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Research Article

Strong and Δ-Convergence Theorems for Common Fixed Points of a Finite Family of Multivalued Demicontractive Mappings in CAT(0) Spaces

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Let K be a nonempty closed and convex subset of a complete CAT(0) space. Let $T_i : K \to \mathrm{CB}(K), i = 1, 2, \ldots, m$, be a family of multivalued demicontractive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. A Krasnoselskii-type iterative sequence is shown to Δ-converge to a common fixed point of the family $\{T_i, i = 1, 2, \ldots, m\}$. Strong convergence theorems are also proved under some additional conditions. Our theorems complement and extend several recent important results on approximation of fixed points of certain nonlinear mappings in CAT(0) spaces. Furthermore, our method of the proof is of special interest.

1. Introduction

A metric space (X, d) is said to be a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean space. It is well known that pre-Hilbert spaces, $\mathbb{R}$-trees (see [1]), and Euclidean buildings (see, e.g., [2]) are among examples of CAT(0) spaces. For a thorough discussion of these spaces and the fundamental role they play in various branches of mathematics see Bridson and Haefliger [1] or Burago et al. [3]. Fixed point theory in CAT(0) spaces was first studied by Kirk (see [4, 5]). He showed that every nonexpansive mapping defined on a nonempty closed convex and bounded subset of a CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multivalued mappings has received much attention (see, e.g., [6–13]). In 1976, Lim [14] introduced a notion of convergence in a general metric space which he called Δ-convergence (see Definition 8). In 2008, Kirk and Panyanak [15] specialized Lim's concept to CAT(0) spaces and showed that many results which involve weak convergence (e.g., Opial property and Kadec-Klee property) have precise analogs in this setting. Later on, Dhompongsa and Panyanak [16] obtained Δ-convergence theorems for the Picard, Mann, and Ishikawa iterations involving one mapping in the CAT(0) space setting.

In [17], Chidume et al. introduced the class of multivalued k-strictly pseudocontractive mappings which is a generalization of the class of multivalued nonexpansive mappings in Hilbert spaces. They constructed a Krasnoselskii-type algorithm sequence and showed that it is an approximate fixed point sequence of the map. In particular, they proved the following theorem.

Theorem 1 (Theorem 3.1 of [17]). *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Suppose that $T : K \to CB(K)$ is a multivalued k-strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. Assume that $T(p) = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$*

$$x_{n+1} = (1 - \lambda) x_n + \lambda y_n, \tag{1}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1)$. Then, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Very recently, Chidume and Ezeora extended the result of Chidume et al. [17] to a finite family of multivalued k-strictly pseudocontractive mappings in real Hilbert spaces. The following theorem is their main result.

Theorem 2 (Theorem 2.2 of [18]). *Let K be a nonempty, closed, and convex subset of a real Hilbert space H, and let $T_i : K \to CB(K)$ be a finite family of multivalued k_i-strictly pseudocontractive mappings, $k_i \in (0,1)$, $i = 1,2,\ldots,m$, such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Assume that, for all $p \in \bigcap_{i=1}^{m} F(T_i)$, $T_i(p) = \{p\}$. Let $\{x_n\}$ be sequence defined by $x_0 \in K$*

$$x_{n+1} = \lambda_0 x_n + \lambda_1 y_n^1 + \lambda_2 y_n^2 + \cdots + \lambda_m y_n^m, \tag{2}$$

where $y_n^i \in T_i x_n$, $n \geqslant 1$, and $\lambda_i \in (k,1)$, $i = 0,1,2,\ldots,m$, such that $\sum_{i=0}^{m} \lambda_i = 1$, where $k = \max\{k_i,\ i = 1,2,\ldots,m\}$. Then, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ $\forall i = 1,2,\ldots,m$.

Remark 3. In Theorem 2.2 of [18], the condition that $\lambda_i \in (0,1)$, $i = 0,1,2,\ldots,m$, such that $\sum_{i=0}^{m} \lambda_i = 1$, where $k = \max\{k_i,\ i = 1,2,\ldots,m\}$, $m \geqslant 2$, restricts the class of operators for which the theorem is applicable. For instance, if $k = (2/3)$, then the theorem is not applicable to the family of the mappings for which $(2/3) = \max\{k_i,\ i = 1,2,\ldots,m\}$, since there is no $\lambda_i \in ((2/3),1)$, $i = 0,1,2,\ldots,m$, such that $\sum_{i=0}^{m} \lambda_i = 1$.

