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Research Article

Existence of Tripled Fixed Points for a Class of Condensing Operators in Banach Spaces

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We give some results concerning the existence of tripled fixed points for a class of condensing operators in Banach spaces. Further, as an application, we study the existence of solutions for a general system of nonlinear integral equations.

1. Introduction and Preliminaries

Measures of noncompactness are very useful tools in functional analysis, for instance, in metric fixed point theory and in the theory of operator equations in Banach spaces. The first measure of noncompactness, denoted by μ, was defined and studied by Kuratowski [1] in 1930. In 1955, Banaś and Goebel [2] used the function μ to prove his fixed point theorem. Darbo's fixed point theorem [2] is a very important generalization of Schauder's fixed point theorem [3] and several authors had used this concept for the resolution of nonlinear equations, some of whom are Aghajani et al. [4, 5], Banaś [6], Banaś and Rzepka [7], Mursaleen and Mohiuddine [8], and many others. Recently in [9], Aghajani et al. give a generalization of Darbo's fixed point theorem. Moreover, they present some results on the existence of coupled fixed points for class of condensing operators. In this paper, we generalize these results to obtain the existence of tripled fixed points for the same class of operators.

Throughout this paper, X is assumed to be a Banach space and $BC(\mathbb{R}^+)$ is the space of all real functions defined, bounded and continuous on $\mathbb{R}^+$. The family of bounded subset, closure, and closed convex hull of X are denoted by $\mathscr{B}_X$, $\overline{X}$, and $\mathrm{Conv}X$, respectively.

Definition 1 (see [10]). Let X be a Banach space and $\mathscr{B}_X$ the family of bounded subset of X. A map

$$\mu : \mathscr{B}_X \longrightarrow [0, \infty) \tag{1}$$

is called measure of noncompactness defined on X if it satisfies the following.

(1) $\mu(A) = 0 \Leftrightarrow A$ is a precompact set.

(2) $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$.

(3) $\mu(A) = \mu(\overline{A})$, $\forall A \in \mathscr{B}_X$.

(4) $\mu(\mathrm{Conv}A) = \mu(A)$.

(5) $\mu(\lambda A + (1-\lambda)B) \leqslant \lambda\mu(A) + (1-\lambda)\mu(B)$, for $\lambda \in [0, 1]$.

(6) Let (A_n) be a sequence of closed sets from $\mathscr{B}_X$ such that $A_{n+1} \subseteq A_n$, $(n \geqslant 1)$, and $\lim_{n\to\infty}\mu(A_n) = 0$. Then, the intersection set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and A_∞ is precompact.

Theorem 2 (see [2]). *Let C be a nonempty closed, bounded, and convex subset of X. If $T : C \to C$ is a continuous mapping*

$$\mu(TA) \leqslant k\mu(A), \quad k \in [0, 1), \tag{2}$$

then T has a fixed point.

Theorem 3 (see [9]). *Let C be a nonempty closed, bounded, and convex subset of X and $T : C \to C$ a continuous mapping such that for any subset A of C*

$$\mu(TA) \leqslant \beta(\mu(A))\mu(A), \tag{3}$$

where $\beta : \mathbb{R}_+ \to [0,1)$*; that is,* $\beta(t_n) \to 1$ *implies* $t_n \to 0$. *Then,* T *has at least one fixed point.*

The following result is a corollary of the previous theorem.

Corollary 4 (see [9]). *Let* C *be a nonempty closed, bounded, and convex subset of* X *and* $T : C \to C$ *a continuous mapping such that for any subset* A *of* C

$$\mu(TA) \leqslant \varphi(\mu(A)), \tag{4}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a nondecreasing and upper semicontinuous functions; that is, for every* $t > 0$, $\varphi(t) < t$. *Then,* T *has at least one fixed point.*

Definition 5 (see [11]). A coupled fixed point of a mapping $G : X \times X \to X$ is an element $(x, y) \in X \times X$ such that $G(x, y) = x$ and $G(y, x) = y$.

Theorem 6 (see [12]). *Let* $\mu_1, \mu_2, \ldots, \mu_n$ *be measures of noncompactness in Banach spaces* $E_1, E_2, \ldots, E_n$, *(respectively).*

Then, the function

$$\widetilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n)), \tag{5}$$

defines a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$, *where* X_i *is the natural projection of* X *on* E_i, *for* $i = 1, 2, \ldots, n$, *and* F *is a convex function defined by*

$$F : [0,\infty)^n \longrightarrow [0,\infty), \tag{6}$$

such that

$$F(x_1, x_2, \ldots, x_n) = 0 \Longleftrightarrow x_i = 0, \quad \text{for } i = 1, 2, \ldots, n. \tag{7}$$

Remark 7. Aghajani and Sabzali [13] illustrated the previous theorem by the following example. Let the mapping F be as follows:

$$F(x, y) = \max\{x, y\}, \quad \text{or} \quad F(x, y) = x + y, \quad \text{for any } (x, y) \in [0,\infty)^2. \tag{8}$$

They showed that

$$\widetilde{\mu}(X) = \max(\mu_1(X_1), \mu_2(X_2)), \tag{9}$$

or

$$\widetilde{\mu}(X) = \mu_1(X_1) + \mu_2(X_2) \tag{10}$$

defines a measure of noncompactness in the space $E_1 \times E_2$, where, for $i = 1, 2$, μ_i are measure of noncompactness in E_i and X_i, $i = 1, 2$ are the natural projections of X on E_i.

