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Stability of k-step fixed point iterative methods for some Prešić type contractive mappings

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Abstract

We introduce the concept of stability of a k-step fixed point iterative method $x_{n+1} = T(x_n, x_{n-1}, \ldots, x_{n-k+1})$, $n \geq k - 1$, and study the stability of this equation for mappings $T: X^k \to X$ satisfying some Prešić type contraction conditions. Our results naturally extend various stability results of fixed point iterative methods in literature, from contractive self-mappings $T: X \to X$ to Prešić type contractive mappings $T: X^k \to X$. Illustrative examples of both stable and unstable fixed point iterations are also presented.

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1 Introduction

Let (X, d) be a metric space and $T: X \to X$ a self-mapping. Denote by $\mathrm{Fix}(T) := \{x \in X : Tx = x\}$ the set of fixed points of T. If (X, d) is complete and T is a contraction, *i.e.*, there exists a constant $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq ad(x, y), \quad \text{for all } x, y \in X, \tag{1}$$

then, by the well-known Banach contraction mapping principle, we know that $\mathrm{Fix}(T) = \{p\}$ and that, for any $x_0 \in X$, the Picard iteration, that is, the sequence defined by $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$, converges to p, as $n \to \infty$.

When applying contraction mapping principle for solving concrete nonlinear problems, because of rounding errors, numerical approximations of functions, derivatives or integrals, discretization etc., instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$, defined by the given iterative method, we practically work with an *approximate sequence* $\{y_n\}_{n=0}^{\infty}$, satisfying the following approximation bounds:

$$y_0 := x_0, \quad d(y_1, Ty_0) \leq \epsilon_1, \quad \ldots, \quad d(y_n, Ty_{n-1}) \leq \epsilon_n, \quad \ldots,$$

where the positive quantity ϵ_n can be interpreted as the 'round-off error' of x_n; see [1].

Under these circumstances, the problem of the *numerical stability* of Picard iteration is whether the approximate sequence $\{y_n\}_{n=0}^{\infty}$ still converges to the fixed point p of T, provided that $\epsilon_n \to 0$ or $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

This question has been answered in the positive, in the case of contraction condition (1), by Ostrowski [2], who established the first stability result for a fixed point iteration procedure, by using the following estimate:

$$d(y_n, p) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1) + \sum_{k=1}^{n} \alpha^{n-k} \epsilon_k, \quad n = 0, 1, 2, \ldots, \tag{2}$$

from which easily follows that $y_n \to p$, provided $\epsilon_n \to 0$.

In 1988, Harder and Hicks [3, 4] introduced the notion of stability for a general fixed point iteration procedure and started a systematic study of this concept, thus obtaining various stability results for Picard iteration that extended Ostrowski's theorem to mappings satisfying more general contractive conditions and also established some stability results for other fixed point iteration procedures (Mann iteration, Ishikawa iteration, Kirk iteration) in the class of Banach contractions, Zamfirescu operators etc.

Further, some authors, see [1, 5–17], continued the study of the stability of Picard and some other fixed point iterations for various general classes of contractive self-mappings T. Very recently, Rus [18, 19] introduced some alternative concepts of stability for fixed point iterative methods.

On the other hand, the contraction mapping principle has been extended by Prešić [20] to mappings $T: X^k \to X$ satisfying a contractive condition that include (1) in the particular case $k = 1$. Some other Prešić type fixed point theorems have been obtained in [21–26], in the case T satisfies more general contractive type conditions, while in [27, 28], and [29], some applications to nonlinear cyclic systems of equations and difference equations are obtained.

As for the k-step fixed point iterative methods associated to Prešić type contractive conditions do not exist corresponding stability results, yet, the main aim of this paper is to fill this gap and to introduce an appropriate concept of stability and then establish some stability results. The relationship of our notion of stability to other existing concepts of stability, mainly drawn from the theory of difference equations, is also discussed.

