

Research Article

New Solutions for System of Fractional Integro-Differential Equations and Abel's Integral Equations by Chebyshev Spectral Method

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Chebyshev spectral method based on operational matrix is applied to both systems of fractional integro-differential equations and Abel's integral equations. Some test problems, for which the exact solution is known, are considered. Numerical results with comparisons are made to confirm the reliability of the method. Chebyshev spectral method may be considered as alternative and efficient technique for finding the approximation of system of fractional integro-differential equations and Abel's integral equations.

1. Introduction

In recent years, the topic of fractional calculus has attracted many scientists because of its several applications in many areas, such as physics, chemistry, biology, and engineering. For a detailed survey with collections of applications in various fields, see, for example, [1–6].

The numerical solution of differential equations of integer order has been a hot topic in numerical and computational mathematics for a long time. There are many different methods and different basis functions have been used to estimate the solution of fractional integro-differential equations or Abel's integral equations, such as Adomian decomposition method [7, 8], fractional differential transform method [9, 10], collocation method [11, 12], homotopy perturbation method [13, 14], homotopy analysis method [15, 16], variational iteration method [17], discrete Galerkin method [18], and Haar wavelet method [19].

Spectral methods provide a computational approach that has achieved substantial popularity over the last four decades. They have gained new popularity in automatic computations for a wide class of physical problems in fluid and heat flow. Their fascinating merit is the high accuracy. So, they have

been applied successfully to numerical simulations of many problems in science and engineering; see [20–24].

The operational matrix of fractional derivatives has been determined for some types of orthogonal polynomials, such as Chebyshev polynomials [25] and Legendre polynomials [26], and for integration has been determined for several types of orthogonal polynomials, such as Chebyshev polynomials [27], Laguerre series [28], and Legendre polynomials [29]. Recently, the Bernstein operational matrix approach is developed for solving a system of high order linear Volterra–Fredholm integro-differential equations in [30].

In the present paper, we use Chebyshev spectral method based on operational matrix to solve system of fractional integro-differential equations:

$$\begin{aligned} {}^C_0D_x^\alpha u_j(x) &= F_j \left(x, u_1(x), u_2(x), \dots, u_m(x), u_1^{(l)}(x), \right. \\ & \left. u_2^{(l)}(x), \dots, u_m^{(l)}(x), \right. \\ & \left. \int_0^x K_j(t, u_1(t), u_2(t), \dots, u_m(t)) dt \right), \\ & j = 1, 2, \dots, m, \end{aligned} \quad (1)$$

with initial conditions

$$u_j^{(\varrho)}(0) = a_{\varrho j}, \tag{2}$$

$$\varrho = 0, 1, 2, \dots, n-1, \quad j = 1, \dots, m, \quad m \in \mathbb{N},$$

where $n-1 < \alpha \leq n, l = 0, 1, \dots, n, n \in \mathbb{N}$.

And we use Abel's integral equation:

$$\lambda u(x) = f(x) + \int_0^x \frac{u(t)}{\sqrt{x-t}} dt, \quad 0 \leq x, t \leq L, \tag{3}$$

where $\lambda = 0$ or $\lambda = 1, f(x)$ is a continuous function, and L is constant.

2. Basic Definitions

In this section, we summarize some basic definitions and properties of fractional calculus theory.

Definition 1. A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0, \infty))$. Clearly $C_\mu \subset C_\beta$ if $\beta \leq \mu$.

Definition 2. A function $f(t), t > 0$, is said to be in space $C_\mu^n, n \in \mathbb{N}$, if $f^{(n)} \in C_\mu$.

Definition 3. The Riemann-Liouville fractional integral operator of order $\alpha (\alpha \geq 0)$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a, \alpha > 0, \tag{4}$$

$$J_a^0 f(t) = f(t).$$

Definition 4. The Caputo fractional derivatives of order α are defined as

$$\begin{aligned} {}_a^C D_t^\alpha f(t) &= J_a^{n-\alpha} D^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} f(s) ds, \end{aligned} \tag{5}$$

$t > a,$

where $n-1 < \alpha \leq n$ and D^n is the classical differential operator of order n .

For Caputo derivative, we have

$${}_0^C D_x^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0, \beta < [\alpha], \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0, \beta \geq [\alpha] \text{ or } \beta \notin \mathbb{N}, \beta > [\alpha]. \end{cases} \tag{6}$$

We use the ceiling function $[\alpha]$ denoting the smallest integer greater than or equal to α and the floor function $\lfloor \alpha \rfloor$ denoting the largest integer less than or equal to α . Also $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Recall that, for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

More properties of the fractional derivatives and the fractional integral can be found in [3, 4].

3. Some Properties of the Shifted Chebyshev Polynomials

The well-known Chebyshev polynomials $\{T_i(t); i = 0, 1, \dots\}$ are defined on the interval $(-1, 1)$ and can be determined with the aid of the following recurrence formula:

$$T_{i+1}(t) = 2tT_i(t) - T_{i-1}(t), \quad i = 1, 2, \dots, \tag{7}$$

where $T_0(t) = 1$ and $T_1(t) = t$. The Chebyshev polynomials are orthogonal on the interval $(-1, 1)$ with respect to the

weight function $w(t) = 1/\sqrt{1-t^2}$. These polynomials satisfy the relation

$$\int_{-1}^1 T_i(t) T_k(t) w(t) dt = \delta_{ik} h_k, \tag{8}$$

where $h_k = (\epsilon_k/2)\pi, \epsilon_0 = 2, \epsilon_k = 1, k \geq 1$.

The analytic form of the Chebyshev polynomial of degree i is given by

$$T_i(t) = i \sum_{k=0}^{[i/2]} (-1)^k 2^{i-2k-1} \frac{(i-k-1)!}{(k!(i-2k)!)} t^{i-2k}, \tag{9}$$

where $[i/2]$ denotes the integer part of $i/2$. The zeros of $T_i(t)$ are denoted by

$$t_k = \cos\left(\left(\frac{2k+1}{2i}\right)\pi\right), \quad k = 0, 1, 2, \dots, i-1. \tag{10}$$

In order to use these polynomials on the interval $x \in (0, L)$, we defined the so-called shifted Chebyshev polynomials by introducing the change of variable $t = 2x/L - 1$. Let the shifted

Chebyshev polynomials $T_i(2x/L - 1)$ be denoted by $T_{L,i}(x)$, satisfying the orthogonality relation

$$\int_0^L T_{L,i}(x) T_{L,k}(x) w_L(x) dx = \delta_{ik} h_k, \quad (11)$$

where $w_L(x) = 1/\sqrt{Lx - x^2}$.