Theorem 8 (see [9]). *Let* Ω *be a nonempty, bounded, closed, and convex subset of a Banach space* E *and let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. *Assume that* φ *is a nondecreasing and upper semicontinuous function. Let* $G : \Omega \times \Omega \to \Omega$ *be a continuous operator satisfying*

$$\mu(G(X_1 \times X_2)) \leqslant \varphi\left(\frac{\mu(X_1) + \mu(X_2)}{2}\right), \quad X_1, X_2 \in \Omega, \tag{11}$$

for any measure of noncompactness μ. *Then,* G *has at least a coupled fixed point.*

2. Main Results

Definition 9 (see [14]). A tripled (x, y, z) of a mapping $G : X \times X \times X \to X$ is called a tripled fixed point if

$$\begin{aligned} G(x, y, z) &= x, \\ G(y, x, z) &= y, \\ G(z, y, x) &= z. \end{aligned} \tag{12}$$

Remark 10. We can notice that by taking

$$F(x, y, z) = \max\{x, y, z\}, \quad \text{for any } (x, y, z) \in [0,\infty)^3, \tag{13}$$

or

$$F(x, y, z) = x + y + z, \quad \text{for any } (x, y, z) \in [0,\infty)^3 \tag{14}$$

F satisfies the conditions of Theorem 6. Thus, for a measure of noncompactness μ_i $(i = 1, 2, 3)$, we have that

$$\widetilde{\mu}(X) = \max(\mu_1(X_1), \mu_2(X_2), \mu_3(X_3)), \tag{15}$$

or

$$\widetilde{\mu}(X) = \mu_1(X_1) + \mu_2(X_2) + \mu_3(X_3) \tag{16}$$

defines a measure of noncompactness in the space $E \times E \times E$ where X_i, $i = 1, 2, 3$ are the natural projections of X on E_i.

So, we obtain the following theorem.

Theorem 11. *Let* Ω *be a nonempty, bounded, closed, and convex subset of a Banach space* E *and let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a nondecreasing and upper semicontinuous function such that* $\varphi(t) < t$ *for all* $t > 0$. *Then, for any measure of noncompactness* μ, *the continuous operator* $G : \Omega \times \Omega \times \Omega \to \Omega$ *satisfying*

$$\mu(G(X_1 \times X_2 \times X_3)) \leqslant \varphi\left(\frac{\mu(X_1) + \mu(X_2) + \mu(X_3)}{3}\right), \quad X_1, X_2, X_3 \in \Omega \tag{17}$$

has at least a tripled fixed point.

Proof. To prove this theorem, let us define the measure of noncompactness $\widetilde{\mu}$ by

$$\widetilde{\mu}(X) = \mu_1(X_1) + \mu_2(X_2) + \mu_3(X_3) \tag{18}$$

and the mapping $\widetilde{G} : \Omega \times \Omega \times \Omega \to \Omega$

$$\widetilde{G}(x, y, z) = (G(x, y, z), G(y, x, z), G(z, y, x)). \tag{19}$$

Since

$$\begin{aligned}\widetilde{\mu}\left(\widetilde{G}(X)\right) &\leqslant \widetilde{\mu}\left(G\left(X_1\times X_2\times X_3\right)\times G\left(X_2\times X_1\times X_3\right)\right.\\&\qquad\left.\times\, G\left(X_3\times X_2\times X_1\right)\right)\\&= \mu\left(G\left(X_1\times X_2\times X_3\right)\right)+\mu\left(G\left(X_2\times X_1\times X_3\right)\right)\\&\quad+\mu\left(G\left(X_3\times X_2\times X_1\right)\right)\\&\leqslant \varphi\left(\frac{\mu(X_1)+\mu(X_2)+\mu(X_3)}{3}\right)\\&\quad+\varphi\left(\frac{\mu(X_1)+\mu(X_2)+\mu(X_3)}{3}\right)\\&\quad+\varphi\left(\frac{\mu(X_1)+\mu(X_2)+\mu(X_3)}{3}\right)\\&= 3\varphi\left(\frac{\mu(X_1)+\mu(X_2)+\mu(X_3)}{3}\right)\end{aligned}\tag{20}$$

and $\widetilde{\mu}' = (1/3)\,\widetilde{\mu}$ is a measure of noncompactness, we get

$$\widetilde{\mu}'\left(\widetilde{G}(X)\right) \leqslant \varphi\left(\widetilde{\mu}'(X)\right).\tag{21}$$

By Corollary 4, we obtain that G has at least a tripled fixed point. □

3. Applications

We can see an application of Theorem 11 in the study of existence of solutions for systems of integral equations defined on the Banach space $BC(\mathbb{R}^+)$ endowed with the norm

$$\|x\| = \sup\{|x(t)| : t > 0\}.\tag{22}$$

The measure of noncompactness on $BC(\mathbb{R}^+)$ for a positive fixed t on $\mathscr{B}_{BC(\mathbb{R}^+)}$ is defined as follows:

$$\mu(X) = \omega_0(X) + \lim_{t\to\infty}\sup\ \operatorname{diam} X(t),\tag{23}$$

such that

$$\begin{aligned}\operatorname{diam} X(t) &= \sup\{|x(t)-y(t)| : x, y \in X\},\\ &\qquad\text{where } X(t) = \{x(t) : x \in X\}.\end{aligned}\tag{24}$$

Before defining $\omega_0(X)$, we need first to introduce the modulus of continuity.