The stability results we shall obtain in this way are extremely general. They unify, extend, generalize, enrich, and complement a multitude of related results from recent literature. Illustrative examples of both stable and unstable fixed point iterative sequences are given.

The paper is organized as follows: in Section 2 we summarize some fixed point theorems for mappings satisfying Prešić type contractive conditions, in Section 3 we present the basic concepts and results concerning the stability of fixed point iteration procedures associated to self-mappings that satisfy explicit contractive conditions. In Section 4, the main stability results of this paper are presented. In Section 5, we end this paper by presenting three detailed examples of stable (Examples 1 and 2) and unstable (Example 3) fixed point iterations.

2 Fixed point theorems for Prešić type mappings

All the stability results mentioned in the previous section were basically established in connection with a corresponding fixed point theorem: Banach, Kannan, Chatterjea, Zamfirescu etc.; see for example [15] for more details.

We start by presenting one of the most interesting generalizations of Banach's contraction mapping principle for mappings $T: X^k \to X$, obtained in 1965 by Prešić [20], which

will be fundamental in establishing our stability results for k-step fixed point iterative methods in the present paper.

Theorem 1 (Prešić [20], 1965) *Let (X, d) be a complete metric space, k a positive integer, $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$, $\sum_{i=1}^{k} \alpha_i = \alpha < 1$ and $T: X^k \to X$ a mapping satisfying*

$$d\big(T(x_0, \ldots, x_{k-1}), T(x_1, \ldots, x_k)\big) \leq \alpha_1 d(x_0, x_1) + \cdots + \alpha_k d(x_{k-1}, x_k), \tag{3}$$

for all $x_0, \ldots, x_k \in X$.

Then:
(1) *T has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that*
    $f(x^*, \ldots, x^*) = x^*$;
(2) *the sequence $\{x_n\}_{n \geq 0}$ defined by*

$$x_{n+1} = T(x_n, \ldots, x_{n-k+1}), \quad n = k-1, k, k+1, \ldots \tag{4}$$

converges to x^*, for any $x_0, \ldots, x_{k-1} \in X$.

It is easy to see that, in the particular case $k = 1$, from Theorem 1 we get exactly the well-known Banach contraction mapping principle, while the k-step iterative method (4) reduces to Picard iteration:

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, 3, \ldots. \tag{5}$$

Theorem 1 and other similar results, like the ones in [21, 23, 24, 26], have important applications in the iterative solution of nonlinear equations, see [30] and [28, 31, 32], as well as in the study of global asymptotic stability of the equilibrium for nonlinear difference equations; see the very recent paper [29].

An important generalization of Theorem 1 was proved by Rus [26], for operators T fulfilling the more general condition

$$d\big(T(x_0, \ldots, x_{k-1}), T(x_1, \ldots, x_k)\big) \leq \varphi\big(d(x_0, x_1), \ldots, d(x_{k-1}, x_k)\big), \tag{6}$$

for any $x_0, \ldots, x_k \in X$, where $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ satisfies certain appropriate conditions.

Another important generalization of Prešić's result was recently obtained by Cirić and Prešić in [21], where, instead of (3) and its generalization (6), the following contraction condition is considered:

$$d\big(T(x_0, \ldots, x_{k-1}), T(x_1, \ldots, x_k)\big) \leq \lambda \max\big\{d(x_0, x_1), \ldots, d(x_{k-1}, x_k)\big\}, \tag{7}$$

for any $x_0, \ldots, x_k \in X$, where $\lambda \in (0, 1)$.

Other general Prešić type fixed point results have been very recently obtained by the second author in [22, 23, 25, 33] based on the contractive condition (18), studied in [23], which is more general than (7), (6) and (3).

3 Some useful lemmas

In order to obtain simple and short proofs for our main results in this paper we shall need some auxiliary lemmas. A proof of the following lemma can be found in [34]; see also [35].

