Cui and Sun *Fixed Point Theory and Applications* 2013, 2013:345
http://www.fixedpointtheoryandapplications.com/content/2013/1/345

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems for a class of nonlinear operators in Hilbert spaces with lattice structure and application

Yujun Cui[1*] and Jingxian Sun[2]

*Correspondence:
cyj720201@163.com
[1]Department of Mathematics, Shandong University of Science and Technology, Qingdao, 266590, P.R. China
Full list of author information is available at the end of the article

Abstract
We discuss the existence of a fixed point for a class of nonlinear operators in Hilbert spaces with lattice structure by a combination of variational and partial ordered methods. An application to second-order ordinary differential equations is included.

1 Introduction

Let E be a Banach space with a cone P. Then E becomes an ordered Banach space under the partial ordering '$\leq$' which is induced by P. For the concepts and properties of the cone, we refer to [1, 2].

We call E a lattice under the partial ordering '$\leq$' if $\sup\{x,y\}$ and $\inf\{x,y\}$ exist for arbitrary $x,y\in E$. For $x\in E$, let

$$x^+ = \sup\{x,\theta\}, \qquad x^- = \inf\{x,\theta\}, \tag{1.1}$$

x^+ and x^- are called the positive part and the negative part of x, respectively, and obviously $x = x^+ + x^-$. Take $|x| = x^+ - x^-$, then $|x|\in P$. One can refer to [3] for the definition and properties of the lattice.

In recent years, many mathematicians have studied a fixed point theorem of nonlinear operators in an ordered Banach space by using topological methods and partial ordered methods (see [4–14] and references therein). However, to the best of our knowledge, few authors have studied the fixed point theorem in Hilbert spaces with lattice structure by applying variational and partial ordered methods. As a result, the goal of this paper is to fill the gap in this area.

Motivated by [7, 8], we obtain some new theorems for nonlinear operators which are not cone mappings by means of variational and partial ordered methods. This paper is organized as follows. Section 2 is devoted to our main results. Section 3 gives examples to indicate the application of our main results.

At the end of this section, we give the following basic concept and lemma from literature which will be used in Section 2.

We consider two real ordered Hilbert spaces in this paper, H with the inner product, norm and cone $(\cdot,\cdot)$, $\|\cdot\|$, P, and X with the inner product, norm and cone P_1, and we assume that $H\subset X$ and $P\subset P_1$, X is dense in H, the injection being continuous.

Definition 1.1 Let $D \subset H$ and $F : D \to X$ be a nonlinear operator. F is said to be quasi-additive on lattice, if there exists $y \in X$ such that

$$Fx = Fx^+ + Fx^- + y, \quad \forall x \in D, \tag{1.2}$$

where x^+ and x^- are defined by (1.1). Note that if $F\theta = \theta$, (1.2) becomes $Fx = Fx^+ + Fx^-$.

Let $B : X \to H \subset X$ be a bounded linear operator. B is said to be positive if $B(P_1) \subset P$. In this case, B is an increase operator, namely for $x, y \in X$, $x \le y$ implies $Bx \le By$. We have the following conclusion.

Lemma 1.1 ([1, 7]) *Suppose that $B : X \to H \subset X$ is a positive bounded linear operator. If the spectral radius $r(B) < 1$, then $(I - B)^{-1}$ exists and is a positive bounded linear operator. Furthermore,*

$$(I - B)^{-1} = I + B + B^2 + \cdots + B^n + \cdots.$$

2 Fixed point theorems

Let H be an ordered real Hilbert space with an ordering given by a closed cone P and suppose that the gradient $\Phi' : H \to H$ of a given functional $\Phi \in C^1(H, R)$ has the expression $\Phi' = I - A$. Obviously, the critical points of the functional Φ are the fixed points of the operator A, and vice versa.

We will show that, under additional assumptions on the operator A, Φ satisfies the Palais-Smale compactness condition on a closed convex set $M \subset H$, which ensures the existence of a critical point of Φ, see [15, 16].

(PS) Every sequence $\{v_m\}_{m\in N} \subset M$, satisfying the conditions
  (i) $\{\Phi(v_m)\}_{m\in N}$ is bounded,
  (ii) $\lim_{m\to+\infty} \|\Phi'(v_m)\| = 0$,
  has a subsequence which converges strongly in M.

