scales. In the present paper, we extend these results obtained in [24] to Eq. (1.1) and establish several sufficient conditions and sufficient and necessary conditions for the limit-point and limit-circle cases for Eq. (1.1).

This paper is organized as follows. In Section 2, some basic concepts and a fundamental theory about time scales are introduced. In Section 3, a family of nested circles which converge to a limiting set is constructed. The dichotomy of the limit-point and limit-circle cases for singular second-order linear equations on time scales is given by the geometric properties of the limiting set. Finally, several criteria of the limit-point case are established, and the invariance of the limit cases is shown under a bounded perturbation for the potential function q in Section 4.

2 Preliminaries

In this section, some basic concepts and fundamental results on time scales are introduced.

Let $\mathbb{T} \subset \mathbb{R}$ be a non-empty closed set. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively, where $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called right-scattered, right-dense, left-scattered, and left-dense if $\sigma(t) > t, \sigma(t) = t, \rho(t) < t$, and $\rho(t) = t$ separately. Denote $\mathbb{T}^k := \mathbb{T}$ if $\mathbb{T}$ is unbounded above and $\mathbb{T}^k := \mathbb{T} \setminus (\rho(\max \mathbb{T}), \max \mathbb{T}]$ otherwise. The graininess $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Let f be a function defined on $\mathbb{T}$. f is said to be Δ-differentiable at $t \in \mathbb{T}^k$ provided there exists a constant a such that, for any $\varepsilon > 0$, there is a neighborhood U of t (*i.e.*, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) with

$$\bigl|f\bigl(\sigma(t)\bigr) - f(s) - a\bigl(\sigma(t) - s\bigr)\bigr| \le \varepsilon \bigl|\sigma(t) - s\bigr| \quad \text{for all } s \in U.$$

In this case, denote $f^\Delta(t) := a$. If f is Δ-differentiable for every $t \in \mathbb{T}^k$, then f is said to be Δ-differentiable on $\mathbb{T}$. If f is Δ-differentiable at $t \in \mathbb{T}^k$, then

$$f^\Delta(t) = \begin{cases} \lim_{\substack{s \to t \\ s \in \mathbb{T}}} \frac{f(t) - f(s)}{t - s}, & \text{if } \mu(t) = 0, \\ \frac{f(\sigma(t)) - f(t)}{\mu(t)}, & \text{if } \mu(t) > 0. \end{cases} \tag{2.1}$$

If $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$, then $F(t)$ is called an anti-derivative of f on $\mathbb{T}$. In this case, define the Δ-integral by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \quad \text{for all } s, t \in \mathbb{T}.$$

For convenience, we introduce the following results ([27, Chapter 1] and [28, Chapter 1]), which are useful in this paper.

Lemma 2.1 *Let $f, g : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$.*

(i) *If f is Δ-differentiable at t, then f is continuous at t.*

(ii) *If f and g are Δ-differentiable at t, then fg is Δ-differentiable at t and*

$$(fg)^\Delta(t) = f^\sigma(t)g^\Delta(t) + f^\Delta(t)g(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t).$$

(iii) *If f and g are Δ-differentiable at t, and $f(t)f^{\sigma}(t) \neq 0$, then $f^{-1}g$ is Δ-differentiable at t and*

$$\left(gf^{-1}\right)^{\Delta}(t) = \left(g^{\Delta}(t)f(t) - g(t)f^{\Delta}(t)\right)\left(f^{\sigma}(t)f(t)\right)^{-1}.$$

A function f defined on $\mathbb{T}$ is said to be rd-continuous if it is continuous at every right-dense point in $\mathbb{T}$ and its left-sided limit exists at every left-dense point in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. The set of kth Δ-differentiable functions with rd-continuous kth derivative is denoted by $C_{rd}^{k}(\mathbb{T}) = C_{rd}^{k}(\mathbb{T}, \mathbb{R})$.

Lemma 2.2 *If f, g are rd-continuous functions on $\mathbb{T}$, then*

(i) *f^{σ} is rd-continuous and f has an anti-derivative on $\mathbb{T}$;*
(ii) *$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)$ for all $t \in \mathbb{T}$.*
(iii) (Integration by parts) $\int_a^b f^{\sigma}(\tau)g^{\Delta}(\tau)\Delta\tau = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(\tau)g(\tau)\Delta\tau$.
(iv) (Hölder's inequality [29, Lemma 2.2(iv)]) *Let $r, s \in \mathbb{T}$ with $r \leq s$, then*

$$\int_r^s \left|f(\tau)g(\tau)\right|\Delta\tau \leq \left\{\int_r^s \left|f(\tau)\right|^p \Delta\tau\right\}^{\frac{1}{p}} \left\{\int_r^s \left|g(\tau)\right|^q \Delta\tau\right\}^{\frac{1}{q}},$$

where $p > 1$ and $q = p/(p-1)$.

Let

$$L_w^2\left(\rho(0), \infty\right) := \left\{y^{\sigma} : \left[\rho(0), \infty\right) \to \mathbb{C} \;\Big|\; \int_{\rho(0)}^{\infty} w(t)\left|y^{\sigma}(t)\right|^2 \Delta t < \infty\right\}.$$

A function $g : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)g(t) \neq 0 \quad \text{for all } t \in \mathbb{T}.$$

Higer [30] showed that for any given $t_0 \in \mathbb{T}$ and for any given rd-continuous and regressive g, the initial value problem

$$y^{\Delta}(t) = g(t)y(t), \quad y(t_0) = 1$$

has a unique solution

$$\begin{aligned} &e_g(t, t_0) = \exp\left\{\int_{t_0}^{t} \xi_{\mu(\tau)}\left(g(\tau)\right)\Delta\tau\right\}, \\ &\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases} \end{aligned} \tag{2.2}$$

Lemma 2.3 ([27, Theorem 6.1]) *Let $y, f \in C_{rd}(\mathbb{T})$ and $g \in \mathcal{R}^{+} := \{g \in C_{rd}(\mathbb{T}) : 1 + \mu(t)g(t) > 0, t \in \mathbb{T}\}$. Then*

$$y^{\Delta}(t) \leq g(t)y(t) + f(t), \quad \forall t \in \mathbb{T},$$

implies

$$y(t) \le y(t_0)e_g(t,t_0) + \int_{t_0}^{t} e_g\big(t,\sigma(\tau)\big)f(\tau)\Delta\tau, \quad \forall t \in \mathbb{T}.$$

We define the Wronskian by

$$W[x,y](t) = p(t)\big[x(t)y^{\Delta}(t) - x^{\Delta}(t)y(t)\big], \quad x,y \in C^2_{rd}(\mathbb{T}). \tag{2.3}$$

The following result is a direct consequence of the Lagrange identity [27, Theorem 4.30].