In [19], Isiogugu and Osilike proved weak and strong convergence theorems for the class of multivalued demicontractive mappings which contains the class of k-strictly pseudocontractive mappings for which the fixed point set $F(T)$ is nonempty. They proved the following theorem in the setting of real Hilbert spaces.

Theorem 4 (Theorem 3.1 of [19]). *Let K be a nonempty closed convex subset of a real Hilbert space H. Suppose that $T : K \to P(K)$ is a demicontractive mapping from K into the family of all proximinal subsets of K with $k \in (0,1)$ and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose (I-T) is weakly demiclosed at zero. Then, the Mann type sequence defined by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n y_n \tag{3}$$

converges weakly to $q \in F(T)$, where $y_n \in Tx_n$ and $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying (i) $\alpha_n \to \alpha < 1 - k$ and (ii) $\alpha > 0$.

It is our purpose in this paper to prove strong and Δ-convergence theorems for a Krasnoselskii-type algorithm sequence to a common fixed point of a finite family of demicontractive mappings in the setting of CAT(0) spaces. In our results, the condition imposed on λ_i, $i = 1,2,3,\ldots,m$, in Theorem 2.2 of [18] is reduced to the condition $\lambda_0 \in (k,1)$, where the rest of the λ_i, $i = 1,2,\ldots,m$, can be chosen arbitrarily in $(0;1)$. Thus, our result is applicable to all classes of demicontractive mappings. Furthermore, our theorems extend and improve the results of Chidume and Ezeora [18], Chidume et al. [17], and Isiogugu and Osilike [19] and complement the results of Dhompongsa and Panyanak [16], Dhompongsa et al. [9], Leustean [11], Shahzad and Markin [13], and Sokhuma [20] and results of a host of other authors on iterative approximation of fixed points in CAT(0) spaces.

2. Preliminaries

Let (X,d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a continuous map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, the mapping c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic segment joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. The space (X,d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and $y \in X$. A subset K of X is said to be convex if, for all x, $y \in K$, the segment $[x,y]$ remains in K.

A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges $\triangle$). A comparison triangle for $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) = \triangle(\overline{x_1},\overline{x_2},\overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d(x_i,x_j) = d_{\mathbb{R}^2}(\overline{x_i},\overline{x_j})$, for $i,j \in \{1,2,3\}$. A geodesic metric space X is called a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then, $\triangle$ is said to satisfy CAT(0) inequality, if, for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \leqslant d_{\mathbb{R}^2}(\overline{x},\overline{y}). \tag{4}$$

If x, y_1, y_2 are points in CAT(0) space, and if y_0 is the midpoint of the segment $[y_1, y_2]$, then, the CAT(0) inequality implies

$$d(x,y_0)^2 \leqslant \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [21]. In fact (cf. [1], p.163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

We now collect some elementary facts about CAT(0) spaces.

Lemma 5 (see, e.g., [16]). *Let (X,d) be a CAT(0) space. Then*

(i) *(X,d) is uniquely geodesic.*

(ii) *For each $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that*

$$d(x,z) = td(x,y), \qquad d(y,z) = (1-t)\,d(x,y). \tag{5}$$

For convenience, from now on, we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (5).

Also, for $\alpha_1,\alpha_2,\alpha_3 \in (0,1)$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $x_1,x_2,x_3 \in X$, we will use the notation $\alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \alpha_3 x_3$ to denote the unique point z satisfying

$$\begin{aligned} d(x_1,z) &= (\alpha_2 + \alpha_3)\, d\left(x_1, \alpha_2' x_2 \oplus \alpha_3' x_3\right), \\ d\left(\alpha_2' x_2 \oplus \alpha_3' x_3, z\right) &= \alpha_1 d\left(x_1, \alpha_2' x_2 \oplus \alpha_3' x_3\right), \\ \alpha_i' &:= \frac{\alpha_i}{(\alpha_2 + \alpha_3)}, \quad i = 2,3. \end{aligned} \tag{6}$$

In particular, taking $\alpha_1 = \alpha_2 = \alpha_3 = (1/3)$, we compute the point $(1/3)x_1 \oplus (1/3)x_2 \oplus (1/3)x_3$ as follows.