The next lemma, the mountain pass lemma [17, 18] on a closed convex subset of a Hilbert space H, is crucial in the proof of our first result.

Lemma 2.1 *Assume that H is a Hilbert space, M is a closed convex subset of H, Φ is a C^1 functional defined on H, $\Phi'(u)$ can be expressed in the form $\Phi'(u) = u - Au$, and $A(M) \subset M$. Assume also that Φ satisfies the PS condition on M, Ω is an open subset of M, and there are two points $u_0 \in \Omega$, $u_1 \in M \backslash \Omega$ such that* $\max\{\Phi(u_0), \Phi(u_1)\} < \inf_{u\in\partial_M\Omega} \Phi(u)$. *Then*

$$c = \inf_{h\in\Phi_M} \max_{t\in[0,1]} f\big(h(t)\big)$$

is a critical value of Φ and there is at least one critical point in M corresponding to this value, where $\Phi_M = \{h \mid h : [0, 1] \to M$ is continuous, and $h(0) = u_0, h(1) = u_1\}$.

Theorem 2.1 *Let X, H be two ordered real Hilbert spaces. Suppose that $\Phi \in C^1(H, R)$ satisfies the following hypotheses.*

(i) *Φ satisfies the PS condition on H and its gradient Φ' admits the decomposition $I - A$ such that $A = BF$, where $F : H \to X$ is quasi-additive on lattice, $B : X \to H$ is a positive bounded linear operator satisfying*:

(1) $(Bx, y) \geq 0$ *for* $\forall x \in P_1$, $y \in P$;

(2) *There exists* $\varphi \in P$ *with* $\|\varphi\| = 1$ *such that*

$$B\varphi = r(B)\varphi,$$

that is, φ *is the normalized first eigenfunction.*

(ii) *There exist* $a_1 > r^{-1}(B)$ *and* $y_1 \in P_1$ *such that*

$$Fx \geq a_1 x - y_1, \quad x \in P. \tag{2.1}$$

(iii) *There exist* $0 < a_2 < r^{-1}(B)$ *and* $y_2 \in P_1$ *such that*

$$Fx \geq a_2 x - y_2, \quad x \in (-P). \tag{2.2}$$

(iv) *There exist* $M_1 \geq M_2 > 0$ *such that* $M_2\|x\| \leq \|x\|_1 \leq M_1\|x\|$, *where* $\|x\|_1$ *denotes the norm of* $|x|$.

(v) *There exist* $0 < a_3 < \frac{1}{M_1 r(B)}$ *and a positive number* r *such that*

$$|Fx| \leq a_3|x|, \quad x \in B_r(\theta), B_r(\theta) = \{x : \|x\| \leq r\}.$$

Then A has a nontrivial fixed point.

Remark 2.1 From Theorem 3.1 below, we point out that conditions (i) and (iv) of Theorem 2.1 appear naturally in the applications for nonlinear differential equations and integral equations.

Proof of Theorem 2.1 It follows from condition (v) that

$$Fx = Fx^+ + Fx^-.$$

Thus, we have

$$Ax = BFx = B(Fx^+ + Fx^-) = Ax^+ + Ax^-.$$

By virtue of (2.1) and (2.2), we have

$$Ax = BFx \geq a_1 Bx - u_1, \quad x \in P \tag{2.3}$$

and

$$Ax = BFx \geq a_2 Bx - u_2, \quad x \in (-P), \tag{2.4}$$

where $u_1 = By_1 \in P$ and $u_2 = By_2 \in P$. Since $0 < a_2 < r^{-1}(B)$, Lemma 1.1 yields that $(I - a_2B)^{-1}$ is a positive linear operator. Let $u_0 = u_1 + u_2$ and $w = -2(I - a_2B)^{-1}u_0$, then $w \in (-P)$ and $w - a_2Bw = -2u_0$. This shows $a_2Bw - u_0 = w + u_0 \geq w$. For $x \geq w$, by $w \in (-P)$ we infer $\theta \geq x^- \geq w$. On account of (2.3) and (2.4), we arrive at

$$Ax = Ax^+ + Ax^- \geq a_1Bx^+ - u_1 + a_2Bx^- - u_2 \geq a_1Bx^+ + a_2Bw - u_0 \geq a_1Bx^+ + w.$$

This implies

$$A(P_w) \subset P_w, \qquad Ax \geq a_1 Bx + w, \quad \forall x \in P_w = \{x \in H, x \geq w\}.$$

It is easy to see that P_w is a closed convex subset of H.