Lemma 2.4 *Let x and y be any two solutions of* (1.1). *Then $W[x,y](t)$ is a constant in* $[\rho(0),\infty) \cap \mathbb{T}$.

3 Classification

In this section, we focus on the classification of the limit cases for singular second-order linear equations on time scales.

Let $y_1(t,\lambda)$ and $y_2(t,\lambda)$ be the two solutions of (1.1) satisfying the following initial conditions:

$$\begin{aligned} &y_1\big(\rho(0),\lambda\big) = p\big(\rho(0)\big)y_2^{\Delta}\big(\rho(0),\lambda\big) = 1, \\ &p\big(\rho(0)\big)y_1^{\Delta}\big(\rho(0),\lambda\big) = y_2\big(\rho(0),\lambda\big) = 0, \end{aligned}$$

respectively. Since their Wronskian is identically equal to 1, these two solutions form a fundamental solution system of (1.1). We form a linear combination of $y_1(t,\lambda)$ and $y_2(t,\lambda)$

$$y(t,\lambda,m) := y_1(t,\lambda) + my_2(t,\lambda). \tag{3.1}$$

Let $b \in (\rho(0),\infty) \cap \mathbb{T}$, $k \in \mathbb{R}$, $\lambda = \mu + i\nu$ with $\nu \neq 0$, and let (3.1) satisfy

$$p(b)y^{\Delta}(b,\lambda,m) + ky(b,\lambda,m) = 0. \tag{3.2}$$

Then

$$m = -\frac{p(b)y_1^{\Delta}(b,\lambda) + ky_1(b,\lambda)}{p(b)y_2^{\Delta}(b,\lambda) + ky_2(b,\lambda)}. \tag{3.3}$$

It can be verified that the integral identity

$$\big[\overline{y(t,\lambda)}p(t)y^{\Delta}(t,\lambda)\big]\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} p(t)\big|y^{\Delta}(t,\lambda)\big|^2\Delta t + \int_{t_1}^{t_2}\big[\lambda w(t) - q(t)\big]\big|y^{\sigma}(t,\lambda)\big|^2\Delta t = 0 \tag{3.4}$$

holds for any solution $y(t,\lambda)$ of (1.1) and for any $t_1, t_2 \in [\rho(0),\infty) \cap \mathbb{T}$. Setting $y(t,\lambda) = y_2(t,\lambda)$, $t_1 = \rho(0)$, $t_2 = b$ in (3.4) and taking its imaginary part, we obtain

$$\Im\big[\overline{y_2(b,\lambda)}p(b)y_2^{\Delta}(b,\lambda)\big] = -\nu\int_{\rho(0)}^{b} w(t)\big|y_2^{\sigma}(t,\lambda)\big|^2\Delta t. \tag{3.5}$$

So

$$\Im\left(\frac{p(b)y_2^\Delta(b,\lambda)}{y_2(b,\lambda)}\right) = \frac{\Im[\overline{y_2(b,\lambda)}p(b)y_2^\Delta(b,\lambda)]}{|y_2(b,\lambda)|^2} \neq 0.$$

It follows from (3.2) and $k \in \mathbb{R}$ that $k \neq -\frac{p(b)y_2^\Delta(b,\lambda)}{y_2(b,\lambda)}$. Hence, the denominator in (3.3) is not equal to zero, and consequently, m is well defined.

Next, we will show that (3.3) describes a circle for any fixed b. It follows from (3.1) and (3.2) that

$$\overline{y(\rho(0),\lambda,m)}p(\rho(0))y^\Delta(\rho(0),\lambda,m) = m$$

and

$$\overline{y(b,\lambda,m)}p(b)y^\Delta(b,\lambda,m) = -k\left|y(b,\lambda,m)\right|^2 \in \mathbb{R}.$$

By (3.4) and the above two relations, we have

$$\Im(m) = \nu \int_{\rho(0)}^{b} w(t)\left|y^\sigma(t,\lambda,m)\right|^2 \Delta t, \tag{3.6}$$

which implies that m lies in the upper half-plane if $\nu > 0$. It follows from (3.2) that

$$k = -\frac{p(b)y^\Delta(b,\lambda,m)}{y(b,\lambda,m)},$$

which, together with $k \in \mathbb{R}$, yields that

$$p(b)\left[y^\Delta(b,\lambda,m)\overline{y(b,\lambda,m)} - \overline{y^\Delta(b,\lambda,m)}y(b,\lambda,m)\right] = 0.$$

It is equivalent to

$$W[y,\overline{y}](b,\lambda,m) = 0. \tag{3.7}$$

By using (3.1), (3.7) can be expanded as

$$|m|^2 W[y_2,\overline{y_2}](b,\lambda) + mW[y_2,\overline{y_1}](b,\lambda) + \bar{m}W[y_1,\overline{y_2}](b,\lambda) + W[y_1,\overline{y_1}](b,\lambda) = 0. \tag{3.8}$$

Moreover, setting $m = u + iv$, we have

$$W[y_2,\overline{y_2}](b,\lambda) = 2iA, \qquad W[y_1,\overline{y_1}](b,\lambda) = 2iD, \qquad -W[y_2,\overline{y_1}](b,\lambda) = B + iC. \tag{3.9}$$

It follows from the last relation in (3.9) that we have $W[y_1,\overline{y_2}](b,\lambda) = B - iC$. By using (2.3) and (3.9), it can be verified that

$$\begin{aligned} &B^2 + C^2 - 4AD \\ &\quad = (B+iC)(B-iC) + (2iA)(2iD) \end{aligned}$$

$$\begin{aligned}
&= \left[W[y_2,\overline{y_2}](b,\lambda)\cdot W[y_1,\overline{y_1}](b,\lambda) - W[y_2,\overline{y_1}](b,\lambda)\cdot W[y_1,\overline{y_2}](b,\lambda)\right] \\
&= \left[W[\overline{y_1},\overline{y_2}](b,\lambda)\cdot W[y_1,y_2](b,\lambda)\right] \\
&= \left|W[y_1,y_2](b,\lambda)\right|^2 > 0.
\end{aligned} \tag{3.10}$$