From the illustration above, $(1/3)x_1 \oplus (1/3)x_2 \oplus (1/3)x_3 := (1/3)x_1 \oplus (2/3)((1/2)x_2 \oplus (1/2)x_3)$, where $(1/2)x_2 \oplus (1/2)x_3$ denotes the unique point $z_1 \in [x_2, x_3]$ such that $d(x_2, z_1) = (1/2)d(x_2, x_3)$, and $d(z_1, x_3) = (1/2)d(x_2, x_3)$.

Thus, we have $(1/3)x_1 \oplus (1/3)x_2 \oplus (1/3)x_3 := (1/3)x_1 \oplus (2/3)z_1$, where $(1/3)x_1 \oplus (2/3)z_1$ denotes the unique point $z_2 \in [x_1, z_1]$ satisfying $d(x_1, z_2) = (2/3)d(x_1, z_1)$, and $d(z_2, z_1) = (1/3)d(x_1, z_1)$. Hence we have $z_2 := (1/3)x_1 \oplus (1/3)x_2 \oplus (1/3)x_3$.

Extending this notation up to some $n \geqslant 3$, we use $\sum_{i=1}^{n} \oplus \alpha_i x_i$ to denote the unique point $z \in [x_1, \sum_{i=2}^{n} \oplus (\alpha_i/\sigma)x_i]$ satisfying

$$d(x_1, z) = \sigma d\left(x_1, \sum_{i=2}^{n} \oplus \frac{\alpha_i}{\sigma} x_i\right),$$
$$d\left(\sum_{i=2}^{n} \oplus \frac{\alpha_i}{\sigma} x_i, z\right) = \alpha_1 d\left(x_1, \sum_{i=2}^{n} \oplus \frac{\alpha_i}{\sigma} x_i\right), \tag{7}$$

where $\alpha_i \in (0,1)$, $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} \alpha_i = 1$, $x_i \in X$, $i = 1, 2, \ldots, n$, $\sigma = \sum_{i=2}^{n} \alpha_i = (1 - \alpha_1)$.

Remark 6. The metric convex combinations defined above in (7) are similar to that defined on a Hilbert ball by Kopecká and Reich in [22], where the authors defined the metric convex combinations for self-maps S_i, $i = 1, 2, 3, \ldots, m$, on a Hilbert ball.

Lemma 7 (see, e.g., Lemmas 2.4 and 2.5 in [16]). *Let (X, d) be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, the following inequalities hold:*

(i) $d((1-t)x \oplus ty, z) \leqslant (1-t)d(x, z) + td(y, z)$,

(ii) $d((1-t)x \oplus ty, z)^2 \leqslant (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$,

where $d(x, z)^2 = (d(x, z))^2$.

We now give the Δ-convergence together with some of its basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\})\}, \tag{8}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{9}$$

It is well known that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

Definition 8. A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim x_n = x$ and x is called the Δ-limit of $\{x_n\}$.

Lemma 9. *(i) (See, e.g., [15]). Every bounded sequence in a complete CAT(0) space has a Δ-convergent subsequence.*

(ii) (See, e.g., [23]). If C is a nonempty closed and convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.

(iii) (See, e.g., [16]). If $\{x_n\}$ is a bounded sequence in a complete CAT(0) X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.

Let (X, d) be a geodesic metric space. We denote by $CB(X)$ the collection of all nonempty closed and bounded subsets of X. Let H be the Hausdorff metric with respect to the metric distance d; that is,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\}, \tag{10}$$

for all $A, B \in CB(X)$, where $d(a, B) = \inf_{b\in B} d(a, b)$ is the distance from the point a to the subset B.

Let $T : X \to 2^X$ be a multivalued mapping on X. A point $x \in X$ is called a fixed point of T if $x \in Tx$. The set $F(T) = \{x \in X : x \in Tx\}$ is called the fixed point set of T.