Let $x \in X$ and $\epsilon > 0$;

$$\omega^T(x,\epsilon) = \sup\{|x(t)-x(s)| : t, s \in [0,T], |t-s| \leqslant \epsilon\},\quad \text{for } T > 0,\tag{25}$$

is the modulus of continuity of x on $[0, T]$ and let

$$\begin{aligned}\omega^T(X,\epsilon) &= \sup\left\{\omega^T(x,\epsilon) : x \in X\right\},\\ \omega_0^T(X) &= \lim_{\epsilon\to 0}\omega^T(X,\epsilon),\\ \omega_0(X) &= \lim_{T\to\infty}\omega_0^T(X).\end{aligned}\tag{26}$$

Assume that

(i) $\xi, \eta, q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\xi(t) \to \infty$ as $t \to \infty$;

(ii) the function $\psi : \mathbb{R} \to \mathbb{R}$ is continuous and there exist positive δ, α such that

$$\left|\psi(t_1)-\psi(t_2)\right| \leqslant \delta\left|t_1-t_2\right|^{\alpha},\tag{27}$$

for any $t_1, t_2 \in \mathbb{R}_+$;

(iii) $f : \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and there exists a nondecareasing continuous function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi(0) = 0$, so that

$$\begin{aligned}&\left|f(t,x_1,x_2,x_3,x_4)-f(t,y_1,y_2,y_3,y_4)\right|\\&\quad\leqslant \frac{1}{2}\left(\varphi\left(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\right)\right)\\&\qquad+\Phi\left(|x_4-y_4|\right);\end{aligned}\tag{28}$$

(iv) the function defined by $|f(t,0,0,0,0)|$ is bounded on $\mathbb{R}_+$; that is,

$$M_1 = \sup\{f(t,0,0,0,0) : t \in \mathbb{R}_+\} < \infty;\tag{29}$$

(v) $h : \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a positive solution r_0 of the inequality

$$\frac{1}{3}\varphi(3r) + M_1 + \Phi(\delta D) \leqslant r,\tag{30}$$

where D is positive constant defined by the equality

$$\begin{aligned}D = \sup\Bigg\{&\left|\int_0^{q(t)}(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\right| :\\ &t, s \in \mathbb{R}_+, x, y, z \in BC(\mathbb{R}_+)\Bigg\},\end{aligned}$$

$$\begin{aligned}\lim_{\epsilon\to\infty}\int_0^{q(t)}&\left[h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\right.\\&\left.-h(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))\right]ds = 0,\end{aligned}\tag{31}$$

uniformly with respect to $x, y, z, u, v, w \in BC(\mathbb{R}_+)$.

Theorem 12. *Suppose that (i)–(v) hold; then the system of integral equations*

$$\begin{aligned} x(t) &= f\Big(t, x(\xi(t)), y(\xi(t)), z(\xi(t)), \\ &\quad \psi\Big(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\Big)\Big), \\ y(t) &= f\Big(t, y(\xi(t)), x(\xi(t)), z(\xi(t)), \\ &\quad \psi\Big(\int_0^{q(t)} h(t,s,y(\eta(s)), x(\eta(s)), z(\eta(s)))\,ds\Big)\Big), \\ z(t) &= f\Big(t, z(\xi(t)), y(\xi(t)), x(\xi(t)), \\ &\quad \psi\Big(\int_0^{q(t)} h(t,s,z(\eta(s)), y(\eta(s)), x(\eta(s)))\,ds\Big)\Big) \end{aligned} \tag{32}$$

has at least one solution in the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

Proof. Let $G : \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+) \to \mathrm{BC}(\mathbb{R}_+)$ be an operator defined by

$$\begin{aligned} G(x,y,z)(t) &= f\Big(t, x(\xi(t)), y(\xi(t)), z(\xi(t)), \\ &\quad \psi\Big(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\Big)\Big). \end{aligned} \tag{33}$$

For $(x,y,z) \in \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+)$, let

$$\begin{aligned} &\|(x,y,z)\|_{\mathrm{BC}(\mathbb{R}_+)\times\mathrm{BC}(\mathbb{R}_+)\times\mathrm{BC}(\mathbb{R}_+)} \\ &\quad = \|x\|_\infty + \|y\|_\infty + \|z\|_\infty. \end{aligned} \tag{34}$$

We can easily prove that the solution of (32) in $\mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+)$ is equivalent to the tripled fixed point of G.