It follows from the first relation in (3.9) and (3.5) that we have $A = \nu\int_{\rho(0)}^{b} w(t)|y_2^{\sigma}(t,\lambda)|^2\Delta t \neq 0$. Then (3.8) becomes

$$\left(u - \frac{C}{2A}\right)^2 + \left(v - \frac{B}{2A}\right)^2 = \frac{B^2 + C^2 - 4AD}{4A^2}, \tag{3.11}$$

which implies that (3.3) forms a circle C_b as k varies. It is evident that the center of C_b is

$$z_0 = \frac{C+iB}{2A} = -\frac{B-iC}{2iA} = -\frac{W[y_1,\overline{y_2}](b,\lambda)}{W[y_2,\overline{y_2}](b,\lambda)}.$$

It follows from Lemma 2.4 and (3.10) that

$$B^2 + C^2 - 4AD = \left|W[y_1,y_2](b,\lambda)\right|^2 = \left|W[y_1,y_2]\big(\rho(0),\lambda\big)\right|^2 = 1.$$

From (3.11), (3.9), (2.3), and (3.5) we have that the radius of C_b is

$$\begin{aligned}
r_b &= \left|\frac{B^2 + C^2 - 4AD}{4A^2}\right|^{\frac{1}{2}} \\
&= |2iA|^{-1} \\
&= \left|W[y_2,\overline{y_2}](b,\lambda)\right|^{-1} \\
&= \left[2|\nu|\int_{\rho(0)}^{b} w(t)\left|y_2^{\sigma}(t,\lambda)\right|^2\Delta t\right]^{-1}.
\end{aligned} \tag{3.12}$$

Let $\overline{C_b}$ denote the closed disk bounded by C_b. We are going to show that the circle sequence $\{\overline{C_b}\}(\rho(0) < b < \infty)$ is nested.

Set

$$U + iV = \nu\int_{\rho(0)}^{b} w(t)y_1^{\sigma}(t,\lambda)\overline{y_2^{\sigma}(t,\lambda)}\Delta t.$$

From the first relation in (3.9), we have

$$A = \nu\int_{\rho(0)}^{b} w(t)\left|y_2^{\sigma}(t,\lambda)\right|^2\Delta t.$$

Similarly,

$$D = \nu\int_{\rho(0)}^{b} w(t)\left|y_1^{\sigma}(t,\lambda)\right|^2\Delta t.$$

So, it follows from (3.6) that

$$v = A\left(u^2 + v^2\right) + 2Uu + 2Vv + D. \tag{3.13}$$

In the case of $\nu > 0$, the point $m = u + i\nu$ is interior to the circle if $\nu > A(u^2 + \nu^2) + 2Uu + 2V\nu + D$. This shows that $m \in \overline{C_b}$ if and only if

$$\Im(m) \geq \nu \int_{\rho(0)}^{b} w(t)\left|y^\sigma(t,\lambda,m)\right|^2 \Delta t.$$

Let $b_1, b_2 \in [\rho(0),\infty) \cap \mathbb{T}$ with $b_1 < b_2$ and consider the corresponding disks $\overline{C_{b_1}}$ and $\overline{C_{b_2}}$. For any $m \in \overline{C_{b_2}}$, we have

$$\Im(m) \geq \nu \int_{\rho(0)}^{b_2} w(t)\left|y^\sigma(t,\lambda,m)\right|^2 \Delta t \geq \nu \int_{\rho(0)}^{b_1} w(t)\left|y^\sigma(t,\lambda,m)\right|^2 \Delta t.$$

Hence, $m \in \overline{C_{b_1}}$. This yields that $\overline{C_{b_2}} \subset \overline{C_{b_1}}$. Therefore, $\{\overline{C_b}\}$ is nested. Consequently, there are the following two alternatives:

(1) $r_b \to 0$ as $b \to \infty$. In this case there is one point $m = m(\lambda)$ which is common to all the disks $\overline{C_b}$, $b \in [\rho(0),\infty) \cap \mathbb{T}$. This is called the limit-point case. It follows from (3.12) that this case occurs if and only if

$$\int_{\rho(0)}^{\infty} w(t)\left|y_2^\sigma(t,\lambda)\right|^2 \Delta t = \infty. \tag{3.14}$$

(2) $r_b \to r_\infty > 0$ as $b \to \infty$. In this case there is a disk $\overline{C_\infty}$ contained in all the disks $\overline{C_b}$, $b \in [\rho(0),\infty) \cap \mathbb{T}$. This is called the limit-circle case. It follows from (3.12) that this case occurs if and only if the integral in (3.14) is convergent, *i.e.*, $y_2(\cdot,\lambda) \in L_w^2(\rho(0),\infty)$.

Theorem 3.1 *For every non-real* $\lambda \in \mathbb{C}$, *Eq.* (1.1) *has at least one non-trivial solution in* $L_w^2(\rho(0),\infty)$.

Proof In the limit-circle case, it follows from the above discussion that $y_2(\cdot,\lambda) \in L_w^2(\rho(0),\infty)$.

Next, we will show that $y_1(\cdot,\lambda) + m(\lambda)y_2(\cdot,\lambda) \in L_w^2(\rho(0),\infty)$ in the limit-point case. Let $\{b_n\} \subset \mathbb{T}$ with $0 < b_n < b_{n+1} \to \infty$ and choose any $m_n \in C_{b_n}$. Then $m_n \to m(\lambda)$ as $n \to \infty$ and $y^\sigma(t,\lambda,m_n)$ uniformly converges to $y^\sigma(t,\lambda,m(\lambda))$ on any finite interval $[\rho(0),\omega] \cap \mathbb{T}$, $\omega \in \mathbb{T}$. Since the sequence $\{\Im(m_n)\}$ is bounded from above and its upper bound is denoted by y_0, then for $b_n > \omega$,

$$y_0 \geq \Im(m_n) = \nu \int_{\rho(0)}^{b_n} w(t)\left|y^\sigma(t,\lambda,m_n)\right|^2 \Delta t \geq \nu \int_{\rho(0)}^{\omega} w(t)\left|y^\sigma(t,\lambda,m_n)\right|^2 \Delta t.$$

