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Research Article

A Generalized Solow-Swan Model

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We set up a generalized Solow-Swan model to study the exogenous impact of population, saving rate, technological change, and labor participation rate on economic growth. By introducing generalized exogenous variables into the classical Solow-Swan model, we obtain a nonautomatic differential equation. It is proved that the solution of the differential equation is asymptotically stable if the generalized exogenous variables converge and does not converge when one of the generalized exogenous variables persistently oscillates.

1. Introduction

In the classical Solow-Swan model, saving rate, technological level, capital depreciation, and population growth rate are assumed to be fixed positive constants [1–3]. However, they vary in the process of the economic growth and appear in different forms in the different periods. The growth rate of population presents an inverted-U form in the demographic transition [4–9], the growth of technology appears in the S-shape in some periods [10], and the saving rate varies with the age structure [11, 12].

In this paper, a generalized Solow-Swan model is set up by introducing generalized exogenous variables into the classical Solow-Swan model, which is described by a nonautomatic differential equation. We use this model to inquire the effect of exogenous short-term shocks and long-term fluctuation on the economic growth.

Firstly, we analyze the dynamics of the generalized Solow-Swan model. It is proved that the right maximal interval of the nonautomatic differential equation is $[0, \infty)$ and a comparison theorem is provided. The solution of the differential equation is Lyapunov asymptotically stable when the generalized exogenous variables converge, which implies that the short-term exogenous shocks have no substantial impact on the long-term economic growth.

The case that the differential equation has oscillational solution is also discussed in this paper. It is obtained that the solution of the equation does not converge when one of the generalized exogenous variables persistently oscillates. Therefore, the economy presents fluctuation when one of the generalized exogenous variables is persistent oscillation.

Secondly, we inquire the impact of the main three factors on the economic growth. The first one is the change of the population and we mainly study the effect of the variable population growth rate and labor participation on the economic growth. It is obtained that the economy with low population growth rate has higher per capita capital than that with high population growth rate and the economy tends stable if the population growth rate tends to a stable level. On the other hand, we obtain that the economy with high labor participation rate has higher per capita capital than that with low labor participation rate. This implies that there exists "demographic dividend" in the late stage of the demographic transition in which the population growth rate declines and the labor force participation rate increases and the population aging will slow down the economic growth. If the population growth or labor participation rate is persistent oscillation, the economic fluctuation will appear.

In the classical Solow-Swan model, the saving rate is a fixed parameter and the shock of the saving rate change on the economic growth is studied by Romer [2]. The saving rate is assumed to be a variable of time in this paper and it is proved that the economy with higher saving rate will grow faster than that with lower saving rate and the economy presents fluctuation when the saving rate is persistent oscillation.

The effect of different types of technological change on the economic growth is inquired by using the generalized Solow-Swan model. It is proved that the economy with higher technological level has higher per capita capital than that with lower technological level under the Hicks or Solow neutral technology. For the Harrod neutral technology, we obtain that the per capita capital of the economy with higher final technological growth rate will exceed that with lower final technological growth whatever how high the initial per capita capital or technological level the latter has. This implies that a developing economy can catch up a developed economy if it keeps a higher technological growth rate. When the technological level (under Hicks or Solow neutral technology) or the technological growth rate (under Harrod neutral technology) is persistent oscillation, the economy presents long-term fluctuation.

Finally, a brief summery is given in Section 6.

2. The Generalized Solow-Swan Model

2.1. Set up the Model. The classical Solow-Swan model is given by

$$\dot{k} = sf(k) - (\delta + n + g)k, \tag{1}$$

where s, n, and g are saving rate, population growth rate, and the technological change rate, respectively, and $f(k)$ is the intensive production function satisfying

$$\begin{aligned} &f(0) = 0, \qquad f(\infty) = \infty, \qquad f'(0) = \infty, \\ &f'(\infty) = 0, \qquad f'(k) > 0, \qquad f''(k) < 0, \\ &\qquad\qquad\qquad\qquad\qquad k > 0. \end{aligned} \tag{2}$$

In this paper, we consider the following nonautomatic differential equation:

$$\dot{k} = a(t) f(k) - b(t) k, \tag{3}$$

where $a(t), b(t)$ are continuous functions on $[-c, \infty)$ and satisfy $a_1 \le a(t) \le a_2$, $b_2 \le b(t) \le b_1$, a_i, b_i, $i = 1, 2$, and c are positive constants.

2.2. The Comparison Theorem

Theorem 1. *If $k(t), \tilde{k}(t)$ are the solution of (3) and*

$$\dot{k} = \tilde{a}(t) f(k) - \tilde{b}(t) k \tag{4}$$

with the initial values $k(0), \tilde{k}(0), \tilde{k}(0) > k(0) > 0$, then

(1) *$k(t) < \tilde{k}(t)$, $t > 0$ on a common interval of existence when $a_1 \le a(t) < \tilde{a}(t) \le a_2$, $b_2 \le \tilde{b}(t) \le b(t) \le b_1$ or $a_1 \le a(t) \le \tilde{a}(t) \le a_2$, $b_2 \le \tilde{b}(t) < b(t) \le b_1$;*

(2) *$k(t) \le \tilde{k}(t)$, $t > 0$, on the common interval of existence when $a_1 \le a(t) \le \tilde{a}(t) \le a_2$, $b_2 \le \tilde{b}(t) \le b(t) \le b_1$.*

Proof. (1) It is only to prove that there exits $\delta > 0$ such that

$$\tilde{k}(t) > k(t), \quad t \in (0, \delta). \tag{5}$$

Let $\phi(t) = \tilde{k}(t) - k(t)$; then $\phi(0) \ge 0$ and $\phi'(0) > 0$. Therefore, (5) holds.