Obviously, $G(x,y,z)(t)$ is continuous for any $(x,y,z) \in \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+)$. Hence, we have

$$\begin{aligned} &|G(x,y,z)(t)| \\ &\leqslant \Big| f\Big(t, x(\xi(t)), y(\xi(t)), z(\xi(t)), \\ &\qquad \psi\Big(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\Big)\Big) \\ &\qquad\quad - f(t,0,0,0)\Big| \\ &\quad + |f(t,0,0,0,0)| \\ &\leqslant \frac{1}{2}\varphi(|x(\xi(t))| + |y(\xi(t))| + |y(\xi(t))|) \\ &\quad + \Phi\Big(\Big|\psi\Big(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\Big)\Big|\Big) \\ &\quad + |f(t,0,0,0,0)| \\ &\leqslant \frac{1}{2}\varphi(|x(\xi(t))| + |y(\xi(t))| + |y(\xi(t))|) \\ &\quad + \Phi\Big(\delta\Big|\Big(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\Big)\Big|^\alpha\Big) \\ &\quad + |f(t,0,0,0,0)|. \end{aligned} \tag{35}$$

Then, by (29) and (30), we get

$$\begin{aligned} &\|G(x,y,z)\|_\infty \\ &\quad \leqslant \frac{1}{3}\varphi(\|x\|_\infty + \|y\|_\infty + \|z\|_\infty) + M_1 + \Phi(\delta D) \leqslant r_0. \end{aligned} \tag{36}$$

So, we obtain

$$G(\overline{B}_{r_0} \times \overline{B}_{r_0} \times \overline{B}_{r_0}) \subset \overline{B}_{r_0}. \tag{37}$$

Now, we prove that $G : \overline{B}_{r_0} \times \overline{B}_{r_0} \times \overline{B}_{r_0} \to \overline{B}_{r_0}$ is continuous. Let $(x,y,z), (u,v,w) \in \overline{B}_{r_0} \times \overline{B}_{r_0} \times \overline{B}_{r_0}$ such that, for $\varepsilon > 0$,

$$\|(x,y,z) - (u,v,w)\|_{\overline{B}_{r_0}\times\overline{B}_{r_0}\times\overline{B}_{r_0}} < \epsilon. \tag{38}$$

Then,

$$\begin{aligned}
&|G(x,y,z)(t) - G(u,v,w)(t)| \\
&= \Big| f\Big(t, x(\xi(t)), y(\xi(t)), z(\xi(t)), \\
&\qquad \psi\Big(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\Big)\Big) \\
&\quad - f\Big(t, x(\xi(t)), y(\xi(t)), z(\xi(t)), \\
&\qquad \psi\Big(\int_0^{q(t)} h(t,s,u(\eta(s)), v(\eta(s)), \\
&\qquad\qquad w(\eta(s)))\,ds\Big)\Big)\Big| \\
&\leqslant \frac{1}{2}\varphi(|x(\xi(t) - u(\xi(t)))| + |y(\xi(t)) - v(\xi(t))| \\
&\qquad + |z(\xi(t)) - w(\xi(t))|) \\
&\quad + \Phi\Big(\psi\Big(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\Big) \\
&\qquad - \psi\Big(\int_0^{q(t)} h(t,s,u(\eta(s)), v(\eta(s)), w(\eta(s)))\,ds\Big)\Big) \\
&\leqslant \frac{1}{2}\varphi(|x(\xi(t) - u(\xi(t)))| + |y(\xi(t)) - v(\xi(t))|) \\
&\qquad + |z(\xi(t)) - w(\xi(t))|) \\
&\quad + \Phi\Big(\delta\Big|\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds \\
&\qquad - \int_0^{q(t)} h(t,s,u(\eta(s)), v(\eta(s)), w(\eta(s)))\,ds\Big|^{\alpha}\Big).
\end{aligned} \tag{39}$$

Using condition (iii) and (29), there exists $T > 0$ such that if $t > T$, then

$$\begin{aligned}
&\Phi\Big(\delta\Big|\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s), z(\eta(s)))\,ds)\Big|^{\alpha}\Big) \\
&\quad \leqslant \frac{1}{3}\epsilon,
\end{aligned} \tag{40}$$

for any $x,y,z \in BC(\mathbb{R}_+)$. We notice two cases.

Case 1. If $t > T$, then from (39) and (40)

$$|G(x,y,z)(t) - G(u,v,w)(t)| \leqslant \frac{1}{3}\varphi(\epsilon) + \frac{1}{3}\epsilon. \tag{41}$$

Case 2. Similarly, for $t \in [0,T]$, we have

$$\begin{aligned}
&|G(x,y,z)(t) - G(u,v,w)(t)| \\
&\leqslant \frac{1}{2}\varphi(|x(\xi(t) - u(\xi(t)))| + |y(\xi(t)) - v(\xi(t))|) \\
&\quad + |z(\xi(t)) - w(\xi(t))|) \\
&\quad + \Phi\Big(\delta\Big|\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds \\
&\qquad - \int_0^{q(t)} h(t,s,u(\eta(s)), v(\eta(s)), w(\eta(s)))\,ds\Big|^{\alpha}\Big) \\
&\leqslant \frac{1}{2}\varphi(\epsilon) + \Phi\big(\delta(q_T\beta(\epsilon))^{\alpha}\big) \\
&< \frac{1}{2}\epsilon + \Phi\big(\delta(q_T\beta(\epsilon))^{\alpha}\big),
\end{aligned} \tag{42}$$

where $q_T = \sup\{q(t) : t \in [0,T]\}$, and

$$\begin{aligned}
\beta(\epsilon) = \sup\{&|h(t,s,x,y,z) - h(t,s,u,v,w)| : \\
&t \in [0,\mathrm{T}], s \in [0,q_T], \\
&x,y,z,u,v,w \in [-r_0, r_0], \\
&\|(x,y,z) - (u,v,w)\| < \epsilon\}.
\end{aligned} \tag{43}$$