Lemma 1 (Cauchy) *Let* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers satisfying*

$$\text{(i)} \quad \lim_{n \to \infty} a_n = 0; \qquad \text{(ii)} \quad \sum_{k=0}^{\infty} b_k < \infty.$$

Then

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k b_{n-k} = 0.$$

We note that by Lemma 1 we find that the second term in (2) converges to zero:

$$\sum_{k=1}^{n} \alpha^{n-k} \epsilon_k \to 0 \quad \text{as } n \to \infty.$$

Lemma 2 *Let k be a positive integer and* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq \alpha_1 a_n + \alpha_2 a_{n-1} + \cdots + \alpha_k a_{n-k+1} + b_n, \quad n \geq k-1, \tag{8}$$

where $\alpha_1, \ldots, \alpha_k \in [0,1)$ and $\alpha_1 + \cdots + \alpha_k < 1$. If $\lim_{n \to \infty} b_n = 0$, then $\lim_{n \to \infty} a_n = 0$.

Proof For simplicity, we prove lemma for $k = 2$, when we denote $\alpha_1 := \alpha$ and $\alpha_2 := \beta$. All arguments used below work unchanged in the general case, too.

We first show that if the sequence $\{u_n\}_{n=0}^{\infty}$ is given by $u_0 = 1$, $u_1 = \alpha$ and

$$u_{n+1} = \alpha u_n + \beta u_{n-1}, \quad n \geq 1, \tag{9}$$

then the series $\sum_{n=0}^{\infty} u_n$ converges.

Indeed, the characteristic equation corresponding to the linear difference equation (9) has two distinct real roots, $r_1, r_2 \in (-1, 1)$, since, if we denote $f(r) := r^2 - \alpha r - \beta$, we have $f(-1) > 0, f(1) > 0$.

Hence, $u_n = c_1 r_1^n + c_2 r_2^n$, where c_1, c_2 are two constants, and so the convergence of $\sum_{n=0}^{\infty} u_n$ follows by the convergence of the geometric progression series $\sum_{n=0}^{\infty} q^n$, with $-1 < q < 1$.

Note that in the general case, the characteristic equation

$$r^k - \alpha_1 r^{k-1} - \cdots - \alpha_{k-1} r - \alpha_k = 0$$

has all its roots (not always real) in absolute value in the interval $[0, 1)$, by virtue of a consequence of the Rouché theorem, see [36], which reads as follows.

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with complex coefficients $a_0, a_1, \ldots, a_n$ such that $a_n \neq 0$. Then there are exactly n (counted with multiplicity) roots of absolute

value less than R, where

$$R = \max\left\{1, \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n|}\right\}.$$

Now, let's come back to the inequality (8) with $k = 2$, $\alpha_1 := \alpha$ and $\alpha_2 := \beta$. Take formally $n := i$ in (8) to obtain

$$a_{i+1} \leq \alpha a_i + \beta a_{i-1} + b_i,$$

then multiply both sides of this inequality by u_{n-i} and sum up all the inequalities obtained for $i = 1, 2, \ldots, n$ to get

$$0 \leq a_{n+1} \leq a_1 u_n + \beta a_0 u_{n-1} + \sum_{k=0}^{n-1} u_k b_{n-k}, \quad n \geq 1.$$

Now apply Lemma 1 to get the desired conclusion. □

Remark 1 Note that for $k = 1$, by Lemma 2 we obtain Theorem 2.1 in [37], a result which is fundamental for obtaining short proofs of stability theorems for one step fixed point iterative methods that has been used in [12, 13, 15], and [17].

4 Stability of *k*-step fixed point iteration procedures

We first recall some concepts of stability for 1-step fixed point iteration procedures, before we consider the stability of a k-step fixed point iteration procedure.

Let (X, d) be a metric space, $T: X \to X$ a self-operator with $\text{Fix}(T) \neq \emptyset$ and let $\{x_n\}_{n=0}^{\infty}$ be a fixed point iteration procedure of the general form

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \ldots, \tag{10}$$

where $f(T, x_n)$ is given, which converges to a fixed point p of T (for example, in the case of Picard iteration we have $f(T, x_n) := T x_n$).