Definition 10. Let (X, d) be a geodesic metric space. A multivalued mapping $T : X \to 2^X$ is said to be

(i) nonexpansive if

$$H(Tx, Ty) \leqslant d(x, y) \quad \forall x, y \in X, \tag{11}$$

(ii) quasinonexpansive if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leqslant d(x, p) \quad \forall x \in X,\ p \in F(T), \tag{12}$$

(iii) demicontractive if $F(T) \neq \emptyset$ and there exists $k \in (0, 1)$ such that

$$H(Tx, Tp)^2 \leqslant d(x, p)^2 + kd(x, Tx)^2 \quad \forall x \in X,\ p \in F(T), \tag{13}$$

where $H(Tx, Tp)^2 = [H(Tx, Tp)]^2$ and $d(x, p)^2 = [d(x, p)]^2$,

(iv) hemicontractive if $k = 1$ in (iii) above; that is,

$$H(Tx, Tp)^2 \leqslant d(x, p)^2 + d(x, Tx)^2 \quad \forall x \in X,\ p \in F(T). \tag{14}$$

It is clear that every multivalued nonexpansive mapping with nonempty fixed point set is quasinonexpansive, and every quasinonexpansive mapping is demicontractive mapping.

The following example shows that the class of demicontractive mappings strictly contains the class of quasinonexpansive mappings.

Example 11. Let $X = \mathbb{R}$ (the set of real numbers with the usual metric). Define $T : X \to 2^X$ by

$$Tx = \begin{cases} \left[-3x, -\dfrac{5x}{2}\right], & x \in [0, \infty), \\ \left[-\dfrac{5x}{2}, -3x\right], & x \in (-\infty, 0]. \end{cases} \tag{15}$$

Then, $F(T) = \{0\}$, and T is demicontractive mapping which is not quasinonexpansive.

Indeed, for each $x \in (-\infty, 0) \cup (0, \infty)$, we have

$$H(Tx, T0)^2 = |-3x - 0|^2 = 9|x - 0|^2, \tag{16}$$

which implies that T is not quasinonexpansive.

We also have that

$$d(x, Tx)^2 = \left|x - \left(-\frac{5}{2}x\right)\right|^2 = \frac{49}{4}|x|^2. \tag{17}$$

Thus,

$$H(Tx, T0)^2 = |x - 0|^2 + 8|x - 0|^2 = |x - 0|^2 + \frac{32}{49}d(x, Tx)^2. \tag{18}$$

Hence, T is a demicontractive mapping with constant $k = (32/49) \in (0, 1)$.

3. Main Results

We start by proving the following lemmas.

Lemma 12. *Let X be a CAT(0) space. Let $\{x_i,\ i = 1, 2, \ldots, n\} \subset X$, and $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} \alpha_i = 1$. Then, the following inequality holds:*

$$d\left(\sum_{i=1}^{n} \oplus \alpha_i x_i, z\right)^2 \leqslant \sum_{i=1}^{n} \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\neq j}^{n} \alpha_i \alpha_j d\left(x_i, x_j\right)^2, \quad \forall z \in X. \tag{19}$$

Proof. The proof is by induction. For $n = 2$, the result follows from Lemma 7(ii). For simplicity, we will give the proof for $n = 3$. From Lemma 7(ii), we have that

$$\begin{aligned}
&d\left(\sum_{i=1}^{3} \oplus \alpha_i x_i, z\right)^2 \\
&= d\left(\alpha_1 x_1 \oplus (\alpha_2 + \alpha_3)\left(\alpha_2' x_2 \oplus \alpha_3' x_3\right), z\right)^2, \\
&\qquad \alpha_i' := \frac{\alpha_i}{(\alpha_2 + \alpha_3)}, \quad i \geqslant 2 \\
&\leqslant \alpha_1 d(x_1, z)^2 + (\alpha_2 + \alpha_3)\, d\left(\alpha_2' x_2 \oplus \alpha_3' x_3, z\right)^2 \\
&\quad - \alpha_1 (\alpha_2 + \alpha_3)\, d\left(x_1, \alpha_2' x_2 \oplus \alpha_3' x_3\right)^2 \\
&\leqslant \alpha_1 d(x_1, z)^2 + (\alpha_2 + \alpha_3)\left[\alpha_2' d(x_2, z)^2 + \alpha_3' d(x_3, z)^2 - \alpha_2' \alpha_3' d(x_2, x_3)^2\right] \\
&\quad - \alpha_1 (\alpha_2 + \alpha_3)\left[\alpha_2' d(x_1, x_2)^2 + \alpha_3' d(x_1, x_3)^2 - \alpha_2' \alpha_3' d(x_2, x_3)^2\right] \\
&= \sum_{i=1}^{3} \alpha_i d(x_i, z)^2 - \alpha_2 \alpha_3' d(x_2, x_3)^2 - \alpha_1 \alpha_2 d(x_1, x_2)^2 \\
&\quad - \alpha_1 \alpha_3 d(x_1, x_3)^2 - \alpha_1 \alpha_2 \alpha_3' d(x_2, x_3)^2 \\
&= \sum_{i=1}^{3} \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\neq j}^{3} \alpha_i \alpha_j d\left(x_i, x_j\right)^2.
\end{aligned} \tag{20}$$