The shifted Chebyshev polynomials are defined as $T_{L,i}(x) = T_i(2x/L - 1) = T_{2i}(\sqrt{x/L})$ and the analytic form is given by

$$T_{L,i}(x) = i \sum_{k=0}^i (-1)^{i-k} 2^{2k} \frac{(i+k-1)!}{(i-k)!(2k)!L^k} x^k, \quad (12)$$

where $T_{L,i}(0) = (-1)^i$ and $T_{L,i}(L) = 1$.

In this form, $T_{L,i}(x)$ may be generated with the aid of the following recurrence formula:

$$T_{L,i+1}(x) = 2 \left(\frac{2x}{L} - 1 \right) T_{L,i}(x) - T_{L,i-1}(x), \quad (13)$$

$$i = 1, 2, \dots,$$

where $T_{L,0}(x) = 1$ and $T_{L,1}(x) = 2x/L - 1$. The zeros of $T_{L,i}(x)$ are denoted by

$$x_k = \frac{L}{2} + \frac{L}{2} \cos \left(\left(\frac{2k+1}{2i} \right) \pi \right), \quad (14)$$

$$k = 0, 1, 2, \dots, i-1.$$

A function $u(x)$, square integrable in $(0, L)$, may be expressed in terms of the shifted Chebyshev polynomials as

$$u(x) = \sum_{j=0}^{\infty} c_j T_{L,j}(x), \quad (15)$$

where the coefficients c_j are given by

$$c_j = \frac{1}{h_j} \int_0^L u(x) T_{L,j}(x) w_L(x) dx, \quad j = 0, 1, 2, \dots \quad (16)$$

In practice, only the first $(N + 1)$ -terms shifted Chebyshev polynomials are considered. Hence, if we write

$$u_N(x) \approx \sum_{j=0}^N c_j T_{L,j}(x) = C^T \phi(x), \quad (17)$$

where the shifted Chebyshev coefficient vector C and the shifted Chebyshev vector $\phi(x)$ are given by

$$C^T = [c_0, c_1, \dots, c_N], \quad (18)$$

$$\phi(x) = [T_{L,0}(x), T_{L,1}(x), \dots, T_{L,N}(x)]^T,$$

then the derivative of the vector $\phi(x)$ can be expressed by

$$\frac{d\phi(x)}{dx} = \mathbf{D}^{(1)} \phi(x), \quad (19)$$

where $\mathbf{D}^{(1)}$ is the $(N + 1) \times (N + 1)$ operational matrix of derivative given by

$$\mathbf{D}^{(1)} = (d_{ij}) = \begin{cases} \frac{4i}{\epsilon_j L}, & j = 0, 1, \dots, i = j + k, \\ 0, & \text{otherwise;} \end{cases} \begin{cases} k = 1, 3, 5, \dots, N, & \text{if } N \text{ is odd,} \\ k = 1, 3, 5, \dots, N - 1, & \text{if } N \text{ is even,} \end{cases} \quad (20)$$

for example, for even N , we have

$$\mathbf{D}^{(1)} = \frac{2}{L} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & \dots & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 & \dots & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & \dots & 0 & 0 \\ 5 & 0 & 10 & 0 & 10 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ N-1 & 0 & 2N-2 & 0 & 2N-2 & \dots & 0 & 0 \\ 0 & 2N & 0 & 2N & 0 & \dots & 2N & 0 \end{pmatrix}. \quad (21)$$

4. The Shifted Chebyshev Operational Matrix (COM) Fractional Derivatives

The main objective of this subsection is to generalize the COM of derivatives for the fractional calculus. By using (19), it is clear that

$$\frac{d^n \phi(x)}{dx^n} = (\mathbf{D}^{(1)})^n \phi(x), \quad (22)$$

where $n \in \mathbb{N}$ and the superscript, in $\mathbf{D}^{(1)}$, denotes matrix powers. Thus

$$\mathbf{D}^{(n)} = (\mathbf{D}^{(1)})^n, \quad n = 1, 2, \dots \quad (23)$$

Lemma 5. Let $T_{L,i}(x)$ be a shifted Chebyshev polynomial; then

$${}_0^C D_x^\alpha T_{L,i}(x) = 0, \quad i = 0, 1, \dots, [\alpha] - 1, \alpha > 0. \quad (24)$$

Theorem 6. Let $\phi(x)$ be the shifted Chebyshev vector defined in (18) and suppose $\alpha > 0$; then

$${}^C_0D_x^\alpha \phi(x) \approx \mathbf{D}^{(\alpha)} \phi(x), \quad (25)$$

where $\mathbf{D}^{(\alpha)}$ is the $(N + 1) \times (N + 1)$ COM of derivatives of order α in the Caputo sense and is defined as follows:

$$\mathbf{D}^{(\alpha)} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ S_\alpha([\alpha], 0) & S_\alpha([\alpha], 1) & S_\alpha([\alpha], 2) & \dots & S_\alpha([\alpha], N) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ S_\alpha(i, 0) & S_\alpha(i, 1) & S_\alpha(i, 2) & \dots & S_\alpha(i, N) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ S_\alpha(N, 0) & S_\alpha(N, 1) & S_\alpha(N, 2) & \dots & S_\alpha(N, N) \end{pmatrix}, \quad (26)$$

where

$$S_\alpha(i, j) = \sum_{k=[\alpha]}^i \frac{(-1)^{i-k} 2i(i+k-1)! \Gamma(k-\alpha+1/2)}{\epsilon_j L^\alpha \Gamma(k+1/2) (i-k)! \Gamma(k-\alpha-j+1) \Gamma(k+j-\alpha+1)}. \quad (27)$$

Note that, in $\mathbf{D}^{(\alpha)}$, the first $[\alpha]$ rows are all zero.

Proof. See [25]. □

Remark 7. If $\alpha = n \in \mathbb{N}$, then Theorem 6 gives the same result as (22).

5. System of Fractional Integro-Differential Equations

In order to use COM for system of fractional integro-differential equations of the form (1), we first approximate $u_j(x)$, ${}^C_0D_x^\alpha u_j(x)$, and $u_j^{(l)}(x)$ by the shifted Chebyshev polynomials as

$$u_j(x) \approx \sum_{i=0}^N c_{j,i} T_{L,i}(x) = C_j^T \phi(x), \quad (28)$$

$${}^C_0D_x^\alpha u_j(x) \approx C_j^T {}^C_0D_x^\alpha \phi(x) \approx C_j^T \mathbf{D}^{(\alpha)} \phi(x), \quad (29)$$

$$u_j^{(l)}(x) \approx C_j^T D^{(l)} \phi(x) \approx C_j^T \mathbf{D}^{(l)} \phi(x). \quad (30)$$

By substituting these equations in (1), we get

$$C_j^T \mathbf{D}^{(\alpha)} \phi(x) \approx F_j \left(x, C_1^T \phi(x), C_2^T \phi(x), \dots, C_m^T \phi(x), C_1^T \mathbf{D}^{(l)} \phi(x), C_2^T \mathbf{D}^{(l)} \phi(x), \dots, C_m^T \mathbf{D}^{(l)} \phi(x), \int_0^x K_j(t, C_1^T \phi(t), \dots, C_m^T \phi(t)) dt \right). \quad (31)$$

Also, by substituting (22) and (28) in (2), we obtain

$$u_j^{(\varrho)}(0) = C_j^T \mathbf{D}^{(\varrho)} \phi(0) = a_{\varrho j}, \quad \varrho = 0, 1, \dots, n-1. \quad (32)$$

Then we have to collocate (31) at the $(N - n + 1)$ shifted Chebyshev roots $x_{L, N-n+1, k}$, $k = 0, 1, \dots, N - n, \forall j = 1, 2, \dots, m$. These equations together with (32) generate $m(N + 1)$ of algebraic equations which can then be solved for the unknown coefficients of the vectors C_j , $j = 1, \dots, m$, using a suitable method. Consequently, the approximate solution $u_j(x)$, $j = 1, 2, \dots, m$, can be obtained.