Hence, by the uniform convergence of $y^\sigma(t,\lambda,m_n)$, we have

$$y_0 \geq \nu \int_{\rho(0)}^{\omega} w(t)\left|y^\sigma\bigl(t,\lambda,m(\lambda)\bigr)\right|^2 \Delta t$$

for all ω. Therefore, $y(\cdot,\lambda,m(\lambda)) = y_1(\cdot,\lambda) + m(\lambda)y_2(\cdot,\lambda) \in L_w^2(\rho(0),\infty)$. This completes the proof. □

Remark 3.1 Similar to the proof of Theorem 3.1, it can be easily verified that $y(\cdot,\lambda,m) \in L_w^2(\rho(0),\infty)$ for any $m \in C_\infty$ with $\Im(\lambda) \neq 0$ in the limit-circle case. Clearly, $y(t,\lambda,m)$ and

$y_2(t,\lambda)$ are linearly independent. Hence, all the solutions of Eq. (1.1) belong to $L^2_w(\rho(0),\infty)$ for any $\lambda \in \mathbb{C}$ with $\Im(\lambda) \neq 0$ in the limit-circle case.

Remark 3.2 It follows from (3.14) and Theorem 3.1 that Eq. (1.1) has exactly one linearly independent solution in $L^2_w(\rho(0),\infty)$ in the limit point case for any $\lambda \in \mathbb{C}$ with $\Im(\lambda) \neq 0$.

Theorem 3.2 *If Eq.* (1.1) *has two linearly independent solutions in* $L^2_w(\rho(0),\infty)$ *for some* $\lambda_0 \in \mathbb{C}$, *then this property holds for all* $\lambda \in \mathbb{C}$.

Proof Suppose that Eq. (1.1) has two linearly independent solutions in $L^2_w(\rho(0),\infty)$ for $\lambda = \lambda_0 \in \mathbb{C}$. Then $y_1(t,\lambda_0)$ and $y_2(t,\lambda_0)$ are in $L^2_w(\rho(0),\infty)$. For briefness, denote

$$u_1(t) = y_1(t,\lambda_0), \qquad u_2(t) = y_2(t,\lambda_0).$$

For any $\lambda \in \mathbb{C}$, let $v(t)$ be an arbitrary non-trivial solution of (1.1), and let $u(t)$ be the solution of (1.1) with $\lambda = \lambda_0$ and with the initial values

$$u(a) = v(a), \qquad u^\Delta(a) = v^\Delta(a), \qquad a \in (0,\infty) \cap \mathbb{T}.$$

From the variation of constants [27, Theorem 3.73], we have

$$v(t) = u(t) + (\lambda - \lambda_0) \int_a^t \big[u_1(t)u_2^\sigma(s) - u_2(t)u_1^\sigma(s)\big] w(s) v^\sigma(s)\Delta s, \quad t \in [a,\infty) \cap \mathbb{T}. \tag{3.15}$$

Replacing t with $\sigma(t)$ in (3.15) and using (ii) of Lemma 2.2, we obtain

$$\begin{aligned} &w^{\frac{1}{2}}(t)v^\sigma(t) \\ &\quad = w^{\frac{1}{2}}(t)u^\sigma(t) + (\lambda - \lambda_0) \int_a^{\sigma(t)} \big[w^{\frac{1}{2}}(t)u_1^\sigma(t)w^{\frac{1}{2}}(s)u_2^\sigma(s) \\ &\qquad - w^{\frac{1}{2}}(t)u_2^\sigma(t)w^{\frac{1}{2}}(s)u_1^\sigma(s)\big] w^{\frac{1}{2}}(s)v^\sigma(s)\Delta s \\ &\quad = w^{\frac{1}{2}}(t)u^\sigma(t) + (\lambda - \lambda_0) \int_a^{t} \big[w^{\frac{1}{2}}(t)u_1^\sigma(t)w^{\frac{1}{2}}(s)u_2^\sigma(s) \\ &\qquad - w^{\frac{1}{2}}(t)u_2^\sigma(t)w^{\frac{1}{2}}(s)u_1^\sigma(s)\big] w^{\frac{1}{2}}(s)v^\sigma(s)\Delta s, \end{aligned}$$

which implies by the Hölder inequality in Lemma 2.2 that

$$\begin{aligned} \big|w^{\frac{1}{2}}(t)v^\sigma(t)\big| \le{}& \big|w^{\frac{1}{2}}(t)u^\sigma(t)\big| + |\lambda - \lambda_0|\big|w^{\frac{1}{2}}(t)u_1^\sigma(t)\big| \\ &\times \left[\int_a^t w(s)\big|u_2^\sigma(s)\big|^2\Delta s \int_a^t w(s)\big|v^\sigma(s)\big|^2\Delta s\right]^{\frac{1}{2}} \\ &+ |\lambda - \lambda_0|\big|w^{\frac{1}{2}}(t)u_2^\sigma(t)\big|\left[\int_a^t w(s)\big|u_1^\sigma(s)\big|^2\Delta s \int_a^t w(s)\big|v^\sigma(s)\big|^2\Delta s\right]^{\frac{1}{2}}. \end{aligned}$$

It follows from the inequality

$$(A + B + C)^2 \le 3\big(A^2 + B^2 + C^2\big),$$

where A, B, C are non-negative numbers, that

$$\frac{1}{3}w(t)\left|v^\sigma(t)\right|^2 \le w(t)\left|u^\sigma(t)\right|^2 + |\lambda-\lambda_0|^2\left[w(t)\left|u_1^\sigma(t)\right|^2\int_a^t w(s)\left|u_2^\sigma(s)\right|^2\Delta s\right.$$
$$\left. + w(t)\left|u_2^\sigma(t)\right|^2\int_a^t w(s)\left|u_1^\sigma(s)\right|^2\Delta s\right]\int_a^t w(s)\left|v^\sigma(s)\right|^2\Delta s.$$