Now, suppose (19) holds up to some $k \geqslant 3$; that is,

$$d\left(\sum_{i=1}^{k} \oplus \alpha_i x_i, z\right)^2 \leqslant \sum_{i=1}^{k} \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\neq j}^{k} \alpha_i \alpha_j d\left(x_i, x_j\right)^2. \tag{21}$$

Then, from Lemma 7 we have

$$\begin{aligned}
&d\left(\sum_{i=1}^{k+1} \oplus \alpha_i x_i, z\right)^2 \\
&= d\left(\alpha_1 x_1 \oplus \sigma\left(\sum_{i=2}^{k+1} \oplus \frac{\alpha_i}{\sigma} x_i\right), z\right)^2, \quad \sigma = \sum_{i=2}^{k+1} \alpha_i, \\
&\leqslant \alpha_1 d(x_1, z)^2 + \sigma d\left(\sum_{i=2}^{k+1} \oplus \frac{\alpha_i}{\sigma} x_i, z\right)^2 \\
&\quad - \alpha_1 \sigma d\left(x_1, \sum_{i=2}^{k+1} \oplus \frac{\alpha_i}{\sigma} x_i\right)^2 \\
&= \alpha_1 d(x_1, z)^2 + \sigma d\left(\sum_{i=1}^{k} \oplus \frac{\alpha_{i+1}}{\sigma} x_{i+1}, z\right)^2 \\
&\quad - \alpha_1 \sigma d\left(x_1, \sum_{i=1}^{k} \oplus \frac{\alpha_{i+1}}{\sigma} x_{i+1}\right)^2.
\end{aligned} \tag{22}$$

Using the induction hypothesis, we have

$$\begin{aligned}
&d\left(\sum_{i=1}^{k+1} \oplus \alpha_i x_i, z\right)^2 \\
&\leqslant \alpha_1 d(x_1, z)^2 + \sum_{i=1}^{k} \alpha_{i+1} d(x_{i+1}, z)^2 - \sum_{i=1, j, i\neq j}^{k} \frac{\alpha_{i+1}\alpha_{j+1}}{\sigma} d\left(x_{i+1}, x_{j+1}\right)^2 \\
&\quad - \sum_{i=1}^{k} \alpha_1 \alpha_{i+1} d(x_1, x_{i+1})^2 + \sum_{i=1, j, i\neq j}^{k} \frac{\alpha_1 \alpha_{i+1} \alpha_{j+1}}{\sigma} d\left(x_{i+1}, x_{j+1}\right)^2 \\
&= \sum_{i=1}^{k+1} \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\neq j}^{k+1} \alpha_i \alpha_j d\left(x_i, x_j\right)^2.
\end{aligned} \tag{23}$$

Hence, by induction we have that (19) is true. The proof is complete. □

Lemma 13. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $T_i : K \to CB(K)$, $i = 1, 2, \ldots, m$, be a family of demicontractive mappings with constants $k_i \in (0, 1)$, $i = 1, 2, \ldots, m$, such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{n} F(T_i)$. For arbitrary $x_1 \in K$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m, \quad n \geq 1, \tag{24}$$

where $y_n^i \in T_i x_n$, $i = 1, 2, \ldots, m$, $\alpha_0 \in (k, 1)$, $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, m$, such that $\sum_{i=0}^{m} \alpha_i = 1$, and $k := \max\{k_i, i = 1, 2, \ldots, m\}$. Then, $\lim_{n\to\infty}\{d(x_n, p)\}$ exists for all $p \in \bigcap_{i=1}^{n} F(T_i)$, and $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, m$.