By the definition of gradient operator and $\Phi \in C^1(P_w, R)$, we have

$$\Phi(x) = \frac{1}{2}\|x\|^2 - \int_0^1 \bigl(A(sx), x\bigr)\, ds.$$

Noticing $\Phi(\theta) = 0$, we claim that there is a mountain surrounding θ and $\exists \alpha > 0$, $\exists \rho > 0$ and $x_0 \in P_w - B_\rho(\theta)$ with $\Phi(x_0) = 0$. Indeed, from (v), we have $r > 0$ such that

$$|Fx| \leq a_3 |x|, \quad x \in B_r(\theta).$$

Replacing x by x^+ in the above inequality, we have $|Fx^+| \leq a_3 x^+$. Thus,

$$-a_3 x^+ \leq Fx^+ \leq a_3 x^+, \quad x \in B_r(\theta).$$

Similarly, we have

$$a_3 x^- \leq Fx^- \leq -a_3 x^-, \quad x \in B_r(\theta).$$

Thus by (i), for $x \in B_r(\theta)$, we have

$$\begin{aligned}
\bigl(A(sx), x\bigr) &= \bigl(A(sx^+) + A(sx^-), x^+ + x^-\bigr) \\
&= \bigl(A(sx^+), x^+\bigr) + \bigl(A(sx^+), x^-\bigr) + \bigl(A(sx^-), x^+\bigr) + \bigl(A(sx^-), x^-\bigr) \\
&\leq \bigl(a_3 B(sx^+), x^+\bigr) - \bigl(a_3 B(sx^+), x^-\bigr) - \bigl(a_3 B(sx^-), x^+\bigr) + \bigl(a_3 B(sx^-), x^-\bigr) \\
&= s a_3 \bigl(B(|x|), |x|\bigr).
\end{aligned}$$

Let $B_1 = a_3 M_1 B$. Then by (iv) we get

$$\begin{aligned}
\Phi(x) &\geq \frac{1}{2}\|x\|^2 - \int_0^1 s a_3 \bigl(B|x|, |x|\bigr)\, ds \\
&= \frac{1}{2}\bigl(\|x\|^2 - \bigl(a_3 B|x|, |x|\bigr)\bigr) \\
&\geq \frac{1}{2M_1}\bigl(\|x\|_1^2 - \bigl(B_1|x|, |x|\bigr)\bigr) = \frac{1}{2M_1}\bigl((I - B_1)|x|, |x|\bigr), \quad x \in B_r(\theta).
\end{aligned}$$

Since $r(B_1) < 1$, Lemma 1.1 yields that $(I - B_1)^{-1}$ exists and

$$(I - B_1)^{-1} = I + B_1 + B_1^2 + \cdots + B_1^n + \cdots. \tag{2.5}$$

It follows from $B_1(P) \subset P$ that $(I - B_1)^{-1}(P) \subset P$. So we know that

$$\|x\|_1 \leq \bigl\|(I - B_1)^{-1}\bigr\| \|y\|, \quad y = (I - B_1)\bigl(|x|\bigr).$$

Combining with (2.5), we have

$$\begin{aligned}\Phi(x) &\geq \frac{1}{2M_1}\big((I-B_1)\big(|x|\big),|x|\big) = \frac{1}{2M_1}\big(y,(I-B_1)^{-1}y\big)\\ &\geq \frac{1}{2M_1}(y,y) \geq \frac{1}{2M_1}\big\|(I-B_1)^{-1}\big\|^{-2}\|x\|_1^2\\ &\geq \frac{M_2^2}{2M_1}\big\|(I-B_1)^{-1}\big\|^{-2}\|x\|^2, \quad x \in B_r(\theta).\end{aligned}$$

Therefore $\exists\alpha > 0$, $\exists\rho' > 0$ such that $\Phi|_{\partial_{P_W}B_{\rho'}(\theta)} \geq \alpha$.