In our computations we used the Gaussian elimination method to solve the resulting linear system of algebraic equations and Newton's iteration method to solve the resulting nonlinear system of algebraic equations.

Now, we can present the following problems.

Example 8. Consider the following system of fractional integro-differential equations [7, 15]:

$$\begin{aligned} {}^C_0D_t^{\alpha_1} x(t) &= 1 + t + t^2 - y(t) - \int_0^t (x(\tau) + y(\tau)) d\tau, \\ {}^C_0D_t^{\alpha_2} y(t) &= -1 - t + x(t) - \int_0^t (x(\tau) - y(\tau)) d\tau, \end{aligned} \quad (33)$$

$$0 < \alpha_1 < \alpha_2 \leq 1,$$

with the initial conditions

$$\begin{aligned} x(0) &= 1, \\ y(0) &= -1. \end{aligned} \quad (34)$$

The exact solutions, when $\alpha_1 = \alpha_2 = 1$, are

$$\begin{aligned} x(t) &= t + e^t, \\ y(t) &= t - e^t. \end{aligned} \quad (35)$$

We use Chebyshev spectral method; we may write the approximate solution

$$\begin{aligned} x(t) &\approx \sum_{i=0}^8 C_{1,i} T_{1,i}(t) = C_1^T \phi(t), \\ y(t) &\approx \sum_{i=0}^8 C_{2,i} T_{1,i}(t) = C_2^T \phi(t), \end{aligned} \quad (36)$$

where

$$\begin{aligned} C_1^T &= (c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17}, c_{18}), \\ C_2^T &= (c_{20}, c_{21}, c_{22}, c_{23}, c_{24}, c_{25}, c_{26}, c_{27}, c_{28}). \end{aligned} \quad (37)$$

Substituting (36) in (33) and (34), for $\alpha_1 = \alpha_2 = 1$, we get

$$C_1^T \mathbf{D}^{(1)} \phi(t) + C_2^T \phi(t) + \int_0^t (C_1^T \phi(\tau) + C_2^T \phi(\tau)) d\tau - 1 - t - t^2 = 0, \tag{38}$$

$$C_2^T \mathbf{D}^{(1)} \phi(t) - C_1^T \phi(t) + \int_0^t (C_1^T \phi(\tau) - C_2^T \phi(\tau)) d\tau + 1 + t = 0,$$

$$C_1^T \phi(0) = 1, \tag{39}$$

$$C_2^T \phi(0) = -1.$$

The roots of the shifted Chebyshev polynomial $T_{1,8}(t)$ are given by

$$t_k = \frac{1}{2} + \frac{1}{2} \cos\left(\left(\frac{2k+1}{16}\right)\pi\right), \quad k = 0, 1, 2, \dots, 7. \tag{40}$$

Now, for calculating the shifted Chebyshev coefficient for $N = 8$, substitute (40) in (38), and solving the resulting linear system of equations and (39), we get

$$\begin{aligned} c_{10} &= 2.25339, \\ c_{11} &= 1.35039, \\ c_{12} &= 0.105209, \\ c_{13} &= 0.0087221, \\ c_{14} &= 0.000543437, \\ c_{15} &= 0.0000271154, \\ c_{16} &= 1.12813 \times 10^{-6}, \\ c_{17} &= 4.02906 \times 10^{-8}, \\ c_{18} &= 1.25909 \times 10^{-9}, \\ c_{20} &= -1.25339, \\ c_{21} &= -0.350392, \\ c_{22} &= -0.105209, \\ c_{23} &= -0.0087221, \\ c_{24} &= -0.000543437, \\ c_{25} &= -2.71154 \times 10^{-5}, \\ c_{26} &= -1.12814 \times 10^{-6}, \\ c_{27} &= -4.02904 \times 10^{-8}, \\ c_{28} &= -1.25409 \times 10^{-9}. \end{aligned} \tag{41}$$

Thus using (36), we get

$$\begin{aligned} x(t) &= 1 + 2t + 0.5t^2 + 0.166666666t^3 + 0.0416732t^4 \\ &\quad + 0.00831337t^5 + 0.00142338t^6 \\ &\quad + 0.000165029t^7 + 0.0000412579t^8, \end{aligned}$$

$$\begin{aligned} y(t) &= -1 + 1.11945 \times 10^{-9}t - 0.5t^2 - 0.16666666t^3 \tag{42} \\ &\quad - 0.0416728t^4 - 0.00831422t^5 \\ &\quad - 0.00142233t^6 - 0.000165683t^7 \\ &\quad - 0.000041094t^8. \end{aligned}$$

Table 1 shows the comparison between the exact solution and the approximate solution with the absolute error at $N = 8$, and Table 2 shows the maximum of absolute error between exact solutions and approximate solutions for various choices of N . Figure 1 shows the graph of the exact solution and the approximate solution at $N = 8$, $\alpha_1 = \alpha_2 = 1$. Figure 2 shows the graph of the exact solutions and the approximate solutions at $N = 8$, $\alpha_1 = \alpha_2 = 0.9, 0.85$, and 0.75 .