Integrating the two sides of the above inequality with respect to t from a to $\tau \in (a,\infty)\cap\mathbb{T}$, we get

$$\frac{1}{3}\int_a^\tau w(t)\left|v^\sigma(t)\right|^2\Delta t \le \int_a^\tau w(t)\left|u^\sigma(t)\right|^2\Delta t$$
$$+ |\lambda-\lambda_0|^2\int_a^\tau\left[w(t)\left|u_1^\sigma(t)\right|^2\int_a^t w(s)|u_2^\sigma(s)|^2\Delta s\right.$$
$$\left. + w(t)\left|u_2^\sigma(t)\right|^2\int_a^t w(s)\left|u_1^\sigma(s)\right|^2\Delta s\right]\int_a^t w(s)\left|v^\sigma(s)\right|^2\Delta s\Delta t,$$

which yields that

$$\frac{1}{3}\int_a^\tau w(t)\left|v^\sigma(t)\right|^2\Delta t$$
$$\le \int_a^\infty w(t)\left|u^\sigma(t)\right|^2\Delta t$$
$$+ 2|\lambda-\lambda_0|^2\int_a^\infty w(t)\left|u_1^\sigma(t)\right|^2\Delta t\int_a^\infty w(t)\left|u_2^\sigma(t)\right|^2\Delta t\int_a^\tau w(t)\left|v^\sigma(t)\right|^2\Delta t.$$

Hence,

$$\left(1-6|\lambda-\lambda_0|^2\int_a^\infty w(t)\left|u_1^\sigma(t)\right|^2\Delta t\int_a^\infty w(t)\left|u_2^\sigma(t)\right|^2\Delta t\right)\int_a^\tau w(t)\left|v^\sigma(t)\right|^2\Delta t$$
$$\le 3\int_a^\infty w(t)\left|u^\sigma(t)\right|^2\Delta t. \tag{3.16}$$

The constant a can be chosen in advance so large that

$$6|\lambda-\lambda_0|^2\int_a^\infty w(t)\left|u_1^\sigma(t)\right|^2\Delta t\int_a^\infty w(t)\left|u_2^\sigma(t)\right|^2\Delta t < 1.$$

It follows from (3.16) that $v \in L_w^2(a,\infty)$ and hence $v \in L_w^2(\rho(0),\infty)$. Therefore, all the solutions of Eq. (1.1) are in $L_w^2(\rho(0),\infty)$. The proof is complete. □

At the end of this section, from the above discussions we present the classification of the limit cases for singular second-order linear equations over the infinite interval $[\rho(0),\infty)\cap\mathbb{T}$ on time scales.

Definition 3.1 If Eq. (1.1) has only one linear independent solution in $L_w^2(\rho(0),\infty)$ for some $\lambda\in\mathbb{C}$, then Eq. (1.1) is said to be in the limit-point case at $t=\infty$. If Eq. (1.1) has two linear independent solutions in $L_w^2(\rho(0),\infty)$ for some $\lambda\in\mathbb{C}$, then Eq. (1.1) is said to be in the limit-circle case at $t=\infty$.

4 Several criteria of the limit-point and limit-circle cases

In this section, we establish several criteria of the limit-point and limit-circle cases for Eq. (1.1).

We first give two criteria of the limit-point case.

Theorem 4.1 *Let* $w(t) \equiv 1$ *and* $p(t) > 0$ *for all* $t \in [\rho(0), \infty) \cap \mathbb{T}$. *If there exists a positive* Δ*-differentiable function* $M(t)$ *on* $[a, \infty) \cap \mathbb{T}$ *for some* $a \geq \rho(0)$ *and two positive constants* k_1 *and* k_2 *such that for all* $t \in [a, \infty) \cap \mathbb{T}$,

(i) $q(t) \geq -k_1 M^\sigma(t)$,

(ii) $p^{\frac{1}{2}}(t)|M^\Delta(t)|(M(t))^{-1}(M^\sigma(t))^{-\frac{1}{2}} \leq k_2$,

(iii) $\int_a^\infty (p(t)M^\sigma(t))^{-\frac{1}{2}}\Delta t = \infty$,

then Eq. (1.1) *is in the limit-point case at* $t = \infty$.

Proof Suppose that Eq. (1.1) is in the limit-circle case at $t = \infty$. By Theorem 3.2, all the solutions of

$$-\left(p(t)y^\Delta(t)\right)^\Delta + q(t)y^\sigma(t) = 0, \quad t \in \left[\rho(0), \infty\right) \cap \mathbb{T} \tag{4.1}$$

are in $L_w^2(\rho(0), \infty)$. Let $y_1(t)$ and $y_2(t)$ be the solutions of (4.1) satisfying the following initial conditions:

$$y_1\left(\rho(0)\right) = p\left(\rho(0)\right)y_2^\Delta\left(\rho(0)\right) = 0, \qquad p\left(\rho(0)\right)y_1^\Delta\left(\rho(0)\right) = y_2\left(\rho(0)\right) = 1. \tag{4.2}$$

It is evident that $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of (4.1) in $L_w^2(\rho(0), \infty)$. By Lemma 2.4, $W[y_1, y_2](t) \equiv 1$ for all $t \in [\rho(0), \infty) \cap \mathbb{T}$. Hence, we have

$$\begin{aligned} &y_1(t)\left\{p^{\frac{1}{2}}(t)y_2^\Delta(t)\left(M^\sigma(t)\right)^{-\frac{1}{2}}\right\} - y_2(t)\left\{p^{\frac{1}{2}}(t)y_1^\Delta(t)\left(M^\sigma(t)\right)^{-\frac{1}{2}}\right\} \\ &\quad = \left(p(t)M^\sigma(t)\right)^{-\frac{1}{2}}, \quad t \in [a, \infty) \cap \mathbb{T}. \end{aligned}$$

It follows from the Hölder inequality and assumption (iii) that

$$\int_a^\infty \frac{p(\tau)(y_1^\Delta(\tau))^2}{M^\sigma(\tau)}\Delta\tau \quad \text{or} \quad \int_a^\infty \frac{p(\tau)(y_2^\Delta(\tau))^2}{M^\sigma(\tau)}\Delta\tau$$

are divergent. Suppose

$$\int_a^\infty \frac{p(\tau)(y_1^\Delta(\tau))^2}{M^\sigma(\tau)}\Delta\tau = \infty.$$