Since β is continuous on $[0,T]\times[0,q_T]\times[-r_0,r_0]\times[-r_0,r_0]$, we have $\beta(\epsilon) \to 0$ and $\epsilon \to 0$. Thus, using (iii), we get

$$\Phi\big(\delta(q_T\beta(\epsilon))^{\alpha}\big) \longrightarrow 0, \quad \text{as } \epsilon \longrightarrow 0. \tag{44}$$

Finally, from (42) and (41), we conclude that G is a continuous function from $\overline{B}_{r_0} \times \overline{B}_{r_0} \times \overline{B}_{r_0}$ into $\overline{B}_{r_0}$.

Now, we show that the map G satisfies all the conditions of Theorem 11. To do this, for an arbitrary fixed $T > 0$ and $\epsilon > 0$, assume that X_1, X_2, and X_3 are nonempty chosen subsets of $\overline{B}_{r_0}$ and $t_1, t_2 \in [0,T]$, with $|t_2 - t_1| \leqslant \epsilon$. Without loss of generality, let

$$q(t_1) < q(t_2). \tag{45}$$

For an arbitrary $(x,y,z) \in X_1 \times X_2 \times X_3$,

$$\begin{aligned}
&|G(x,y,z)(t_1) - G(x,y,z)(t_2)| \\
&\leqslant \Big| f\Big(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \\
&\qquad \psi\Big(\int_0^{q(t_1)} h(t_1,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\Big)\Big) \\
&\quad - f\Big(t_1, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2)), \\
&\qquad \psi\Big(\int_0^{q(t_1)} h(t_1,s,x(\eta(s)), \\
&\qquad\qquad y(\eta(s)), z(\eta(s)))\,ds\Big)\Big)\Big|
\end{aligned}$$

$$+\Bigg|f\Bigg(t_1, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2)), \psi\Bigg(\int_0^{q(t_1)} h(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\Bigg)\Bigg) - f\Bigg(t_2, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2)), \psi\Bigg(\int_0^{q(t_1)} h(t_1,s,x(\eta(s)), y(\eta(s)),z(\eta(s)))\,ds\Bigg)\Bigg)\Bigg|$$

$$+\Bigg|f\Bigg(t_2, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2)), \psi\Bigg(\int_0^{q(t_1)} h(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\Bigg)\Bigg) - f\Bigg(t_2, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2)), \psi\Bigg(\int_0^{q(t_1)} h(t_1,s,x(\eta(s)), y(\eta(s)),z(\eta(s)))\,ds\Bigg)\Bigg)\Bigg|$$

$$+\Bigg|f\Bigg(t_2, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2)), \psi\Bigg(\int_0^{q(t_1)} h(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\Bigg)\Bigg) - f\Bigg(t_2, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2)), \psi\Bigg(\int_0^{q(t_2)} h(t_2,s,x(\eta(s)), y(\eta(s)),z(\eta(s)))\,ds\Bigg)\Bigg)\Bigg|$$

$$\leqslant \frac{1}{3}\varphi\big(|x(\xi(t_1)-x(\xi(t_2)))| + |y(\xi(t_1)-y(\xi(t_2)))| + |z(\xi(t_1)-z(\xi(t_2)))| + \omega_{r_0,D_1}^T(f,\epsilon)\big)$$
$$+\Phi\Bigg(\delta\Bigg|\int_0^{q(t_2)}[h(t_2,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) - h(t_1,s,x(\eta(s)),y(\eta(s)), z(\eta(s)))]\,ds\Bigg|^\alpha\Bigg)$$
$$+\Phi\Bigg(\delta\Bigg|\int_{q(t_1)}^{q(t_2)} h(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\Bigg|^\alpha\Bigg)$$
$$\leqslant \frac{1}{3}\varphi\big(\omega^T(x,\omega^T(\xi,\epsilon)) + \omega^T(y,\omega^T(\xi,\epsilon)) + \omega^T(z,\omega^T(\xi,\epsilon)) + \omega_{r_0,D_1}^T(f,\epsilon)\big)$$
$$+\Phi\big(\delta\big(q_T\omega_{r_0}^T(h,\epsilon)\big)^\alpha\big) + \Phi\big(\delta\big(U_{r_0}^T\omega^T(q,\epsilon)\big)^\alpha\big), \quad (46)$$