Example 9. Consider the following nonlinear fractional system of integro-differential equations [15]:

$$\begin{aligned} {}_0^C D_t^{\alpha_1} x(t) &= 1 - \frac{1}{3}t^3 - \frac{1}{2}y'^2(t) \\ &\quad + \frac{1}{2} \int_0^t (x^2(\tau) + y^2(\tau)) d\tau, \\ {}_0^C D_t^{\alpha_2} y(t) &= -1 + t^2 - tx(t) \\ &\quad + \frac{1}{4} \int_0^t (x^2(\tau) - y^2(\tau)) d\tau, \end{aligned} \tag{43}$$

$$1 < \alpha_1, \alpha_2 \leq 2,$$

with the initial conditions

$$\begin{aligned} x(0) &= 1, \\ x'(0) &= 2, \\ y(0) &= -1, \\ y'(0) &= 0. \end{aligned} \tag{44}$$

The exact solutions, when $\alpha_1 = \alpha_2 = 2$, are

$$\begin{aligned} x(t) &= t + e^t, \\ y(t) &= t - e^t. \end{aligned} \tag{45}$$

We use Chebyshev spectral method; we may write the approximate solution

$$\begin{aligned} x(t) &\approx \sum_{i=0}^3 C_{1,i} T_{1,i}(t) = C_1^T \phi(t), \\ y(t) &\approx \sum_{i=0}^3 C_{2,i} T_{1,i}(t) = C_2^T \phi(t), \end{aligned} \tag{46}$$

TABLE 1

t_i	x_{Exact}	$x_{\text{Approx.}}$	$ x_{\text{Exact}} - x_{\text{Approx.}} $	y_{Exact}	$y_{\text{Approx.}}$	$ y_{\text{Exact}} - y_{\text{Approx.}} $
0	1	1	0	-1	-1	6.66134×10^{-16}
0.1	1.20517	1.20517	5.23892×10^{-11}	-1.00517	-1.00517	4.22691×10^{-11}
0.2	1.42140	1.42140	4.84666×10^{-11}	-1.02140	-1.02140	5.00378×10^{-11}
0.3	1.64986	1.64986	1.00231×10^{-12}	-1.04986	-1.04986	4.82325×10^{-12}
0.4	1.89182	1.89182	9.13603×10^{-11}	-1.09182	-1.09182	8.16736×10^{-11}
0.5	2.14872	2.14872	1.82374×10^{-11}	-1.14872	-1.14872	1.17573×10^{-12}
0.6	2.42212	2.42212	6.51625×10^{-11}	-1.22212	-1.22212	7.4986×10^{-11}
0.7	2.71375	2.71375	1.55334×10^{-11}	-1.31375	-1.31375	1.77258×10^{-11}
0.8	3.02554	3.02554	7.43441×10^{-11}	-1.42554	-1.42554	6.87448×10^{-11}
0.9	3.35960	3.35960	2.49547×10^{-11}	-1.55960	-1.55960	4.03237×10^{-11}
1	3.71828	3.71828	2.28355×10^{-11}	-1.71828	-1.71828	1.71747×10^{-11}

TABLE 2

N	Maximum of absolute error of $x(t)$	Maximum of absolute error of $y(t)$
2	5.35939×10^{-2}	3.73645×10^{-2}
4	9.25152×10^{-5}	6.71757×10^{-5}
6	7.59171×10^{-7}	5.37812×10^{-7}
8	9.13603×10^{-11}	8.16736×10^{-11}

where

$$\begin{aligned} C_1^T &= (c_{10}, c_{11}, c_{12}, c_{13}), \\ C_2^T &= (c_{20}, c_{21}, c_{22}, c_{23}). \end{aligned} \tag{47}$$

Substituting (46) in (43) and (44), for $\alpha_1 = \alpha_2 = 2$, we get

$$\begin{aligned} &C_1^T \mathbf{D}^{(2)} \phi(t) + \frac{1}{2} [C_2^T \mathbf{D}^{(1)} \phi(t)]^2 \\ &- \frac{1}{2} \int_0^t ([C_1^T \phi(\tau)]^2 + [C_2^T \phi(\tau)]^2) d\tau - 1 \\ &+ \frac{1}{3} t^3 = 0, \end{aligned} \tag{48}$$

$$\begin{aligned} &C_2^T \mathbf{D}^{(2)} \phi(t) + t C_1^T \phi(t) \\ &- \frac{1}{4} \int_0^t ([C_1^T \phi(\tau)]^2 - [C_2^T \phi(\tau)]^2) d\tau + 1 \\ &- t^2 = 0, \\ &C_1^T \phi(0) = 1, \\ &C_1^T \mathbf{D}^{(1)} \phi(0) = 2, \\ &C_2^T \phi(0) = -1, \\ &C_2^T \mathbf{D}^{(1)} \phi(0) = 0. \end{aligned} \tag{49}$$

The roots of the shifted Chebyshev polynomial $T_{1,2}(x)$ are

$$\begin{aligned} t_0 &= \frac{1}{4} (2 + \sqrt{2}), \\ t_1 &= \frac{1}{4} (2 - \sqrt{2}). \end{aligned} \tag{50}$$

Now, for calculating the shifted Chebyshev coefficient for $N = 3$, substitute (50) in (48), and solving the resulting nonlinear system of equations and (49), we get

$$\begin{aligned} c_{10} &= 2.2567036, \\ c_{11} &= 1.3561825, \\ c_{12} &= 0.10782546, \\ c_{13} &= 0.00834659, \\ c_{20} &= -1.2587356, \\ c_{21} &= -0.35967315, \\ c_{22} &= -0.10975298, \\ c_{23} &= -0.00881542. \end{aligned} \tag{51}$$

Thus using (46), we get

$$\begin{aligned} x(t) &= 1 + 2t + 0.461967t^2 + 0.267091t^3, \\ y(t) &= -1 + 2.77556 \times 10^{-17}t - 0.454884t^2 \\ &- 0.282093t^3. \end{aligned} \tag{52}$$

Table 3 shows the comparison between the exact solution and the approximate solution with the absolute error at $N = 3$. Figure 3 shows the graph of the exact solution and the approximate solution at $N = 3, \alpha_1 = \alpha_2 = 2$. Figure 4 shows the graph of the exact solutions and the approximate solutions at $N = 3, \alpha_1 = \alpha_2 = 1.9, 1.8, \text{ and } 1.7$.

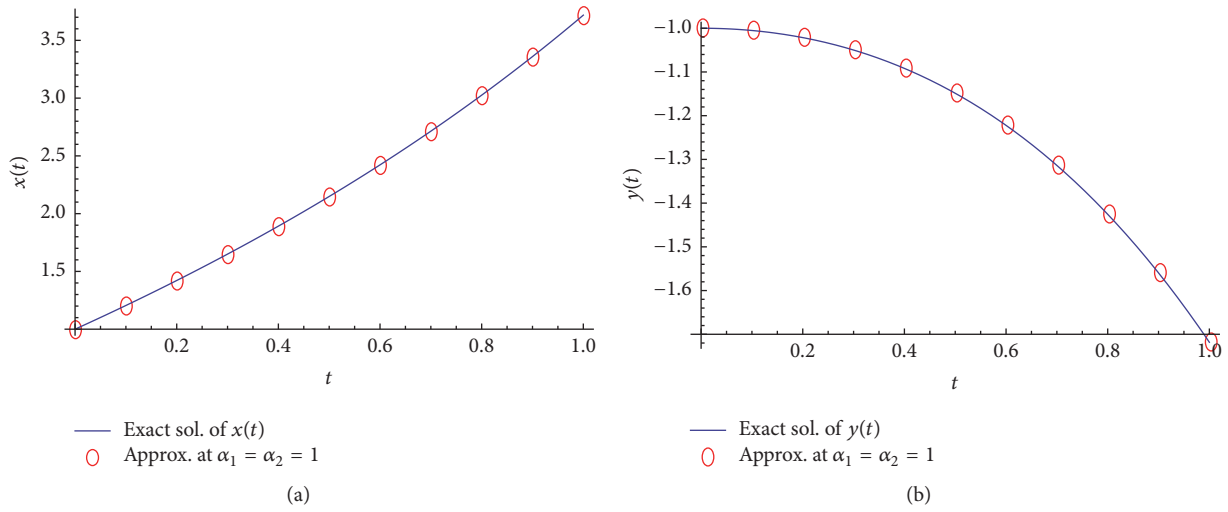


FIGURE 1: The graphs of the exact solutions and the approximate solutions at $N = 8, \alpha_1 = \alpha_2 = 1$.