Proof. Let $p \in \bigcap_{i=1}^{n} F(T_i)$. By Lemma 12 and Definition 10(iv), we have

$$\begin{aligned}
&d(x_{n+1}, p)^2 \\
&= d\left(\alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m, p\right)^2 \\
&\leq \alpha_0 d(x_n, p)^2 + \sum_{i=1}^{m} \alpha_i d\left(y_n^i, p\right)^2 - \sum_{i=1}^{m} \alpha_0 \alpha_i d\left(x_{n_0}, y_n^i\right) \\
&\quad - \sum_{i,j=2, i\neq j}^{m} \alpha_i \alpha_j d\left(y_n^i, y_n^j\right) \\
&\leq \alpha_0 d(x_n, p)^2 + \sum_{i=1}^{m} \alpha_i \left(H\left(T_i x_n, Tp\right)\right)^2 - \sum_{i=1}^{m} \alpha_0 \alpha_i d\left(x_n, y_n^i\right)^2 \\
&\leq \alpha_0 d(x_n, p)^2 + \sum_{i=1}^{m} \alpha_i d(x_n, p)^2 + \sum_{i=1}^{m} k_i \alpha_i d\left(x_n, y_n^i\right)^2 \\
&\quad - \sum_{i=1}^{m} \alpha_i \alpha_0 d\left(x_n, y_n^i\right)^2 \\
&\leq d(x_n, p)^2 - (\alpha_0 - k) \sum_{i=1}^{m} \alpha_i d\left(x_n, y_n^i\right)^2 \\
&\leq d(x_n, p)^2 - (\alpha_0 - k) \sum_{i=1}^{m} \alpha d\left(x_n, y_n^i\right)^2 \\
&\leq d(x_n, p)^2, \quad \alpha = \min_{0\leq i\leq m} \alpha_i,
\end{aligned} \tag{25}$$

which shows that $\{d(x_n, p)\}$ is nonincreasing and bounded. Hence, its limit exists.

Moreover, we have that

$$\alpha (\alpha_0 - k) \sum_{i=1}^{m} d\left(x_n, y_n^i\right)^2 \leq d(x_1, p)^2 < \infty. \tag{26}$$

Therefore, $\lim_{n\to\infty} d(x_n, y_n^i) = 0 \ \forall i = 1, 2, \ldots, m$. Consequently,

$$\lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0 \quad \forall i = 0, 1, \ldots, m. \tag{27}$$

□

Theorem 14. *Let K be a nonempty closed convex subset of a complete CAT(0) space. Let $T_i : K \to CB(K)$, $i = 1, 2, \ldots, m$, be a family of demicontractive mappings with constants $k_i \in (0, 1)$, $i = 1, 2, \ldots, m$, such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that T_i is Δ-demiclosed at 0 for all $i = 1, 2, \ldots, m$, and $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{n} F(T_i)$. For arbitrary $x_1 \in K$, define a sequence x_n by*

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m, \quad n \geq 1, \tag{28}$$

where $y_n^i \in T_i x_n$, $i = 1, 2, \ldots, m$, $\alpha_0 \in (k, 1)$, $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, m$, such that $\sum_{i=0}^{m} \alpha_i = 1$ and $k := \max\{k_i, i = 1, 2, \ldots, m\}$. Then, $\{x_n\}$ Δ-converges to a point $p \in \bigcap_{i=1}^{m} F(T_i)$.

Proof. Define $W_\Delta(x_n) := \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We will show that $W_\Delta(x_n) \subseteq \bigcap_{i=1}^{m} F(T_i)$ and that $W_\Delta(x_n)$ consists of exactly one point.

Let $u \in W_\Delta(x_n)$; this implies that there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$ since by Lemma 13 $\{u_n\}$ is bounded. This implies from Lemma 9 ((i) and (ii)) that there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} v_n = v \in K$.

Using Lemma 13 and the fact that T_i, $i = 1, 2, \ldots, m$ is Δ-demiclosed at 0 for all $i = 1, 2, \ldots, m$ we have that $v \in \bigcap_{i=1}^{m} F(T_i)$, and hence, $d(u_n, v)$ converges. Lemma 9 (iii) implies that $u = v$. Thus, we have $W_\Delta(x_n) \subseteq \bigcap_{i=1}^{m} F(T_i)$.