Next, we take $x_0 = s\varphi$, where $\varphi > \theta$ is the normalized first eigenfunction of B, and $s > 0$ is to be determined. Set

$$g(s) = \Phi(s\varphi) = \frac{s^2}{2} - \int_0^1 \big(A(\tau s\varphi), s\varphi\big)\,d\tau.$$

From (2.3), we have

$$\begin{aligned}g(s) &\leq \frac{s^2}{2} - \int_0^1 \big(a_1B(\tau s\varphi), s\varphi\big)\,d\tau + s\|By_1\|\\ &= \frac{s^2}{2}\big(1-(a_1B\varphi,\varphi)\big) + s\|By_1\|\\ &= \frac{s^2}{2}\big(1-a_1r(B)\big) + s\|By_1\|.\end{aligned}$$

So we have $g(s) \to -\infty$ as $s \to +\infty$. Therefore there exists $s_0 > 0$ satisfying $g(s_0) = 0$, set $s = s_0$, x_0 is as required. Hence Theorem 2.1 holds by Lemma 2.1. □

3 Application

Consider the two-point boundary value problem

$$\begin{cases} x''(t) = f(t,x(t)), & t \in (0,1),\\ x(0) = x(1) = 0.\end{cases} \tag{3.1}$$

Here f is a continuous function from $\mathbb{R}$ into $\mathbb{R}$.

Let $H = H_0^1(0,1)$ with the inner product and norm

$$(u,v) = \int_0^1 u'v'\,dt, \qquad \|u\| = \left(\int_0^1 \big(u'\big)^2\,dt\right)^{\frac{1}{2}},$$

and $P = \{u \in H \mid u(t) \geq 0\}$, then H is a lattice under the partial ordering induced by P.

Let $X = L^2(0,1)$ with the cone

$$P_1 = \big\{u \in X \mid u(t) \geq 0\big\}.$$

For any $x \in H$, it is evident that

$$\big(\big|x(t)\big|\big)' = \begin{cases} x'(t) & \text{if } x(t) > 0,\\ 0 & \text{if } x(t) = 0,\\ -x'(t) & \text{if } x(t) < 0,\end{cases}$$

and hence $\|x\|_1 = \|x\|$ and so condition (iv) of Theorem 2.1 holds with $M_1 = M_2 = 1$.

Let

$$(Bx)(t) = \int_0^1 G(t,s)x(s)\,ds \tag{3.2}$$

with

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

It is easy to see that $B : P_1 \to P$ is a positive bounded linear operator satisfying $(Bx)''(t) = -x$. For any $x \in P_1$, $y \in P$, we have

$$\begin{aligned} (Bx, y) &= \int_0^1 (Bx)'(t)y'(t)\,dt \\ &= -\int_0^1 (Bx)''(t)y(t)\,dt \\ &= \int_0^1 x(t)y(t)\,dt \ge 0 \end{aligned}$$

and so condition (1) of Theorem 2.1 holds.

As is well known, B has an unbounded sequence of eigenvalues:

$$\lambda_n = n^2\pi, \quad n = 1, 2, \ldots.$$

The algebraic multiplicities of every eigenvalue are simple, and the spectral radius $r(B) = \lambda_1^{-1} = \frac{1}{\pi^2}$ and $\lambda_1 B(\sin \pi t) = \sin \pi t$.

Theorem 3.1 *Let* $f(t,u) : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be continuous. Suppose that there exists* $\varepsilon > 0$ *such that*

$$\limsup_{u\to-\infty} \frac{f(t,u)}{u} \le \lambda_1 - \varepsilon, \quad \textit{uniformly on } t \in [0,1], \tag{3.3}$$

$$\lambda_1 + \varepsilon \le \liminf_{u\to+\infty} \frac{f(t,u)}{u} \le \limsup_{u\to+\infty} \frac{f(t,u)}{u} < +\infty, \quad \textit{uniformly on } t \in [0,1], \tag{3.4}$$

$$\limsup_{u\to 0} \left|\frac{f(t,u)}{u}\right| \le \lambda_1 - \varepsilon, \quad \textit{uniformly on } t \in [0,1]. \tag{3.5}$$

Then (3.1) *has at least one nontrivial solution.*

Proof We define the functional $\Phi : H \to \mathbb{R}$ as

$$\Phi(u) = \int_0^1 \left(\frac{1}{2}\left|u'(t)\right|^2 - \int_0^{u(t)} f(t,\tau)\,d\tau\right) dt. \tag{3.6}$$

One has that $\Phi'(x)y = \int_0^1 x'y'\,dt - \int_0^1 (Fx)(t)y\,dt = \int_0^1 (x' - (BFx)'(t))y'\,dt$, which implies that $\Phi'(x) = x - BFx$ in H, where B is given by (3.2) and $(Fx)(t) = f(t, x(t))$. Furthermore, from the properties of H, the regularity properties of Φ, it is easy to assert that the solutions to (3.1) are precisely the critical points of Φ.