From (4.1) and assumption (i), we have

$$\begin{aligned} \int_a^t \frac{y_1^\sigma(\tau)[p(\tau)y_1^\Delta(\tau)]^\Delta}{M^\sigma(\tau)}\Delta\tau &= \int_a^t \frac{q(\tau)(y_1^\sigma(\tau))^2}{M^\sigma(\tau)}\Delta\tau \\ &\geq -k_1\int_a^t \left(y_1^\sigma(\tau)\right)^2\Delta\tau. \end{aligned} \tag{4.3}$$

Applying integration by parts in Lemma 2.2, by (iii) in Lemma 2.1, we get

$$\begin{aligned}\int_a^t &\frac{y_1^\sigma(\tau)[p(\tau)y_1^\Delta(\tau)]^\Delta}{M^\sigma(\tau)}\Delta\tau \\ &= \left[\frac{y_1(t)p(t)y_1^\Delta(t)}{M(t)}\right]\bigg|_a^t - \int_a^t \frac{p(\tau)(y_1^\Delta(\tau))^2}{M^\sigma(\tau)}\Delta\tau + \int_a^t \frac{y_1(\tau)p(\tau)y_1^\Delta(\tau)M^\Delta(\tau)}{M(\tau)M^\sigma(\tau)}\Delta\tau. \qquad (4.4)\end{aligned}$$

Again applying the Hölder inequality, from condition (ii), we have

$$\begin{aligned}\left|\int_a^t \frac{y_1(\tau)p(\tau)y_1^\Delta(\tau)M^\Delta(\tau)}{M(\tau)M^\sigma(\tau)}\Delta\tau\right| &\le \left\{\int_a^t \frac{p(\tau)(M^\Delta(\tau))^2 y_1^2(\tau)}{M^2(\tau)M^\sigma(\tau)}\Delta\tau\right\}^{\frac{1}{2}} H^{\frac{1}{2}}(t) \\ &\le k_2\left\{\int_a^\infty y_1^2(\tau)\Delta\tau\right\}^{\frac{1}{2}} H^{\frac{1}{2}}(t), \qquad (4.5)\end{aligned}$$

where

$$H(t) := \int_a^t \frac{p(\tau)(y_1^\Delta(\tau))^2}{M^\sigma(\tau)}\Delta\tau.$$

Since

$$\int_a^\infty \left(y_1^\sigma(\tau)\right)^2\Delta\tau > \int_a^t \left(y_1^\sigma(\tau)\right)^2\Delta\tau,$$

it follows from (4.3)-(4.5) that

$$\begin{aligned}&\frac{y_1(t)p(t)y_1^\Delta(t)}{M(t)} \\ &\quad > \frac{y_1(a)p(a)y_1^\Delta(a)}{M(a)} + H(t) - k_2\left\{\int_a^\infty y_1^2(\tau)\Delta\tau\right\}^{\frac{1}{2}} H^{\frac{1}{2}}(t) - k_1\int_a^\infty \left(y_1^\sigma(\tau)\right)^2\Delta\tau.\end{aligned}$$

It follows from the assumption that $H(t) \to \infty$ as $t \to \infty$. From the above relation and $p(t) > 0$ for all $t \in [\rho(0), \infty) \cap \mathbb{T}$, we have that $y_1(t)y_1^\Delta(t)$ is ultimately positive. Therefore, $y_1(t) \nrightarrow 0$ as $t \to \infty$; and consequently, $y_1(t)$ does not belong to $L_w^2(\rho(0), \infty)$. This contradicts the assumption that all the solutions of (4.1) are in $L_w^2(\rho(0), \infty)$. Then Eq. (4.1) has at least one non-trivial solution outside of $L_w^2(\rho(0), \infty)$. It follows from Theorem 3.2 that Eq. (1.1) is in the limit-point case at $t = \infty$. This completes the proof. □

Remark 4.1 Since $\mathbb{R}$ and $\mathbb{N}$ are two special time scales, Theorem 4.1 not only contains the criterion of the limit-point case for second-order differential equations [5, Chapter 9, Theorem 2.4], but also the criterion of the limit-point case for second-order difference equation (1.3) [15, Theorem 3.3].

The following corollary is a direct consequence of Theorem 4.1 by setting $M(t) \equiv 1$ for $t \in [\rho(0), \infty) \cap \mathbb{T}$.

Corollary 4.1 *If* $w(t) \equiv 1$, $p(t) > 0$, $q(t)$ *is bounded below in* $[\rho(0), \infty) \cap \mathbb{T}$, *and* $\int_{\rho(0)}^\infty (p(t))^{-\frac{1}{2}}\Delta t = \infty$, *then Eq.* (1.1) *is in the limit-point case at* $t = \infty$.

Theorem 4.2 *If*

$$\int_{\rho(0)}^{\infty} \frac{\mu^\sigma(t)[w(t)w^\sigma(t)]^{\frac{1}{2}}}{|p^\sigma(t)|}\Delta t = \infty, \tag{4.6}$$

then Eq. (1.1) *is in the limit-point case at* $t = \infty$.

Proof On the contrary, suppose that Eq. (1.1) is in the limit-circle case at $t = \infty$. Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of (1.1) in $L_w^2(\rho(0),\infty)$ satisfying the initial conditions (4.2). By Lemma 2.4, we have

$$W[y_1,y_2](t) = W[y_1,y_2]\big(\rho(0)\big) \equiv 1, \quad t \in [\rho(0),\infty) \cap \mathbb{T},$$

which, together with (2.1), implies that

$$\begin{aligned} &p(t)y_1(t)y_2^\sigma(t) - y_2(t)y_1^\sigma(t) \\ &\quad = y_1(t)p(t)\big(\mu(t)y_2^\Delta(t) + y_2(t)\big) - y_2(t)p(t)\big(\mu(t)y_1^\Delta(t) + y_1(t)\big) \\ &\quad = \mu(t)p(t)\big[y_1(t)y_2^\Delta(t) - y_2(t)y_1^\Delta(t)\big] \\ &\quad = \mu(t)W[y_1,y_2](t) = \mu(t), \quad t \in [\rho(0),\infty) \cap \mathbb{T}. \end{aligned}$$