where

$$\omega^T(\xi,\epsilon) = \sup\{|(\xi(t_2)-\xi(t_1))| : t_1,t_2 \leqslant \epsilon, |t_2-t_1| \leqslant \epsilon\},$$
$$\omega^T(x,\omega^T(\xi,\epsilon)) = \sup\big\{|(x(t_2)-x(t_1))| : t_1,t_2 \in [0,T], |t_2-t_1| \leqslant \omega^T(\xi,\epsilon)\big\},$$
$$D_1 = q_T\sup\{|h(t,s,x,y,z)|, t\in[0,T], s\in[0,q_T], x,y,z\in[-r_0,r_0]\}$$
$$\omega_{r_0,D_1}^T(f,\epsilon) = \sup\{|f(t_2,x,y,z,d) - f(t_1,x,y,z,d)|, t_1,t_2\in[0,T], |t_2-t_1|\leqslant\epsilon, x,y,z\in[-r_0,r_0], d\in[-D_1,D_1]\},$$
$$\omega_{r_0}^T(f,\epsilon) = \sup\{|h(t_2,s,x,y,z) - f(t_1,s,x,y,z)|, t_1,t_2\in[0,T], |t_2-t_1|\leqslant\epsilon, s\in[0,q_T], x,y,z\in[-r_0,r_0]\},$$
$$U_{r_0}^T = \sup\{|h(t_1,s,x,y,z)| : t_1\in[0,T], s\in[0,q_T], x,y,z\in[-r_0,r_0]\}, \quad (47)$$

we obtain

$$\omega^T(G(X_1\times X_2\times X_3),\epsilon) \leqslant \frac{1}{3}\varphi\big(\omega^T(X_1,\omega^T(\xi,\epsilon)) + \omega^T(X_2,\omega^T(\xi,\epsilon)) + \omega^T(X_3,\omega^T(\xi,\epsilon))\big) + \omega_{r_0,D_1}^T(f,\epsilon)$$
$$+\Phi\big(\delta\big(q_T\omega_{r_0}^T(h,\epsilon)\big)^\alpha\big) + \Phi\big(\delta\big(U_{r_0}^T\omega^T(q,\epsilon)\big)^\alpha\big). \quad (48)$$

Further, by the uniform continuity of f and h on the compact sets $[0,T]\times[-r_0,r_0]\times[-r_0,r_0]\times[-r_0,r_0]\times[-D_1,D_1]$ and $[0,T]\times[0,q_T]\times[-r_0,r_0]\times[-r_0,r_0]\times[-r_0,r_0]$, respectively, we get $\omega_{r_0,D_1}^T(f,\epsilon)\to 0$ and $\omega_{r_0}^T(h,\epsilon)\to 0$ as $\epsilon\to 0$.

Moreover, Φ is a nondecreasing continuous function with $\Phi(0) = 0$ and (iii), and we obtain

$$\Phi\left(\delta\left(q_T\omega_{r_0}^T(h,\epsilon)\right)^{\alpha}\right) + \Phi\left(\delta\left(U_{r_0}^T\omega^T(q,\epsilon)\right)^{\alpha}\right) \longrightarrow 0, \quad \epsilon \longrightarrow 0. \tag{49}$$

By (48), we get

$$\omega_0^T(G(X_1 \times X_2 \times X_3)) \leqslant \frac{1}{3}\varphi\left(\omega_0^T(X_1) + \omega_0^T(X_2) + \omega_0^T(X_3)\right). \tag{50}$$

Taking the limit $T \to \infty$ in (50), we obtain

$$\omega_0(G(X_1 \times X_2 \times X_3)) \leqslant \frac{1}{3}\varphi(\omega_0(X_1) + \omega_0(X_2) + \omega_0(X_3)). \tag{51}$$

Then, for arbitrary (x, y, z), $(u, v, w) \in X_1 \times X_2 \times X_3$, and $t \in \mathbb{R}_+$, we have

$$\begin{aligned} &|G(x,y,z)(t) - G(u,v,w)(t)| \\ &\leqslant \frac{1}{3}\varphi(|x(\xi(t)) - u(\xi(t))| + |y(\xi(t)) - v(\xi(t))| \\ &\quad + |z(\xi(t)) - w(\xi(t))|) \\ &\quad + \Phi\left(\delta\left|\int_0^{q(t)}[h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) - h(t,s,u(\eta(s)), v(\eta(s)),w(\eta(s)))]\,ds\right|^{\alpha}\right) \\ &\leqslant \frac{1}{2}\varphi(\operatorname{diam} X_1(\xi(t)) + \operatorname{diam} X_2(\xi(t)) + \operatorname{diam} X_3(\xi(t))) \\ &\quad + \Phi\left(\delta\left|\int_0^{q(t)}[h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) - h(t,s,u(\eta(s)), v(\eta(s)),w(\eta(s)))]\,ds\right|^{\alpha}\right). \end{aligned} \tag{52}$$

Since (x, y, z), (u, v, w), and t are arbitrary in (52),

$$\begin{aligned} &\operatorname{diam} G(X_1 \times X_2 \times X_3)(t) \\ &\leqslant \frac{1}{3}\varphi(\operatorname{diam} X_1(\xi(t)) + \operatorname{diam} X_2(\xi(t)) + \operatorname{diam} X_3(\xi(t))) \\ &\quad + \Phi\left(\delta\left|\int_0^{q(t)}[h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) - h(t,s,u(\eta(s)),v(\eta(s)), w(\eta(s)))(t,s,x(\eta(s)), y(\eta(s)),z(\eta(s)))]\,ds\right|^{\alpha}\right). \end{aligned} \tag{53}$$