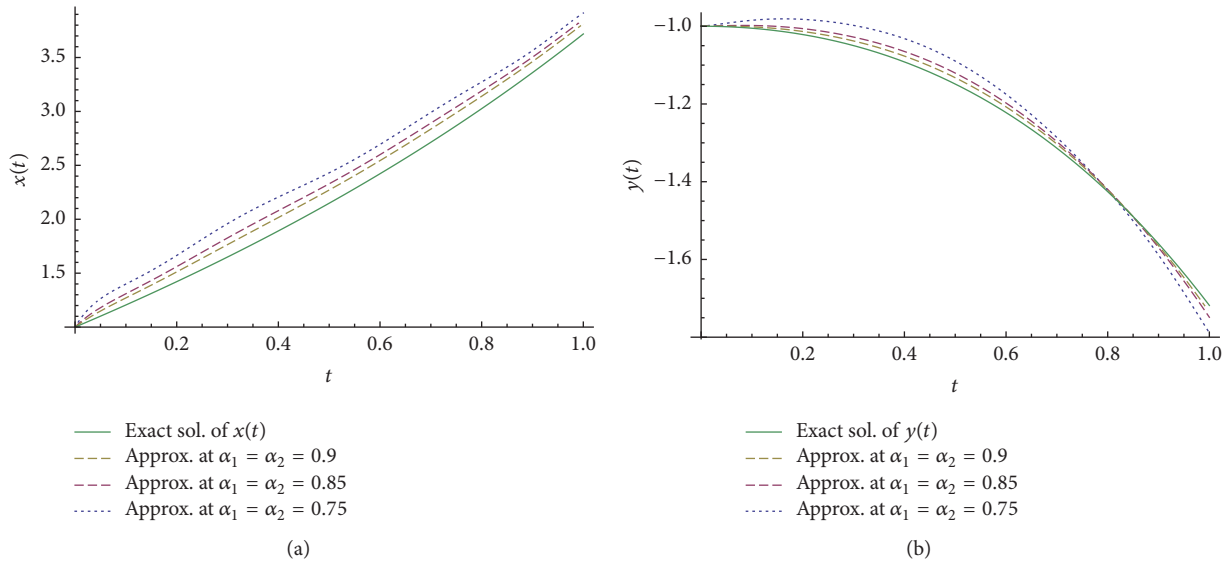


FIGURE 2: The graphs of the exact solutions and the approximate solutions at $N = 8, \alpha_1 = \alpha_2 = 0.9, 0.85, \text{ and } 0.75$.

TABLE 3

t_i	x_{Exact}	$x_{\text{Approx.}}$	$ x_{\text{Exact}} - x_{\text{Approx.}} $	y_{Exact}	$y_{\text{Approx.}}$	$ y_{\text{Exact}} - y_{\text{Approx.}} $
0	1	1	2.22045×10^{-16}	-1	-1	0
0.1	1.2051709	1.2048868	2.84157×10^{-4}	-1.0051709	-1.0048309	3.39987×10^{-4}
0.2	1.4214028	1.4206154	7.87348×10^{-4}	-1.0214028	-1.0204521	9.50662×10^{-4}
0.3	1.6498588	1.6487885	1.07031×10^{-3}	-1.0498588	-1.0485561	1.30275×10^{-3}
0.4	1.8918247	1.8910086	8.16141×10^{-4}	-1.0918247	-1.0908354	9.89323×10^{-4}
0.5	2.1487213	2.1488781	1.56876×10^{-4}	-1.1487213	-1.1489826	2.61337×10^{-4}
0.6	2.4221188	2.4239998	1.88101×10^{-3}	-1.2221188	-1.2246903	2.57152×10^{-3}
0.7	2.7137527	2.7179761	4.22339×10^{-3}	-1.3137527	-1.3196511	5.89836×10^{-3}
0.8	3.0255409	3.0324095	6.86862×10^{-3}	-1.42554092	-1.4355574	1.00165×10^{-2}
0.9	3.3596031	3.3689027	9.2996×10^{-3}	-1.5596031	-1.5741019	1.44988×10^{-2}
1	3.7182818	3.7290581	1.07763×10^{-2}	-1.7182818	-1.7369771	1.86953×10^{-2}

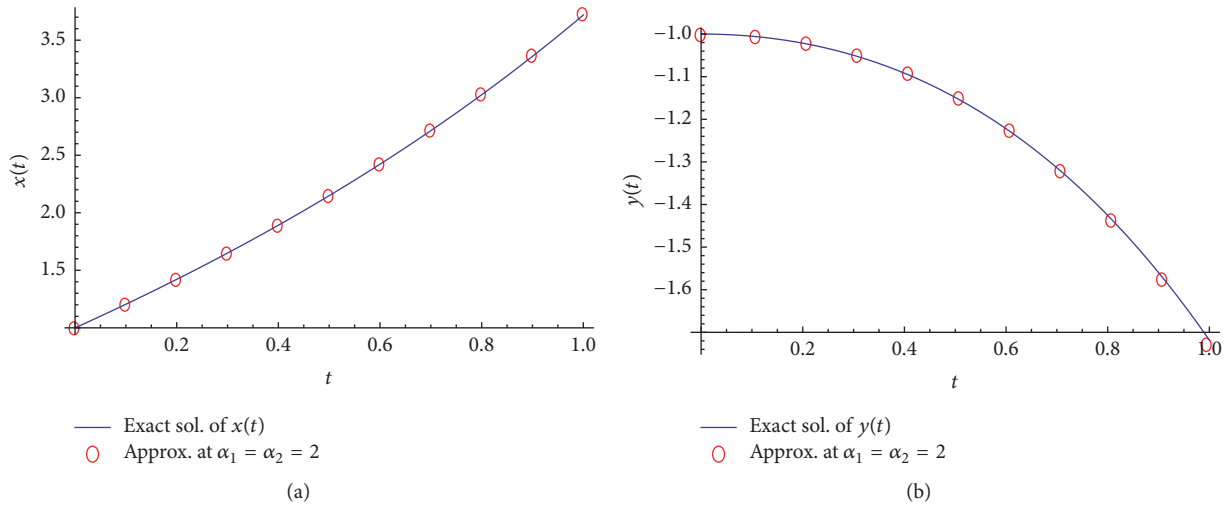


FIGURE 3: The graphs of the exact solutions and the approximate solutions at $N = 3, \alpha_1 = \alpha_2 = 2$.