So, we get

$$\big|y_1(t)\big|\big|y_2^\sigma(t)\big| + \big|y_2(t)\big|\big|y_1^\sigma(t)\big| \ge \frac{\mu(t)}{|p(t)|}, \quad t \in [\rho(0),\infty) \cap \mathbb{T},$$

which implies

$$\begin{aligned} &\big[w(t)w^\sigma(t)\big]^{\frac{1}{2}}\big[\big|y_1^\sigma(t)\big|\big|y_2^{\sigma^2}(t)\big| + \big|y_2^\sigma(t)\big|\big|y_1^{\sigma^2}(t)\big|\big] \\ &\quad \ge \frac{\mu^\sigma(t)[w(t)w^\sigma(t)]^{\frac{1}{2}}}{|p^\sigma(t)|}, \quad t \in [\rho(0),\infty) \cap \mathbb{T}, \end{aligned} \tag{4.7}$$

where $y^{\sigma^2}(t) = y^\sigma(\sigma(t))$. By the Hölder inequality and the assumption that $y_1, y_2 \in L_w^2(\rho(0),\infty)$, one has

$$\begin{aligned} &\int_{\rho(0)}^{\infty}\big[w(t)w^\sigma(t)\big]^{\frac{1}{2}}\big[\big|y_1^\sigma(t)\big|\big|y_2^{\sigma^2}(t)\big| + \big|y_2^\sigma(t)\big|\big|y_1^{\sigma^2}(t)\big|\big]\Delta t \\ &\quad \le \left(\int_{\rho(0)}^{\infty} w(t)\big|y_1^\sigma(t)\big|^2\Delta t\right)^{\frac{1}{2}}\left(\int_{\rho(0)}^{\infty} w^\sigma(t)\big|y_2^{\sigma^2}(t)\big|^2\Delta t\right)^{\frac{1}{2}} \\ &\qquad + \left(\int_{\rho(0)}^{\infty} w(t)\big|y_2^\sigma(t)\big|^2\Delta t\right)^{\frac{1}{2}}\left(\int_{\rho(0)}^{\infty} w^\sigma(t)\big|y_1^{\sigma^2}(t)\big|^2\Delta t\right)^{\frac{1}{2}} < \infty. \end{aligned}$$

Hence, it follows from (4.7) that

$$\int_{\rho(0)}^{\infty} \frac{\mu^\sigma(t)[w(t)w^\sigma(t)]^{\frac{1}{2}}}{|p^\sigma(t)|}\Delta t < \infty,$$

which is a contradiction to the assumption (4.6). Therefore, Eq. (1.1) is in the limit-point case at $t = \infty$. This completes the proof. □

Remark 4.2 Let $\mathbb{T} = \mathbb{N}$, Theorem 4.2 is the same as that obtained by Chen and Shi for second-order difference equations [13, Corollary 3.1].

Next, we study the invariance of the limit cases under a bounded perturbation for the potential function q. Let $f(t) = M$ and $p(t) = f(t)$ in [27, Theorem 2.4(i)]. It follows from [27, Theorem 2.36(i)], [27, Theorem 2.39(i)], and [27, Theorem 2.4(i)] that we have the following lemma, which is useful in the subsequent discussion.

Lemma 4.1 (Gronwall's inequality) *Let $y, f \in C_{rd}(\mathbb{T})$ be two non-negative functions on $[\rho(0),\infty)\cap\mathbb{T}$ and M be a non-negative constant. If*

$$y(t) \le M + \int_{\rho(0)}^{t} f(\tau)y(\tau)\Delta\tau \quad \textit{for all } t \in \bigl[\rho(0),\infty\bigr)\cap\mathbb{T}, \tag{4.8}$$

then

$$y(t) \le Me_f\bigl(t,\rho(0)\bigr) \quad \textit{for all } t \in \bigl[\rho(0),\infty\bigr)\cap\mathbb{T},$$

where $e_f(t,s)$ is defined as in (2.2).

The following result shows that if Eq. (1.1) is in the limit-circle case, so is it under a bounded perturbation for the potential function q.

Lemma 4.2 *Let $q(t) = d(t) + e(t)$ for all $t \in [\rho(0),\infty)\cap\mathbb{T}$ and $e(t)$ be bounded with respect to $w(t)$ on $[\rho(0),\infty)\cap\mathbb{T}$; that is, there exists a positive constant M such that*

$$\bigl|e(t)\bigr| \le Mw(t), \quad t \in \bigl[\rho(0),\infty\bigr)\cap\mathbb{T}. \tag{4.9}$$

Then Eq. (1.1) *is in the limit-circle case at $t = \infty$ if and only if the equation*

$$-\bigl(p(t)y^{\Delta}(t)\bigr)^{\Delta} + d(t)y^{\sigma}(t) = \lambda w(t)y^{\sigma}(t) \tag{4.10}$$

is in the limit-circle case at $t = \infty$.

Proof Suppose that (4.10) is in the limit-circle case at $t = \infty$. To show that Eq. (1.1) is in the limit-circle case, it suffices to show that each solution (4.1) is in $L^2_w(\rho(0),\infty)$ by Theorem 3.2.

Let $y_1(t)$ and $y_2(t)$ be two solutions of the equation

$$-\bigl(p(t)y^{\Delta}(t)\bigr)^{\Delta} + d(t)y^{\sigma}(t) = 0 \tag{4.11}$$

satisfying the initial conditions (4.2). Then $y_1(t)$, $y_2(t)$ are two linearly independent solutions in $L^2_w(\rho(0),\infty)$ by Theorem 3.2.

Let $y(t)$ be any solution of (4.1). Then

$$-\big(p(t)y^{\Delta}(t)\big)^{\Delta} + d(t)y^{\sigma}(t) = r(t) \quad \text{for all } t \in \big[\rho(0), \infty\big) \cap \mathbb{T},$$

where $r(t) := -e(t)y^{\sigma}(t)$. By the variation of constants [27, Theorem 3.73] there exist two constants α and β such that

$$y(t) = \alpha y_1(t) + \beta y_2(t) + \int_{\rho(0)}^{t} r(\tau)\big(y_1^{\sigma}(\tau)y_2(t) - y_2^{\sigma}(\tau)y_1(t)\big)\Delta\tau \quad \text{for all } t \in \big[\rho(0), \infty\big) \cap \mathbb{T}.$$

Hence, replacing t by $\sigma(t)$ and by (ii) in Lemma 2.2, we get

$$y^{\sigma}(t) = \alpha y_1^{\sigma}(t) + \beta y_2^{\sigma}(t) + \int_{\rho(0)}^{t} r(\tau)\big(y_1^{\sigma}(\tau)y_2^{\sigma}(t) - y_2^{\sigma}(\tau)y_1^{\sigma}(t)\big)\Delta\tau. \tag{4.12}$$