Definition 1 (Harder and Hicks, [3]) Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\varepsilon_n = d\big(y_{n+1}, f(T, y_n)\big), \quad \text{for } n = 0, 1, 2, \ldots. \tag{11}$$

We shall say that the fixed point iteration procedure (10) is *T-stable* or *stable with respect to T* if

$$\lim_{n \to \infty} \varepsilon_n = 0 \quad \Rightarrow \quad \lim_{n \to \infty} y_n = p. \tag{12}$$

As Picard iteration and other fixed point iteration procedures are not stable with respect to some classes of contractive operators, various weak stability concepts have also been introduced; see [9, 11, 12, 15]. For example Osilike [9] introduced the concept of *almost stability*, while Berinde [12] introduced the concept of *summable almost stability*, two notions which are presented in the following.

Definition 2 (Osilike, [9]) Let (X, d) be a metric space, $T\colon X \to X$ a self-operator with
$\mathrm{Fix}(T) \neq \emptyset$ and let $\{x_n\}_{n=0}^{\infty}$ be a fixed point iteration procedure given by (10), supposed to
converge to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and let $\{\varepsilon_n\}$ be
defined by (11). We shall say that the fixed point iteration procedure (10) is *almost T-stable*
or *almost stable with respect to T* if

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty \quad \Rightarrow \quad \lim_{n \to \infty} y_n = p. \tag{13}$$

Definition 3 (Berinde, [13]) Let (X, d) be a metric space, $T\colon X \to X$ a self-operator with
$\mathrm{Fix}(T) \neq \emptyset$ and let $\{x_n\}_{n=0}^{\infty}$ be a fixed point iteration procedure given by (10), supposed to
converge to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and let $\{\varepsilon_n\}$
be defined by (11). We shall say that the fixed point iteration procedure (10) is *summable
almost T-stable* or *summable almost stable with respect to T* if

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} d(y_n, p) < \infty. \tag{14}$$

It is clear from Definitions 1-3 that:

(1) any stable iteration procedure is almost stable;

(2) any summable almost stable procedure is almost stable,

but the reverses of these assertions are not generally true; see Example 1 in [13]. Moreover,
in general, the class of stable iteration procedures is independent of the class of summable
almost stable procedures.

We introduce now the concept of stability for a k-step fixed point iteration procedure.

Definition 4 Let (X, d) be a metric space, k a positive integer, $T\colon X^k \to X$ a mapping
with $\mathrm{Fix}(T) = \{x \in X : T(x, \ldots, x) = x\} \neq \emptyset$ and let $\{x_n\}_{n=0}^{\infty}$ be a k-step fixed point iteration
procedure of the general form

$$x_{n+1} = f(T, x_n, x_{n-1}, \ldots, x_{n-k+1}), \quad n = k-1, k, k+1, \ldots, \tag{15}$$

which converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\varepsilon_n = d(y_{n+1}, f(T, y_n, y_{n-1}, \ldots, y_{n-k+1})), \quad \text{for } n = k-1, k, k+1, \ldots. \tag{16}$$

We shall say that the fixed point iteration procedure (15) is *T-stable* or *stable with respect
to T* if

$$\lim_{n \to \infty} \varepsilon_n = 0 \quad \Rightarrow \quad \lim_{n \to \infty} y_n = p. \tag{17}$$

Remark 2 For $k = 1$, by Definition 4 we get exactly Definition 1.

The main result of this paper is the following theorem, which establish a stability result
that corresponds to the fixed point theorem for Prešić-Kannan contractive mappings ([23],
Theorem 2).

Theorem 2 *Let (X, d) be a complete metric space, k a positive integer, $a \in \mathbb{R}$ a constant such that $0 < ak(k + 1) < 1$ and $T: X^k \to X$ a mapping satisfying the following contractive type condition*:

$$d\big(T(x_0, \ldots, x_{k-1}), T(x_1, \ldots, x_k)\big) \leq a\sum_{i=0}^{k} d\big(x_i, T(x_i, \ldots, x_i)\big), \tag{18}$$

for any $x_0, x_1, \ldots, x_k \in X$.