By means of (3.3), (3.4) and (3.5), we know that there exist $R_0 > r > 0$ such that

$$f(u) \geq \left(\lambda_1 - \frac{\varepsilon}{2}\right)u, \quad u \leq -R_0,$$

$$f(u) \geq \left(\lambda_1 + \frac{\varepsilon}{2}\right)u, \quad u \geq R_0,$$

$$\left|f(u)\right| \leq \left(\lambda_1 - \frac{\varepsilon}{2}\right)|u|, \quad |u| \leq r,$$

and therefore there exists a constant $M > 0$ such that

$$Fx \geq \left(\lambda_1 - \frac{\varepsilon}{2}\right)x - M, \quad -x \in P,$$

$$Fx \geq \left(\lambda_1 + \frac{\varepsilon}{2}\right)x - M, \quad x \in P,$$

$$|Fx| \leq \left(\lambda_1 - \frac{\varepsilon}{2}\right)|x|, \quad \|x\| \leq r,$$

and so conditions (ii), (iii) and (iv) of Theorem 2.1 hold.

From the above discussion, in order to apply Theorem 2.1, we only need to verify that Φ satisfies the PS condition. Suppose that $\{x_n\} \subset H$ satisfies $\Phi'(x_n) \to 0$ as $n \to \infty$ and $|\Phi(x_n)| \leq C$. Taking the inner product of $\Phi'(x_n)$ and x_n^-, we have

$$\circ(1)\left\|x_n^-\right\| = \left\|x_n^-\right\|^2 - \int_0^1 f(t, x_n)x_n^-\,dt.$$

We have

$$\begin{aligned} -\int_0^1 f(t, x_n)x_n^-\,dt &\geq -\int_{x(t)\leq R_0} f(t, x_n)x_n^-\,dt \\ &\geq -\int_{x(t)\leq -R_0} f(t, x_n)x_n^-\,dt - C_1\left\|x_n^-\right\|, \end{aligned}$$

which implies

$$-\int_0^1 f(t, x_n)x_n^-\,dt \geq -\left(\lambda_1 - \frac{1}{2}\varepsilon\right)\int_0^1 \left|x_n^-\right|^2 dt - C_1\left\|x_n^-\right\|.$$

Then we have

$$\circ(1)\left\|x_n^-\right\| \geq \left\|x_n^-\right\|^2 - \left(\lambda_1 - \frac{1}{2}\varepsilon\right)\int_0^1 \left|x_n^-\right|^2 dt - C_1\left\|x_n^-\right\| \geq \frac{\varepsilon}{2\lambda_1}\left\|x_n^-\right\|^2 - C_1\left\|x_n^-\right\|,$$

which gives a bound for $\{x_n^-\}$. Then both $\Phi(x_n^-)$ and $\Phi(x_n^+)$ are bounded. In order to find a bound for $\{x_n^+\}$, we use a contradiction argument and assume that $\|x_n^+\| \to \infty$ as $n \to \infty$. Defining $v_n = \frac{x_n^+}{\|x_n^+\|}$ and selecting a subsequence if necessary, we have $v_n \to v$ weakly in H and strongly in X as $n \to \infty$ for some $v \in H$, $v \geq 0$. Since $\int_0^u f(t, \tau)\,d\tau \leq C_2|u|^2$ for $u \geq 0$ and $\Phi(x_n^+)$ is bounded, it follows from (3.6) that

$$\frac{1}{2} = \frac{\Phi(x_n^+)}{\|x_n^+\|^2} + \int_0^1 \frac{\int_0^{x_n^+} f(t, \tau)\,d\tau}{\|x_n^+\|^2}\,dt \leq \circ(1) + C_2\int_0^1 v_n^2\,dt,$$

which implies that $v \neq 0$. The embedding theorem and the boundedness of $\{x_n^-\}$ in H guarantee that there exists $M_1 > 0$ such that $x_n^- \geq -M_1$ for all n. So $f(t,u)$ is bounded for $-M_1 \leq u \leq R_0$. Taking the inner product of $\frac{\Phi'(x_n)}{\|x_n^+\|}$ and $\sin\pi t$, we see that