Taking again $T \to \infty$ in (53), we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \operatorname{diam} G(X_1 \times X_2 \times X_3)(t) + \omega_0 \\ &\leqslant \frac{1}{3}\varphi\left(\limsup_{t\to\infty} \operatorname{diam} X_1(\xi(t)) + \limsup_{t\to\infty} \operatorname{diam} X_2(\xi(t)) + \limsup_{t\to\infty} \operatorname{diam} X_3(\xi(t))\right). \end{aligned} \tag{54}$$

We conclude from (51) and (54) that

$$\begin{aligned} &\limsup_{t\to\infty} \omega_0(G(X_1 \times X_2 \times X_3))(t) + \omega_0(G(X_1 \times X_2 \times X_3)) \\ &\leqslant \frac{1}{3}\varphi\left(\limsup_{t\to\infty} \operatorname{diam} X_1(\xi(t)) + \limsup_{t\to\infty} \operatorname{diam} X_2(\xi(t)) + \limsup_{t\to\infty} \operatorname{diam} X_3(\xi(t))\right) \\ &\quad + \frac{1}{3}\varphi(\omega_0(X_1) + \omega_0(X_2) + \omega_0(X_3)). \end{aligned} \tag{55}$$

Since φ is a concave function, (55) implies

$$\begin{aligned} &\limsup_{t\to\infty} \operatorname{diam} G(X_1 \times X_2 \times X_3)(t) + \omega_0(G(X_1 \times X_2 \times X_3)) \\ &\leqslant \varphi\left(\frac{1}{3}\left[\limsup_{t\to\infty} \operatorname{diam} X_1(\xi(t)) + \omega_0(X_1)\right]\right) \\ &\quad + \varphi\left(\frac{1}{3}\left[\limsup_{t\to\infty} \operatorname{diam} X_2(\xi(t)) + \omega_0(X_2)\right]\right) \\ &\quad + \varphi\left(\frac{1}{3}\left[\limsup_{t\to\infty} \operatorname{diam} X_3(\xi(t)) + \omega_0(X_3)\right]\right). \end{aligned} \tag{56}$$

Finally, since μ is defined by

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t), \tag{57}$$

we get

$$\mu(G(X_1 \times X_2 \times X_3)) \leqslant \varphi\left(\frac{\mu(X_1) + \mu(X_2) + \mu(X_3)}{3}\right). \quad (58)$$

Hence, by Theorem 11, T has at least a tripled fixed point in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$. □

Example 1. We consider the following system of integral equations

$$\begin{aligned}
x(t) = {} & \frac{1}{3+t^2} x(t) + y(t) + z(t) \\
& + \int_0^T \Big(x(s)\, s \left|\sin y(t)\right| \left|\cos z(t)\right| \\
& \quad + e^s \left(1 + x^2(s)\right)\left(1 + \sin^2 y(s)\right) \\
& \quad \cdot \left(1 + \cos^2 z(s)\right)\Big) \\
& \quad \cdot \Big(e^t \left(1 + x^2(s)\right)\left(1 + \sin^2 y(s)\right) \\
& \quad \cdot \left(1 + \cos^2 z(s)\right)\Big)^{-1} ds, \\
y(t) = {} & \frac{1}{3+t^2} y(t) + x(t) + z(t) \\
& + \int_0^T \Big(y(s)\, s \left|\sin x(t)\right| \left|\cos z(t)\right| \\
& \quad + e^s \left(1 + y^2(s)\right)\left(1 + \sin^2 x(s)\right) \\
& \quad \cdot \left(1 + \cos^2 z(s)\right)\Big) \\
& \quad \cdot \Big(e^t \left(1 + y^2(s)\right)\left(1 + \sin^2 x(s)\right) \\
& \quad \cdot \left(1 + \cos^2 z(s)\right)\Big)^{-1} ds, \\
z(t) = {} & \frac{1}{3+t^2} z(t) + y(t) + x(t) \\
& + \int_0^T \Big(z(s)\, s \left|\sin y(t)\right| \left|\cos x(t)\right| \\
& \quad + e^s \left(1 + z^2(s)\right)\left(1 + \sin^2 y(s)\right) \\
& \quad \cdot \left(1 + \cos^2 x(s)\right)\Big) \\
& \quad \cdot \Big(e^t \left(1 + z^2(s)\right)\left(1 + \sin^2 y(s)\right) \\
& \quad \cdot \left(1 + \cos^2 x(s)\right)\Big)^{-1} ds.
\end{aligned} \quad (59)$$

We notice that by taking

$$\begin{aligned}
f(t, x, y, z, p) &= \frac{1}{3+t^2} x + \frac{1}{3} y + \frac{1}{3} z + p, \\
h(t, s, x, y, z) &= \frac{xs\left|\sin y\right| \left|\cos z\right| + e^s \left(1 + x^2\right)\left(1 + \sin^2 y\right)\left(1 + \cos^2 z\right)}{e^t \left(1 + x^2\right)\left(1 + \sin^2 y\right)\left(1 + \cos^2 z\right)}, \\
\eta(t) &= \xi(t) = q(t) = \Psi(t) = \Phi(t) = t, \\
\varphi(t) &= t - 3,
\end{aligned} \quad (60)$$

we get the system integral equations (32).