   Then:

(1) *T has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $T(x^*, \ldots, x^*) = x^*$;*

(2) *The sequence $\{x_n\}_{n\geq 0}$ with $x_0, \ldots, x_{k-1} \in X$ and*

$$x_n = T(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), \quad n \geq k, \tag{19}$$

   converges to x^*;

(3) *The k-step iteration $\{x_n\}_{n\geq 0}$ given by (19) is T-stable.*

Proof Items (1) and (2) follow by Theorem 2 in [23]. Now let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and denote

$$\varepsilon_n = d\big(y_{n+1}, T(y_n, y_{n-1}, \ldots, y_{n-k+1})\big), \quad \text{for } n = k-1, k, k+1, \ldots. \tag{20}$$

First, by triangle inequality, we have

$$d\big(y_{n+1}, x^*\big) \leq d\big(y_{n+1}, T(y_n, y_{n-1}, \ldots, y_{n-k+1})\big) + d\big(T(y_n, y_{n-1}, \ldots, y_{n-k+1}), x^*\big). \tag{21}$$

In view of the contractive condition (18), we have

$$
\begin{aligned}
&d\big(T(y_n, y_{n-1}, \ldots, y_{n-k+1}), x^*\big) \\
&\quad = d\big(T(y_n, \ldots, y_{n-k+1}), T(x^*, \ldots, x^*)\big) \\
&\quad \leq d\big(T(y_n, \ldots, y_{n-k+1}), T(y_{n-1}, \ldots, y_{n-k+1}, x^*)\big) \\
&\qquad + d\big(T(y_{n-1}, \ldots, y_{n-k+1}, x^*), T(y_{n-2}, \ldots, y_{n-k+1}, x^*, x^*)\big) \\
&\qquad + \cdots + d\big(T(y_{n-k+1}, \ldots, x^*, x^*, x^*), T(x^*, \ldots, x^*, x^*, x^*)\big).
\end{aligned}
$$

If we denote $F(x) := T(x, x, \ldots, x)$, then by the above inequality we get

$$
\begin{aligned}
&d\big(T(y_n, y_{n-1}, \ldots, y_{n-k+1}), x^*\big) \\
&\quad \leq a\big[d\big(y_n, T(y_n)\big) + \cdots + d\big(y_{n-k+1}, x^*\big) + d\big(x^*, x^*\big)\big] \\
&\qquad + a\big[d\big(y_{n-1}, T(y_{n-1})\big) + \cdots + d\big(y_{n-k+1}, x^*\big) + d\big(x^*, x^*\big) + d\big(x^*, x^*\big)\big] \\
&\qquad + \cdots + a\big[d\big(y_{n-k+1}, T(y_{n-k+1})\big) + d\big(x^*, x^*\big) + \cdots + d\big(x^*, x^*\big)\big] \\
&\quad = a\big[d\big(y_n, T(y_n)\big) + 2d\big(y_{n-1}, T(y_{n-1})\big) + \cdots + kd\big(y_{n-k+1}, T(y_{n-k+1})\big)\big].
\end{aligned} \tag{22}
$$

On the other hand, for any j we have, by the triangle inequality,

$$d(x_j, F(x_j)) \leq d(x_j, x^*) + d(x^*, F(x_j)). \tag{23}$$

Similarly to the way we obtained (22), we get for any $x, y \in X$,

$$d(F(x), F(y)) = d(T(x, x, \ldots, x), T(y, y, \ldots, y)) \leq A[d(x, F(x)) + d(y, F(y))], \tag{24}$$

where, by hypothesis, $A = \frac{ak(k+1)}{2} < \frac{1}{2}$.