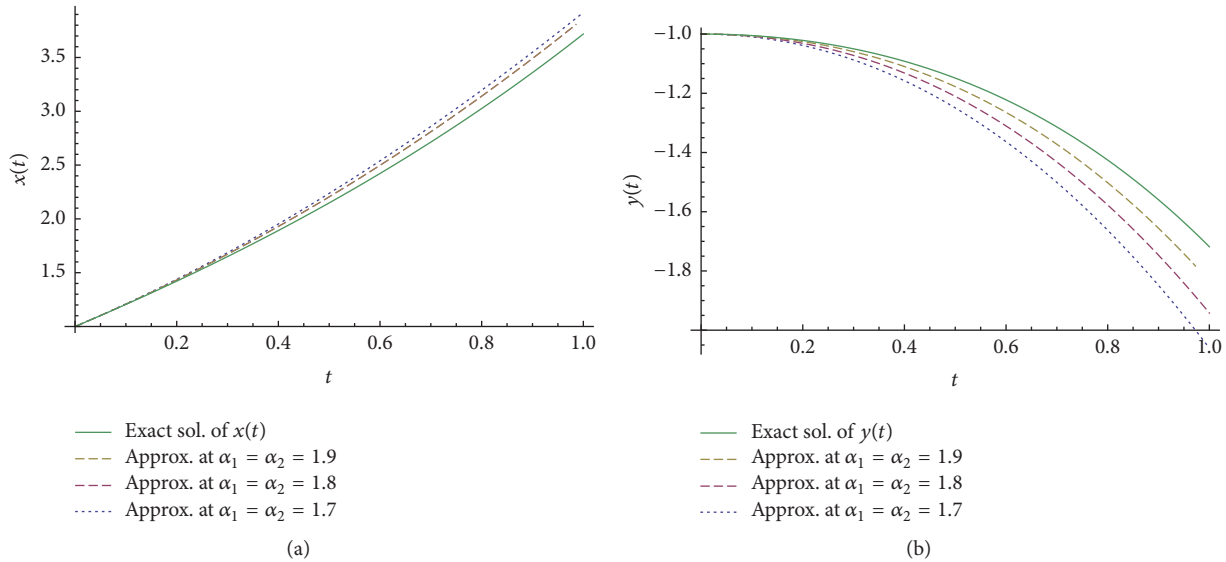


FIGURE 4: The graphs of the exact solutions and the approximate solutions at $N = 3, \alpha_1 = \alpha_2 = 1.9, 1.8, \text{ and } 1.7$.

6. Abel's Integral Equation

In order to use COM for Abel's integral equation of the form (3), we first approximate $u(x)$ by the shifted Chebyshev polynomials as

$$u(x) \approx \sum_{i=0}^N c_i T_{L,i}(x) = C^T \phi(x). \tag{53}$$

By substituting (53) in (3), we get

$$\lambda C^T \phi(x) = f(x) + \int_0^x \frac{C^T \phi(t)}{\sqrt{x-t}} dt. \tag{54}$$

Then we have to collocate (54) at the $(N + 1)$ shifted Chebyshev roots $x_{L,N+1,k}, k = 0, 1, \dots, N$. These equations

generate $(N + 1)$ linear algebraic equations which can be solved for the unknown coefficients of the vector C , using a suitable method. Consequently, $u(x)$ given in (53) can be calculated, which gives a solution of (3).

Example 10. Consider Abel's integral equation of the first kind [31, 32]

$$\int_0^x \frac{u(t)}{\sqrt{x-t}} dt = \frac{2}{105} \sqrt{x} (105 - 56x^2 + 48x^3), \tag{55}$$

$x \in [0, 1]$

which has the exact solution $u(x) = x^3 - x^2 + 1$.

By applying the Chebyshev spectral method, we may write the approximate solution

$$u(x) = \sum_{i=0}^3 C_i T_{1,i}(x) = C^T \phi(x). \tag{56}$$

Substituting (56) in (55), we get

$$\int_0^x \frac{C^T \phi(t)}{\sqrt{x-t}} dt - \frac{2}{105} \sqrt{x} (105 - 56x^2 + 48x^3) = 0. \tag{57}$$

The roots of the shifted Chebyshev polynomial $T_{1,4}(x)$ are

$$\begin{aligned} x_0 &= \frac{1}{4} \left(2 - \sqrt{2 - \sqrt{2}} \right), \\ x_1 &= \frac{1}{4} \left(2 + \sqrt{2 - \sqrt{2}} \right), \\ x_2 &= \frac{1}{4} \left(2 - \sqrt{2 + \sqrt{2}} \right), \\ x_3 &= \frac{1}{4} \left(2 + \sqrt{2 + \sqrt{2}} \right). \end{aligned} \tag{58}$$

Now, calculating the shifted Chebyshev coefficient for $N = 3$ by substituting (58) in (57) and solving four equations yields

$$\begin{aligned} c_0 &= \frac{15}{16}, \\ c_1 &= -\frac{1}{32}, \\ c_2 &= \frac{1}{16}, \\ c_3 &= \frac{1}{32}. \end{aligned} \tag{59}$$

Therefore, we have

$$\begin{aligned} u(x) &= \left(\frac{15}{16}, -\frac{1}{32}, \frac{1}{16}, \frac{1}{32} \right) \begin{pmatrix} 1 \\ 2x - 1 \\ 8x^2 - 8x + 1 \\ 32x^3 - 48x^2 + 18x - 1 \end{pmatrix} \\ &= x^3 - x^2 + 1, \end{aligned} \tag{60}$$

which is the exact solution.

Example 11. Consider Abel's integral equation of the first kind [32]

$$\int_0^x \frac{u(t)}{\sqrt{x-t}} dt = \frac{3\pi}{8} x^2, \quad x \in [0, 1] \tag{61}$$

which has the exact solution $u(x) = x\sqrt{x}$.