We now show that $W_\Delta(x_n)$ consists of exactly one point. Let $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ be arbitrary subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Since $u \in W_\Delta(x_n) \subseteq \bigcap_{i=1}^{m} F(T_i)$, we have by Lemma 13 that $d(x_n, u)$ converges. Lemma 9 (iii) implies that $u = v$. The proof is complete. □

Corollary 15. *Let K, X, T_i, $i = 1, 2, \ldots, m$, and $\{x_n\}$ be as in Theorem 14. Suppose there exists $i_0 \in \{1, 2, \ldots, m\}$ such that T_{i_0} is semicompact; then $\{x_n\}$ converges strongly to a common fixed point of T_i, $i = 1, 2, \ldots, m$.*

Proof. Since by Lemma 13 $d(x_n, T_{i_0} x_n) \to 0$ and T_{i_0} is semicompact, then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $\{u_n\} \to u \in K$, which implies $\Delta\text{-}\lim_{n\to\infty} u_n = u \in K$. By Theorem 14 we have that $u \in \bigcap_{i=1}^{n} F(T_i)$, which implies by Lemma 13 that $x_n \to u$. □

Corollary 16. *Let K be a nonempty compact convex subset of a complete CAT(0) space. Let $T_i : K \to CB(K)$, $i = 1, 2, \ldots, m$, be a family of demicontractive mappings with constants $k_i \in (0, 1)$, $i = 1, 2, \ldots, m$, such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that T_i is Δ-demiclosed at 0 for all $i = 1, 2, \ldots, m$, and $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{n} F(T_i)$. For arbitrary $x_1 \in K$, define a sequence x_n by*

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m, \quad n \geq 1, \tag{29}$$

where $y_n^i \in T_i x_n$, $i = 1, 2, \ldots, m$, $\alpha_0 \in (k, 1)$, $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, m$, such that $\sum_{i=0}^{m} \alpha_i = 1$ and $k := \max\{k_i, i = 1, 2, \ldots, m\}$. Then, $\{x_n\}$ converges strongly to some point $p \in \bigcap_{i=1}^{m} F(T_i)$.

Proof. The proof follows from the fact that if K is compact, then every multivalued mapping $T : K \to CB(K)$ is semicompact. Thus, the conclusion follows from Corollary 15. □

Corollary 17. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space. Let $T_i : K \to CB(K), i = 1, 2, \ldots, m$, be a family of quasinonexpansive mappings such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that T_i is Δ-demiclosed at 0 for all $i = 1, 2, \ldots, m$, $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{n} F(T_i)$, and there exists $i_0 \in \{1, 2, \ldots, m\}$ such that T_{i_0} is semicompact. For arbitrary $x_1 \in K$, define a sequence x_n by*

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m, \quad n \geq 1, \tag{30}$$

where $y_n^i \in T_i x_n$, $i = 1, 2, \ldots, m$, $\alpha_0 \in (k, 1), \alpha_i \in (0, 1)$, $i = 1, 2, \ldots, m$, such that $\sum_{i=0}^{m} \alpha_i = 1$. Then, $\{x_n\}$ converges strongly to some point $p \in \bigcap_{i=1}^{m} F(T_i)$.

Remark 18. It is worth mentioning that our result is true for all $\mathrm{CAT}(k)$ spaces, $k \leqslant 0$, since, for $k \leqslant k'$, $\mathrm{CAT}(k) \subseteq \mathrm{CAT}(k')$ (see Bridson and Haefliger [1]).

Remark 19. Our results extend the results of Chidume and Ezeora [18] to a more general space than Hilbert space (CAT(0) spaces). Furthermore, the condition imposed on λ_i, $i = 0, 1, 2, \ldots, m$, in Theorem 2.2 of [18] ($\lambda_i \in (k, 1)$, $i = 0, 1, 2, \ldots, m$, such that $\sum_{i=0}^{m} \lambda_i = 1$) restricts the class of operators for which the theorem is applicable. In our result, the condition is reduced to $\lambda_0 \in (k, 1)$, $\lambda_i \in (0, 1)$, $i = 1, 2, \ldots, m$, such that $\sum_{i=0}^{m} \lambda_i = 1$, thereby making our results applicable to all classes of demicontractive mappings.

Remark 20. It is worth mentioning that the result proved in Lemma 12 is of special interest.

Remark 21. The results of Chidume et al. (Theorem 3.1 of [17]) and Isiogugu and Osilike (Theorem 3.1 of [19]) are special cases of our results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, Germany, 1999.