Since x^* is a fixed point of T, by taking $x := x^*$ and $y := x_j$ in (24) we have

$$d(x^*, F(x_j)) = d(T(x^*, x^*, \ldots, x^*), T(x_j, x_j, \ldots, x_j)) \leq A d(x_j, F(x_j)).$$

Now, by (23) we get

$$d(x_j, F(x_j)) \leq d(x_j, x^*) + A d(x_j, F(x_j)),$$

that is,

$$d(x_j, F(x_j)) \leq \frac{1}{1 - A} d(x_j, x^*). \tag{25}$$

Now, by (21), (22), and (25) we find that the sequence of non-negative real numbers $\{d(y_n, x^*)\}$ satisfies the recurrence inequality

$$d(y_{n+1}, x^*) \leq \alpha_1 d(y_n, x^*) + \cdots + \alpha_k d(y_{n-k+1}, x^*) + \epsilon_n, \quad n \geq k - 1, \tag{26}$$

where $\alpha_j = \frac{ja}{1-A}$. It is a simple task to show that

$$\alpha_1 + \cdots + \alpha_k = \frac{a(1 + 2 + \cdots + k)}{1 - A} = \frac{A}{1 - A} < 1,$$

since $A = \frac{ak(k+1)}{2} < \frac{1}{2}$, by hypothesis.

Now, assume $\lim_{n \to \infty} \varepsilon_n = 0$. By applying Lemma 2 with $a_n := d(y_n, x^*)$ and $b_n := \varepsilon_n$, by virtue of (26), we conclude that $d(y_{n+1}, x^*) \to 0$ as $n \to \infty$, that is, the k-step iteration $\{x_n\}_{n \geq 0}$ defined by (19) is T-stable. □

A similar stability result can be obtained for mappings T satisfying a more general contractive condition than (18).

Denote $\text{Fix}(T) = \{x \in X : T(x, x, \ldots, x) = x\}$.

Theorem 3 *Let (X, d) be a metric space, k a positive integer and $T : X^k \to X$ a mapping with the property that $\text{Fix}(T) \neq \emptyset$ for which there exist $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$, $\sum_{i=1}^{k} \alpha_i = \alpha < 1$ satisfying*

$$d(T(x_0, \ldots, x_{k-1}), x^*) \leq \alpha_1 d(x_0, x^*) + \cdots + \alpha_k d(x_{k-1}, x^*), \tag{27}$$

for all $x_0, \ldots, x_{k-1} \in X$ and some $x^* \in \text{Fix}(T)$. Then the k-step iteration $\{x_n\}_{n \geq 0}$ defined by (4) is T-stable.

Proof Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and denote

$$\varepsilon_n = d\big(y_{n+1}, T(y_n, y_{n-1}, \dots, y_{n-k+1})\big), \quad \text{for } n = k-1, k, k+1, \dots.$$

By triangle inequality, we have

$$d\big(y_{n+1}, x^*\big) \leq d\big(y_{n+1}, T(y_n, y_{n-1}, \dots, y_{n-k+1})\big) + d\big(T(y_n, y_{n-1}, \dots, y_{n-k+1}), x^*\big) \qquad (28)$$

and by (27) we obtain

$$d\big(T(y_n, y_{n-1}, \dots, y_{n-k+1}), x^*\big) \leq \alpha_1 d\big(y_n, x^*\big) + \cdots + \alpha_k d\big(y_{n-k+1}, x^*\big)$$

which together with (28) yields

$$d\big(y_{n+1}, x^*\big) \leq \alpha_1 d\big(y_n, x^*\big) + \cdots + \alpha_k d\big(y_{n-k+1}, x^*\big) + \epsilon_n, \quad n \geq k-1, \qquad (29)$$

and the rest of the proof is similar to that of Theorem 2. □

Remark 3 It is clear by the proof of Theorem 2 that if T satisfies (18) then T also satisfies (27), but the reverse is not true, as shown by Example 2.

Another general stability result is given by the next theorem.