TABLE 4

x_i	u_{Exact}	$u_{\text{Approx.}}$	$ u_{\text{Exact}} - u_{\text{Approx.}} $
0	0	$-1.9732143095 \times 10^{-5}$	1.97321×10^{-5}
0.1	0.0316227766	0.03162341809	6.41487×10^{-7}
0.2	0.0894427191	0.08944237293	3.46169×10^{-7}
0.3	0.1643167673	0.16431491828	1.84897×10^{-6}
0.4	0.2529822128	0.25298265879	4.45981×10^{-7}
0.5	0.3535533906	0.35355438911	9.98519×10^{-7}
0.6	0.4647580015	0.46475676796	1.23323×10^{-6}
0.7	0.5856620186	0.58566199942	1.77613×10^{-8}
0.8	0.7155417528	0.71554297116	1.21836×10^{-6}
0.9	0.8538149682	0.85381621495	1.2467×10^{-6}
1	1	1.00000610948	6.10948×10^{-6}

TABLE 5

N	Maximum of absolute error
4	1.90301×10^{-3}
8	2.57352×10^{-4}
12	8.39682×10^{-5}
16	3.73104×10^{-5}
20	1.97321×10^{-5}

Table 4 shows the comparison between the exact solution and the approximate solution with the absolute error at $N = 20$, and Table 5 shows the maximum of absolute error between exact solution and approximate solution for various choices of N . Figure 5 shows the graph of the exact solution and the approximate solution at $N = 20$.

Example 12. Consider Abel's integral equation of the second kind [32, 33]

$$u(x) = x - \frac{4}{3} x^{3/2} + \int_0^x \frac{u(t)}{\sqrt{x-t}} dt, \quad x \in [0, L] \tag{62}$$

which has the exact solution $u(x) = x$.

By applying the Chebyshev spectral method, we may write the approximate solution

$$u(x) \approx \sum_{i=0}^2 C_i T_{L,i}(x) = C^T \phi(x). \tag{63}$$

Substituting (63) in (62), we get

$$C^T \phi(x) - \int_0^x \frac{C^T \phi(t)}{\sqrt{x-t}} dt - x + \frac{4}{3} x^{3/2} = 0. \tag{64}$$

The roots of the shifted Chebyshev polynomial $T_{L,3}(x)$ are

$$\begin{aligned} x_0 &= \frac{L}{2}, \\ x_1 &= \frac{L}{4} (2 - \sqrt{3}), \\ x_2 &= \frac{L}{4} (2 + \sqrt{3}). \end{aligned} \tag{65}$$

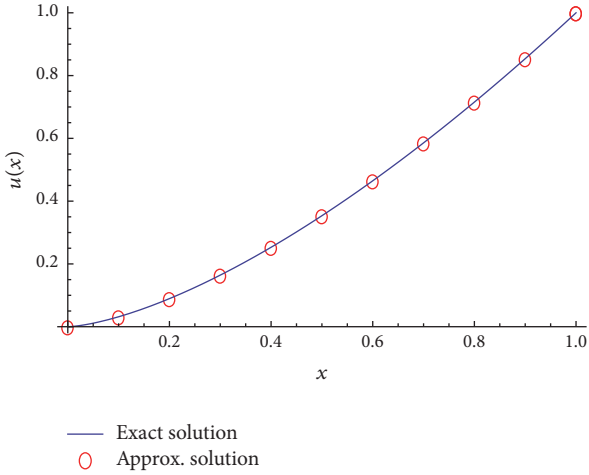


FIGURE 5: The graph of the exact solution and the approximate solution at $N = 20$.

Now, calculating the shifted Chebyshev coefficient for $N = 2$ by substituting (65) in (64) and solving three equations yields

$$\begin{aligned} c_0 &= \frac{L}{2}, \\ c_1 &= \frac{L}{2}, \\ c_2 &= 0. \end{aligned} \tag{66}$$

Therefore, we have

$$u(x) = \left(\frac{L}{2}, \frac{L}{2}, 0\right) \begin{pmatrix} 1 \\ \frac{2x}{L} - 1 \\ \frac{8x^2}{L^2} - \frac{8x}{L} + 1 \end{pmatrix} = x, \tag{67}$$

which is the exact solution.

Example 13. Consider Abel's integral equation of the second kind [32, 34]

$$u(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{u(t)}{\sqrt{x-t}} dt, \quad x \in [0, L] \tag{68}$$

which has the exact solution $u(x) = x^2$.

We use Chebyshev spectral method; we may write the approximate solution

$$u(x) \approx \sum_{i=0}^2 C_i T_{L,i}(x) = C^T \phi(x). \tag{69}$$

Substituting (69) in (68), we get

$$C^T \phi(x) - \int_0^x \frac{C^T \phi(t)}{\sqrt{x-t}} dt - x^2 - \frac{16}{15}x^{5/2} = 0. \tag{70}$$

TABLE 6

x_i	u_{Exact}	$u_{\text{Approx.}}$	$ u_{\text{Exact}} - u_{\text{Approx.}} $
0	1	1	2.22045×10^{-16}
0.1	0.9090909091	0.9090909091	2.22045×10^{-16}
0.2	0.8333333333	0.8333333333	1.77636×10^{-15}
0.3	0.7692307692	0.76923076923	6.66134×10^{-16}
0.4	0.7142857143	0.7142857143	3.33067×10^{-16}
0.5	0.6666666667	0.6666666667	3.55271×10^{-15}
0.6	0.625	0.625	2.22045×10^{-16}
0.7	0.588235294	0.588235294	6.66134×10^{-16}
0.8	0.555555556	0.555555556	1.11022×10^{-16}
0.9	0.526315790	0.526315790	1.55431×10^{-15}
1	0.5	0.5	2.44249×10^{-15}

The roots of the shifted Chebyshev polynomial $T_{L,3}(x)$ are

$$\begin{aligned} x_0 &= \frac{L}{2}, \\ x_1 &= \frac{L}{4} (2 - \sqrt{3}), \\ x_2 &= \frac{L}{4} (2 + \sqrt{3}). \end{aligned} \tag{71}$$

Now, calculating the shifted Chebyshev coefficient for $N = 2$ by substituting (71) in (70) and solving three equations yields

$$\begin{aligned} c_0 &= \frac{3L^2}{8}, \\ c_1 &= \frac{L^2}{2}, \\ c_2 &= \frac{L^2}{8}. \end{aligned} \tag{72}$$

Therefore, we have

$$u(x) = \left(\frac{3L^2}{8}, \frac{L^2}{2}, \frac{L^2}{8}\right) \begin{pmatrix} 1 \\ \frac{2x}{L} - 1 \\ \frac{8x^2}{L^2} - \frac{8x}{L} + 1 \end{pmatrix} = x^2, \tag{73}$$

which is the exact solution.

Example 14. Consider Abel's integral equation of the second kind [32, 34]

$$u(x) = \frac{1}{x+1} + \frac{2 \operatorname{arc sinh}(\sqrt{x})}{\sqrt{1+x}} - \int_0^x \frac{u(t)}{\sqrt{x-t}} dt, \tag{74}$$

$x \in [0, 1]$

which has the exact solution $u(x) = 1/(x+1)$.