[2] K. S. Brown, *Buildings*, Springer, New York, NY, USA, 1989.

[3] D. Burago, Y. Burago, and S. Ivanov, "A course in metric geometry," in *Graduate Studies in Math.*, vol. 33, American Mathematical Society, Providence, RI, USA, 2001.

[4] W. A. Kirk, "Geodesic geometry and fixed point theory," in *Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003)*, vol. 64 of *Colección Abierta*, pp. 195–225, Universidad de Sevilla, Sevilla, Spain, 2003.

[5] W. A. Kirk, "Geometry and fixed point theory II," in *Proceedings of the International Conference of Fixed Point Theory and Application*, pp. 113–142, Yokohama Publishers, Yokohama, Japan, 2004.

[6] P. Chaoha and A. Phon-on, "A note on fixed point sets in $CAT(0)$ spaces," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 983–987, 2006.

[7] S. Dhompongsa, A. Kaewkhao, and B. Panyanak, "Lim's theorems for multivalued mappings in *CAT (0)* spaces," *Journal of Mathematical Analysis and Applications*, vol. 312, no. 2, pp. 478–487, 2005.

[8] S. Dhompongsa, W. A. Kirk, and B. Sims, "Fixed points of uniformly Lipschitzian mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 4, pp. 762–772, 2006.

[9] S. Dhompongsa, W. Fupinwong, and A. Kaewkhao, "A common fixed points of a nonexpansive semigroup and a convergence theorem for Mann iterations in geodesic metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 12, pp. 4268–4273, 2009.

[10] K. Fujiwara, K. Nagano, and T. Shioya, "Fixed point sets of parabolic isometries of *CAT(0)*-spaces," *Commentarii Mathematici Helvetici*, vol. 81, no. 2, pp. 305–335, 2006.

[11] L. Leustean, "A quadratic rate of asymptotic regularity for *CAT(0)*-spaces," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 386–399, 2007.

[12] N. Shahzad, "Invariant approximations in $CAT(0)$ spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 12, pp. 4338–4340, 2009.

[13] N. Shahzad and J. Markin, "Invariant approximations for commuting mappings in *CAT(0)* and hyperconvex spaces," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1457–1464, 2008.

[14] T. C. Lim, "Remarks on some fixed point theorems," *Proceedings of the American Mathematical Society*, vol. 60, pp. 179–182, 1976.

[15] W. A. Kirk and B. Panyanak, "A concept of convergence in geodesic spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 12, pp. 3689–3696, 2008.

[16] D. Dhompongsa and B. Panyanak, "On Δ-convergence theorems in CAT(0) spaces," *Computers & Mathematics with Applications*, vol. 56, pp. 2572–2579, 2008.

[17] C. E. Chidume, C. O. Chidume, N. Djitté, and M. S. Minjibir, "Convergence theorems for fixed points of multivalued strictly pseudocontractive mappings in Hilbert spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 629468, 10 pages, 2013.

[18] C. E. Chidume and J. N. Ezeora, "Krasnoselskii-type algorithm for family of multi-valued strictly pseudo-conttractive mappings," *Fixed Point Theory and Applications*, vol. 2014, article 111, 2014.

[19] F. O. Isiogugu and M. O. Osilike, "Convergence theorems for new classes of multi-valued hemicontractive-type mapping," *Fixed Point Theory and Applications*, vol. 2014, article 93, 2014.

[20] K. Sokhuma, "On Δ-convergence theorems for a pair of single-valued and multivalued nonexpansive mappings in *CAT(0)* spaces," *Journal of Mathematical Analysis*, vol. 4, no. 2, pp. 23–31, 2013.

[21] F. Bruhat and J. Tits, *Groupes Réductifs sur un Corp Local. I. Donneés Radicielles Valuées*, vol. 41, Institut des Hautes Études Scientifiques, 1972.

[22] E. Kopecká and S. Reich, "Asymptotic behavior of resolvents of coaccretive operators in the Hilbert ball," *Nonlinear Analysis*, vol. 70, no. 9, pp. 3187–3194, 2009.

[23] S. Dhompongsa, W. A. Kirk, and B. Panyanak, "Nonexpansive set-valued mappings in metric and Banach spaces," *Journal of Nonlinear and Convex Analysis*, vol. 8, no. 1, pp. 35–45, 2007.