Theorem 4 *Let (X, d) be a metric space, k a positive integer and $T: X^k \to X$ a mapping with the property that $\mathrm{Fix}(T) \neq \emptyset$ and for which there exist $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}_+$, $\sum_{i=1}^{k} \alpha_i = \alpha < 1$ and a constant $L > 0$ such that*

$$d\big(T(x_0, \dots, x_{k-1}), T(x, \dots, x)\big)$$
$$\leq \alpha_1 d(x_0, x) + \cdots + \alpha_k d(x_{k-1}, x) + L d\big(x, T(x, \dots, x)\big), \qquad (30)$$

for all $x_0, \dots, x_k, x \in X$. Then the k-step iteration $\{x_n\}_{n \geq 0}$ defined by (4) is T-stable.

Proof Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and denote

$$\varepsilon_n = d\big(y_{n+1}, T(y_n, y_{n-1}, \dots, y_{n-k+1})\big), \quad \text{for } n = k-1, k, k+1, \dots.$$

If $x := x^* \in \mathrm{Fix}(T)$ then by triangle inequality we get (28) and by taking $x_0 := y_n, \dots, x_{k-1} := y_{n-k+1}$ and $x := x^* \in \mathrm{Fix}(T)$ in (30) we obtain

$$d\big(T(y_n, y_{n-1}, \dots, y_{n-k+1}), x^*\big) \leq \alpha_1 d\big(y_n, x^*\big) + \cdots + \alpha_k d\big(y_{n-k+1}, x^*\big),$$

since $x^* = T(x^*, \dots, x^*)$. The rest of the proof follows similarly to the proof of Theorem 2.

□

5 Examples and concluding remarks

Example 1 ([23], Example 1) Let $X = [0,1]$ with the usual metric and $T: X^2 \to X$ be defined by

$$T(x,y) = \frac{1}{6}, \quad (x,y) \in \left[0, \frac{3}{4}\right) \times [0,1]; \qquad T(x,y) = \frac{1}{15}, \quad (x,y) \in \left[\frac{3}{4}, 1\right] \times [0,1].$$

Then, see the detailed proof in [23], T satisfies (18) but does not satisfy (3), (6) and (7). Hence by Theorem 2 it follows that the k-step fixed point iteration (19) is T-stable.

To show that (18) holds, while (3), (6), (7) do not hold; see the detailed proof in [23].

Example 2 Let $X = \mathbb{R}$ with the usual metric and $T: X^2 \to X$ be defined by

$$T(x,y) = \frac{x + 2y}{4}, \quad \forall(x,y) \in X^2.$$

It is easy to check that T satisfies condition (3) (with $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$), condition (6) (with $\varphi(t_1, t_2) = \frac{1}{4}t_1 + \frac{1}{2}t_2$), condition (7) (with $\lambda = \frac{3}{4}$) and condition (27) (with $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$) but does not satisfy (18).

Assume condition (18) does hold and take $x_0 := \frac{1}{2}$, $x_1 := \frac{1}{2}$ and $x_2 := 0$. We get

$$\frac{1}{4} \leq \frac{a}{4} \quad \Leftrightarrow \quad 1 \leq a,$$

a contradiction, since, by hypothesis, $a < \frac{1}{6}$. Hence by Theorem 3 or Theorem 4 (but not by Theorem 2) it follows that the k-step fixed point iteration (19) is T-stable.

The next example illustrates unstable fixed point iterations, in the case neither (18) nor (27) is satisfied.

Example 3 Consider the case $k = 1$ (in order to graph the orbits), $X = [-\frac{3}{2}, \frac{1}{2}]$ with the Euclidean norm and let $T: X \to X$ be given by $Tx = 2x^2 + 2x - 1$, for all $x \in X$.

Then T does not satisfy (3), (18), (27), and (30), $\text{Fix}(T) = \{-1, \frac{1}{2}\}$ and the orbits of $x_0 = 0.49$ under T behave chaotic; see Figure 1.