$$\begin{aligned}(v_n, \sin\pi t) &= \left(\frac{\Phi'(x_n)}{\|x_n^+\|}, \sin\pi t\right) - \left(\frac{x_n^-}{\|x_n^+\|}, \sin\pi t\right) + \int_0^1 \frac{f(t,x_n)}{\|x_n^+\|}\sin\pi t\,dt \\ &\geq \circ(1) + \int_0^1 \frac{(\lambda_1 + \frac{\varepsilon}{2})x_n^+ - M}{\|x_n^+\|}\sin\pi t\,dt \\ &\geq \circ(1) + \left(\lambda_1 + \frac{\varepsilon}{2}\right)\int_0^1 v_n \sin\pi t\,dt.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\lambda_1 \int_0^1 v_n \sin\pi t\,dt \geq \left(\lambda_1 + \frac{\varepsilon}{2}\right)\int_0^1 v_n \sin\pi t\,dt,$$

which is a contradiction. Thus, $\{x_n\}$ is bounded in H and it has a convergent subsequence, as a standard consequence. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly to this research work. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Shandong University of Science and Technology, Qingdao, 266590, P.R. China.
[2]Department of Mathematics, Jiangsu Normal University, Xuzhou, 221116, P.R. China.

Acknowledgements
The authors would like to thank the referees for carefully reading this article and making valuable comments and suggestions. This work is supported by the Foundation items: NSFC (10971179, 11371221, 11071141, 61201431), the Postdoctoral Science Foundation of Shandong Province, NSF (BS2010SF023, BS2012SF022) of Shandong Province.

Received: 14 June 2013 Accepted: 1 December 2013 Published: 17 Dec 2013

References
1. Deimling, K: Nonlinear Functional Analysis. Springer, New York (1985)
2. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, San Diego (1988)
3. Luxemburg, WAJ, Zaanen, AC: Riesz Space, vol. 1. North-Holland, London (1971)
4. Cui, Y, Sun, J: Fixed point theorems for a class of nonlinear operators in Hilbert spaces and applications. Positivity **15**, 455-464 (2011)
5. Krasnoselskii, MA: Positive Solutions of Operator Equations. Noordhoff, Groningen (1964)
6. Liu, X, Sun, J: Computation of topological degree of unilaterally asymptotically linear operators and its applications. Nonlinear Anal. **71**, 96-106 (2009)
7. Sun, J, Liu, X: Computation of topological degree for nonlinear operators and applications. Nonlinear Anal. **69**, 4121-4130 (2008)
8. Sun, J, Liu, X: Computation of topological degree in ordered Banach spaces with lattice structure and its application to superlinear differential equations. J. Math. Anal. Appl. **348**, 927-937 (2008)
9. Sun, J, Liu, X: Computation for topological degree and its applications. J. Math. Anal. Appl. **202**, 785-796 (1996)
10. Nieto, JJ, Pouso, RL, Rodríguez-López, R: Fixed point theorems in ordered abstract spaces. Proc. Am. Math. Soc. **135**, 2505-2517 (2007)
11. O'Regan, D, Petruşel, A: Fixed point theorems for generalized contractions in ordered metric spaces. J. Math. Anal. Appl. **341**, 1241-1252 (2008)
12. Nieto, JJ, Rodríguez-López, R: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. **23**, 2205-2212 (2007)
13. Chifu, C, Petruşel, G: Fixed-point results for generalized contractions on ordered gauge spaces with applications. Fixed Point Theory Appl. **2011**, Article ID 979586 (2011)
14. Siegfried, C, Seppo, H: Fixed Point Theory in Ordered Sets and Applications. From Differential and Integral Equations to Game Theory. Springer, New York (2011)

15. Chong, KG: Infinite Dimensional Theory and Multiple Solution Problems. Birkhäuser, Boston (1993)
16. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Reg. Conf. Ser. Math., vol. 65. Conference Board of the Mathematical Sciences, Washington (1986)
17. Liu, Z, Sun, J: Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations. J. Differ. Equ. **172**, 257-299 (2001)
18. Guo, D, Sun, J, Qi, G: Some extensions of the mountain pass lemma. Differ. Integral Equ. **1**, 351-358 (1988)

10.1186/1687-1812-2013-345
Cite this article as: Cui and Sun: Fixed point theorems for a class of nonlinear operators in Hilbert spaces with lattice structure and application. *Fixed Point Theory and Applications* 2013, 2013:345