To solve this system, we need to verify conditions (i)–(v).

Obviously, $\xi, \eta, q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\xi \to \infty$ as $t \to \infty$. Further, the function $\psi : \mathbb{R} \to \mathbb{R}$ is continuous for $\delta = \alpha = 1$, and we have

$$\left|\psi(t_1) - \psi(t_2)\right| \leqslant \delta \left|t_1 - t_2\right|^{\alpha}, \quad (61)$$

for any $t_1, t_2 \in \mathbb{R}_+$. Conditions (i) and (ii) hold.

Now, let

$$\begin{aligned}
&\left|f(t, x, y, z, p) - f(t, u, v, w, \rho)\right| \\
&= \left|\frac{1}{3+t^2} x + \frac{1}{3} y + \frac{1}{3} z + p - \left(\frac{1}{3+t^2} u + \frac{1}{3} v + \frac{1}{3} w + \rho\right)\right| \\
&\leqslant \frac{1}{3}\left[|x-u| + |y-v| + |z-w|\right] + |p - \rho| \\
&= \frac{1}{3}\varphi\left(|x-u| + |y-v| + |z-w|\right) + \Phi\left(|p-\rho|\right).
\end{aligned} \quad (62)$$

Then, (iii) also holds.

Moreover,

$$M_1 = \sup \left|\{f(t, 0, 0, 0, 0) : t \in \mathbb{R}_+\}\right| = 0; \quad (63)$$

then, (iv) is valid.

Let us verify the last condition (v). First,

$$\begin{aligned}
&\left|h(t, s, x, y, z) - h(t, s, u, v, w)\right| \\
&= \Bigg|\frac{xs\left|\sin y\right| \left|\cos z\right| + e^s \left(1 + x^2\right)\left(1 + \sin^2 y\right)\left(1 + \cos^2 z\right)}{e^t \left(1 + x^2\right)\left(1 + \sin^2 y\right)\left(1 + \cos^2 z\right)} \\
&\quad - \frac{us\left|\sin v\right| \left|\cos w\right| + e^s \left(1 + u^2\right)\left(1 + \sin^2 v\right)\left(1 + \cos^2 w\right)}{e^t \left(1 + u^2\right)\left(1 + \sin^2 v\right)\left(1 + \cos^2 w\right)}\Bigg| \\
&= \Bigg|\frac{xs\left|\sin y\right| \left|\cos z\right|}{e^t \left(1 + x^2\right)\left(1 + \sin^2 y\right)\left(1 + \cos^2 z\right)} \\
&\quad - \frac{us\left|\sin v\right| \left|\cos w\right|}{e^t \left(1 + u^2\right)\left(1 + \sin^2 v\right)\left(1 + \cos^2 w\right)}\Bigg| \\
&\leqslant \left|\frac{x}{1+x^2}\frac{s}{e^t} - \frac{u}{1+u^2}\frac{s}{e^t}\right| \leqslant \frac{1}{2}\frac{s}{e^t} + \frac{1}{2}\frac{s}{e^t} \leqslant \frac{s}{e^t}.
\end{aligned} \quad (64)$$

Hence,

$$\lim_{t\to\infty}\int_0^t \left|h\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)\right.$$
$$\left.-h\left(t,s,u\left(\eta\left(s\right)\right),v\left(\eta\left(s\right)\right),w\left(\eta\left(s\right)\right)\right)\right|ds \quad (65)$$
$$\leqslant \lim_{t\to\infty}\int_0^t \frac{s}{e^t}ds=0.$$

Furthermore, for any $x, y, z \in \mathrm{BC}(\mathbb{R}_+)\times \mathrm{BC}(\mathbb{R}_+)\times \mathrm{BC}(\mathbb{R}_+)$,

$$\begin{aligned}\left|\int_0^t h\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right| &\leqslant \int_0^t \left|h\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)\right|ds \\ &\leqslant \int_0^t\left(\frac{s}{2e^t}+\frac{e^s}{e^t}\right)ds=\frac{t^2}{4e^t}+1-\frac{1}{e^t} \\ &= \frac{t^2+4e^t-4}{4e^t}.\end{aligned} \quad (66)$$

Thus,

$$\sup\left\{\left|\int_0^t h\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right|,\right.$$
$$\left. t,s\in\mathbb{R}_+, x,y,z\in \mathrm{BC}(\mathbb{R}_+)\times \mathrm{BC}(\mathbb{R}_+)\times \mathrm{BC}(\mathbb{R}_+)\right\}$$
$$=\sup\left\{\frac{t^2+4e^t-4}{4e^t}, t\in\mathbb{R}_+\right\}=1. \quad (67)$$

It is easy to see that, for any $r > 0$, we have that

$$\frac{1}{3}\varphi(3r)+M_1+\Phi(\delta D)\leqslant r \quad (68)$$

holds and condition (v) is valid.

Consequently, the system has at least one solution in $\mathrm{BC}(\mathbb{R}_+)\times \mathrm{BC}(\mathbb{R}_+)\times \mathrm{BC}(\mathbb{R}_+)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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