From (4.9) and (4.12), we have

$$\begin{aligned} \big|y^{\sigma}(t)\big| &\le |\alpha|\big|y_1^{\sigma}(t)\big| + |\beta|\big|y_2^{\sigma}(t)\big| \\ &\quad + M\int_{\rho(0)}^{t} \big(\big|y_1^{\sigma}(\tau)\big|\big|y_2^{\sigma}(t)\big| + \big|y_2^{\sigma}(\tau)\big|\big|y_1^{\sigma}(t)\big|\big)w(\tau)\big|y^{\sigma}(\tau)\big|\Delta\tau. \end{aligned} \tag{4.13}$$

Since $y_1(t)$, $y_2(t)$ are solutions of Eq. (4.11), which satisfy the initial conditions (4.2), it follows from the existence-uniqueness theorem that $|y_1^{\sigma}(t)| + |y_2^{\sigma}(t)| \neq 0$ for all $t \in [\rho(0), \infty) \cap \mathbb{T}$. Let

$$y_0^{\sigma}(t) := \frac{|y^{\sigma}(t)|}{|y_1^{\sigma}(t)| + |y_2^{\sigma}(t)|} \quad \text{for all } t \in \big[\rho(0), \infty\big) \cap \mathbb{T}.$$

From (4.13), we have

$$\begin{aligned} y_0^{\sigma}(t) &\le \frac{|\alpha||y_1^{\sigma}(t)| + |\beta||y_2^{\sigma}(t)|}{|y_1^{\sigma}(t)| + |y_2^{\sigma}(t)|} \\ &\quad + M\int_{\rho(0)}^{t} \frac{(|y_1^{\sigma}(\tau)||y_2^{\sigma}(t)| + |y_2^{\sigma}(\tau)||y_1^{\sigma}(t)|)w(\tau)|y^{\sigma}(\tau)|}{|y_1^{\sigma}(t)| + |y_2^{\sigma}(t)|}\Delta\tau \\ &\le |\alpha| + |\beta| + M\int_{\rho(0)}^{t} \big(\big|y_1^{\sigma}(\tau)\big| + \big|y_2^{\sigma}(\tau)\big|\big)w(\tau)\big|y^{\sigma}(\tau)\big|\Delta\tau \\ &= |\alpha| + |\beta| + M\int_{\rho(0)}^{t} \big(\big|y_1^{\sigma}(\tau)\big| + \big|y_2^{\sigma}(\tau)\big|\big)^2 w(\tau)\big|y_0^{\sigma}(\tau)\big|\Delta\tau \\ &\le |\alpha| + |\beta| + 2M\int_{\rho(0)}^{t} \big(\big|y_1^{\sigma}(\tau)\big|^2 + \big|y_2^{\sigma}(\tau)\big|^2\big)w(\tau)\big|y_0^{\sigma}(\tau)\big|\Delta\tau. \end{aligned}$$

It follows from (i) of Lemma 2.2 that $y_0^{\sigma}(\cdot) \in C_{rd}(\mathbb{T})$. By Lemma 4.1, we have

$$\begin{aligned} y_0^{\sigma}(t) &\le \big(|\alpha| + |\beta|\big)e_{[2Mw(|y_1^{\sigma}|^2+|y_2^{\sigma}|^2)]}\big(t, \rho(0)\big) \\ &= \big(|\alpha| + |\beta|\big)\exp\bigg[\int_{\rho(0)}^{t} \xi_{\mu(\tau)}\big(2Mw(\tau)\big(\big|y_1^{\sigma}(\tau)\big|^2 + \big|y_2^{\sigma}(\tau)\big|^2\big)\big)\Delta\tau\bigg] \\ &= \big(|\alpha| + |\beta|\big)\exp\bigg[\int_{\rho(0)}^{t} \frac{1}{\mu(\tau)}\operatorname{Log}\big(1 + \mu(\tau)2Mw(\tau)\big(\big|y_1^{\sigma}(\tau)\big|^2 + \big|y_2^{\sigma}(\tau)\big|^2\big)\big)\Delta\tau\bigg] \end{aligned}$$

$$\le \left(|\alpha| + |\beta|\right) \exp\left[\int_{\rho(0)}^{t} 2Mw(\tau)\left(\left|y_1^\sigma(\tau)\right|^2 + \left|y_2^\sigma(\tau)\right|^2\right)\Delta\tau\right]$$
$$\le \left(|\alpha| + |\beta|\right) \exp\left[\int_{\rho(0)}^{\infty} 2Mw(\tau)\left(\left|y_1^\sigma(\tau)\right|^2 + \left|y_2^\sigma(\tau)\right|^2\right)\Delta\tau\right] =: C < \infty,$$

which implies that $|y^\sigma(t)| \le C(|y_1^\sigma(t)| + |y_2^\sigma(t)|)$. Hence, $y(\cdot) \in L_w^2(\rho(0),\infty)$; and consequently, Eq. (1.1) is in the limit-circle case at $t = \infty$.

On the other hand, using

$$-\left(p(t)y^\Delta(t)\right)^\Delta + d(t)y^\sigma(t) = -\left(p(t)y^\Delta(t)\right)^\Delta + \left(q(t) - e(t)\right)y^\sigma(t)$$

one can easily conclude that if Eq. (1.1) is in the limit-circle case, then Eq. (4.10) is in the limit-circle case. This completes the proof. □

Theorem 4.3 *Let $q(t) = d(t) + e(t)$ for all $t \in [\rho(0),\infty) \cap \mathbb{T}$ and $e(t)$ be bounded with respect to $w(t)$ on $[\rho(0),\infty) \cap \mathbb{T}$. Then the limit cases for Eq.* (1.1) *are invariant.*

Remark 4.3 Lemma 4.2 extends the related result [13, Lemma 2.4] for the singular second-order difference equation to the time scales. In addition, let $\mathbb{T} = \mathbb{R}$ in Lemma 4.2, then we can directly prove [31, Theorem 6.1] with the similar method.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YS supervised the study and helped the revision. CZ carried out the main results of this article and drafted the manuscript. All the authors have read and approved the final manuscript.

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