Indeed, assume that (3) holds, *i.e.*,

$$|Tx - Ty| \leq \alpha_1|x - y|, \quad \forall x, y \in X,$$

and take $x = -\frac{3}{2}$, $y = -1$ to get the contradiction $3 \leq \alpha_1 < 1$. For the same values of x and y, (18) and (30) are not satisfied. Now, assume that (27) holds for $x^* = -1$, *i.e.*,

$$|Tx + 1| \leq \alpha_1|x + 1|, \quad \forall x \in X,$$

and take $x = 0$, to get the contradiction $1 \leq \alpha_1 < 1$.

Note that for $x_0 = 0.5$ we obtain a convergent (and constant) sequence, $x_n = 0.5$, $n \geq 0$, while for x_0 as close as possible to 0.5 but different of 0.5 we have always unstable orbits $\{x_n\}$.

Figure 1 Unstable orbits of $x_0 = 0.49$ under T in Example 3.
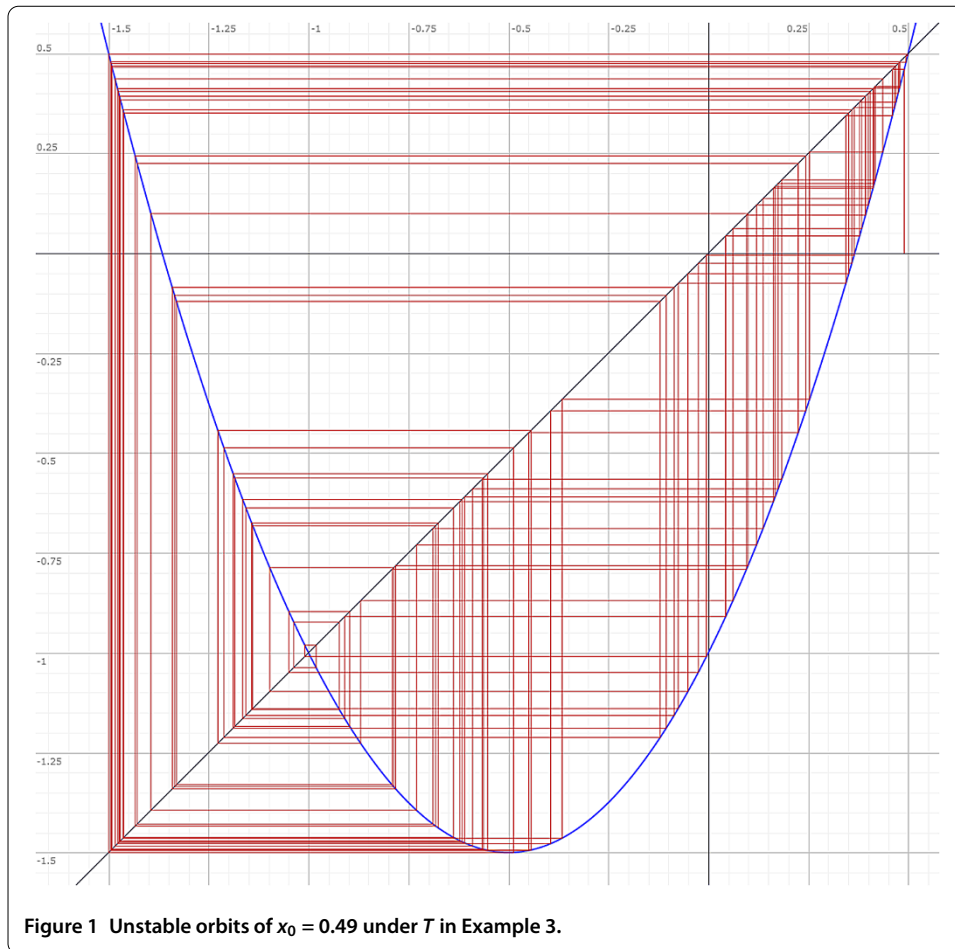
Similar approaches to those in the present paper can be done for mappings defined on product spaces but adapted from the ones in the usual case; see the recent related fixed point results [38–42], and [43].

Authors' contributions
The authors read and approved the final manuscript.

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