Table 6 shows the comparison between the exact solution and the approximate solution with the absolute error at

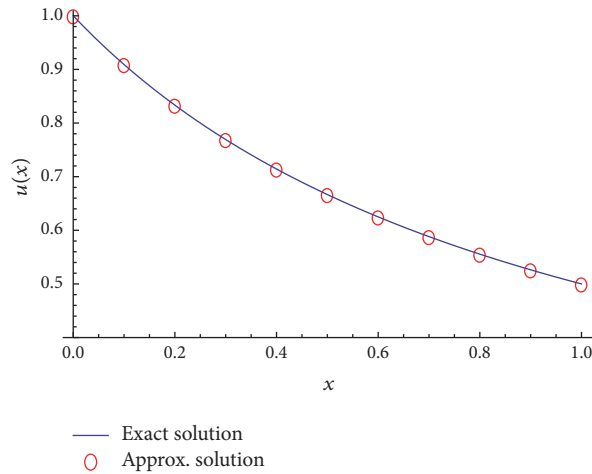


FIGURE 6: The graph of the exact solution and the approximate solution at $N = 20$.

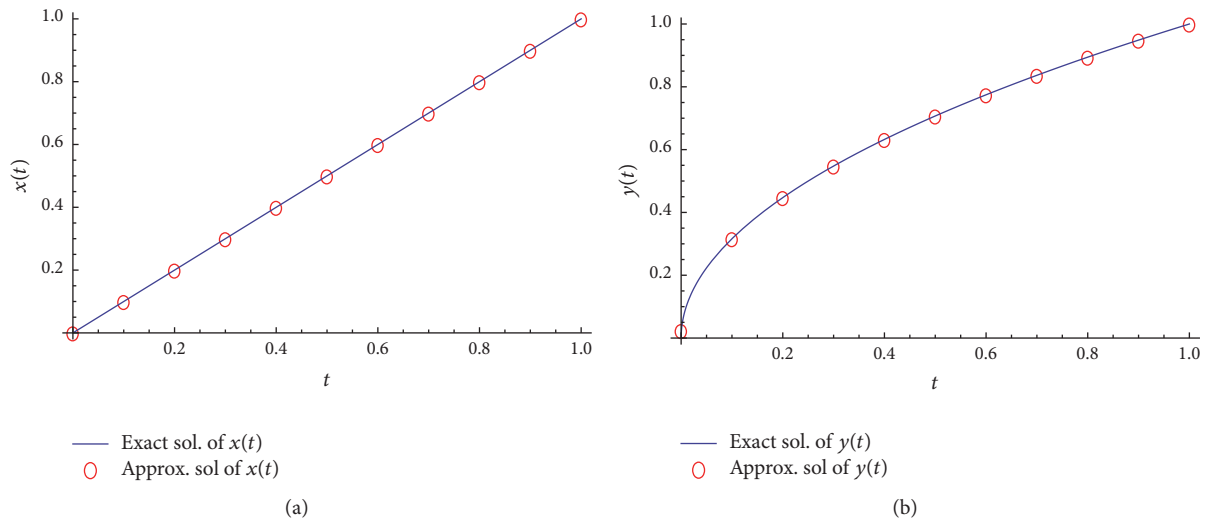


FIGURE 7: The graphs of the exact solutions and the approximate solutions at $N = 20$.

$N = 20$, and Table 7 shows the maximum of absolute error between exact solution and approximate solution for various choices of N . Figure 6 shows the graph of the exact solution and the approximate solution at $N = 20$.

Example 15. Consider the linear system of singular Volterra integral equations [10]

$$\pi x(t) - \int_0^t \frac{1}{\sqrt{t-s}} [x(s) + y(s)] ds = \frac{\pi}{2}t - \frac{4}{3}t^{3/2}, \tag{75}$$

$$y(t) - \int_0^t \frac{1}{\sqrt{t-s}} x(s) ds = \sqrt{t} - \frac{4}{3}t^{3/2}.$$

The exact solutions are

$$\begin{aligned} x(t) &= t, \\ y(t) &= \sqrt{t}. \end{aligned} \tag{76}$$

TABLE 7

N	Maximum of absolute error
4	2.47780×10^{-4}
8	2.05220×10^{-7}
12	1.80403×10^{-10}
16	1.55986×10^{-13}
20	3.55271×10^{-15}

Table 8 shows the comparison between the exact solution and the approximate solution with the absolute error at $N = 20$, and Table 9 shows the maximum of absolute error between exact solution and approximate solution for various choices of N . Figure 7 shows the graph of the exact solution and the approximate solution at $N = 20$.

TABLE 8

t_i	x_{Exact}	$x_{\text{Approx.}}$	$ x_{\text{Exact}} - x_{\text{Approx.}} $	y_{Exact}	$y_{\text{Approx.}}$	$ y_{\text{Exact}} - y_{\text{Approx.}} $
0	0	1.62145×10^{-4}	1.62145×10^{-4}	0	2.38416×10^{-2}	2.38416×10^{-2}
0.1	0.1	0.0999982	1.76511×10^{-6}	0.316228	0.316404	1.7626×10^{-4}
0.2	0.2	0.19999	9.70125×10^{-6}	0.447214	0.447335	1.21557×10^{-4}
0.3	0.3	0.299981	1.92749×10^{-5}	0.547723	0.547788	6.49502×10^{-5}
0.4	0.4	0.399991	9.32098×10^{-6}	0.632456	0.632369	8.61806×10^{-5}
0.5	0.5	0.49999	9.57004×10^{-6}	0.707107	0.70709	1.69007×10^{-5}
0.6	0.6	0.599978	2.1643×10^{-5}	0.774597	0.774625	2.87108×10^{-5}
0.7	0.7	0.699983	1.74306×10^{-5}	0.83666	0.836603	5.70565×10^{-5}
0.8	0.8	0.799983	1.74232×10^{-5}	0.894427	0.894364	6.33597×10^{-5}
0.9	0.9	0.899978	2.16216×10^{-5}	0.948683	0.948623	6.03199×10^{-5}
1	1	0.999976	2.44081×10^{-5}	1	0.999923	7.72476×10^{-5}

TABLE 9

N	Maximum of absolute error of $x(t)$	Maximum of absolute error of $y(t)$
4	3.08564×10^{-3}	1.02329×10^{-1}
8	9.08442×10^{-4}	5.59578×10^{-2}
12	4.28561×10^{-4}	3.85957×10^{-2}
16	2.486×10^{-4}	2.9472×10^{-2}
20	1.62145×10^{-4}	2.38416×10^{-2}

7. Conclusion

In this article, we develop the Chebyshev spectral method based on operational matrix for solving linear and nonlinear system of fractional integro-differential equations and Abel's integral equations. Our approach was based on the shifted Chebyshev collocation methods. It can be concluded that the Chebyshev spectral method is very powerful and efficient technique for finding exact solutions for wide classes of problems.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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