(2) It is directly derived by the differential inequality [13]. □

2.3. The Right Maximal Interval of the Solution. For the classical Solow Model, we have the following lemma.

Lemma 2 (see [1]). *The right maximal intervals of the solutions, $k_i(t)$, $i = 1, 2$, of the initial value problems*

$$\dot{k}_i = a_i f(k_i) - b_i k_i, \quad k_i(0) = k_0 > 0, \quad i = 1, 2 \tag{6}$$

are $[0, \infty)$ and

$$\lim_{t \to \infty} k_i(t) = k_i^*, \qquad \min\{k_0, k_i^*\} \le k_i(t) \le \max\{k_0, k_i^*\}, \qquad i = 1, 2, \tag{7}$$

where k_i^*, $i = 1, 2$, are the nonzero solutions of the equations $a_i f(k_i) - b_i k_i = 0$, $i = 1, 2$.

Theorem 3. *The solution of the initial value problem*

$$\begin{gathered} \dot{k} = a(t) f(k) - b(t) k, \\ k(0) = k_0 > 0 \end{gathered} \tag{8}$$

exists on the interval $[0, \infty)$.

Proof. Let $k(t)$ be the solution of the initial value problem (8) with the right maximal interval $[0, b)$; then, by

$$a_1 f(k) - b_1 k \le a(t) f(k) - b(t) k \le a_2 f(k) - b_2 k \tag{9}$$

and Theorem 1, we have $k_1(t) \le k(t) \le k_2(t)$, $t \in [0, b)$, and it stays in the set $\Omega = \{(t, k) \mid 0 \le t < b, k_1(t) \le k \le k_2(t)\}$ for $t > 0$.

If $b < \infty$, let $E_\epsilon = \{(t, k) \mid -\epsilon < t < b + \epsilon, \min\{k_0, k_1^*, k_2^*\} - \epsilon < k < \max\{k_0, k_1^*, k_2^*\} + \epsilon\}$; then, for small enough ϵ, the function $a(t) f(k) - b(t) k$ satisfies Lipschitz condition on E_ϵ. By Extension Theorem [13], the solution of (8) will reach the boundary of E_ϵ, which is a contradiction since Ω is a proper subset of E_ϵ and $k(t) \in \Omega$, $t > 0$. Therefore, $b = \infty$. □

2.4. The Asymptotic Stability

Theorem 4. *If $\lim_{t \to \infty} a(t) = a$, $\lim_{t \to \infty} b(t) = b$, and a, b are two positive constants, then the solution of (8) converges to k^*, where k^* is the nonzero solution of the equation $af(k) - bk = 0$.*

Proof. For any given $\epsilon > 0$, there exist $\overline{a}_i, \overline{b}_i$, $i = 1, 2$, such that $0 < \overline{a}_1 < a < \overline{a}_2$, $0 < \overline{b}_2 < b < \overline{b}_1$, and the nonzero solutions of $a_i f(k) - b_i k = 0$, k_i^*, $i = 1, 2$, satisfy $k_1^* - \epsilon/2 < k^* < k_2^* + \epsilon/2$, where k_i^*, $i = 1, 2$, are the nonzero solutions of $\overline{a}_i f(k) - \overline{b}_i k = 0$, $i = 1, 2$.

Since $\lim_{t\to\infty} a(t) = a$, $\lim_{t\to\infty} b(t) = b$, there exists T_1 such that $\overline{a}_1 < a(t) < \overline{a}_2$, $\overline{b}_2 < b(t) < \overline{b}_1$, $t \geq T_1$. Let $\widetilde{k}_i(t)$, $i = 1, 2$, and $\widetilde{k}(t)$ be the solutions of (6) and (8) with the initial $\widetilde{k}_1(0) = \widetilde{k}_2(0) = \widetilde{k}(0) = k(T_1)$; then, from $\lim_{t\to\infty} \widetilde{k}_1(t) = k_1^*$, $\lim_{t\to\infty} \widetilde{k}_2(t) = k_2^*$, and $\widetilde{k}_2(t) < \widetilde{k}(t) < \widetilde{k}_1(t)$, $t > 0$, there exists T_2 such that

$$\begin{aligned} k^* - \epsilon < k_1^* - \frac{\epsilon}{2} &< \widetilde{k}_1(t) \leq \widetilde{k}(t) \leq \widetilde{k}_2(t) \\ &< k_2^* + \frac{\epsilon}{2} < k^* + \epsilon, \quad t > T_2; \end{aligned} \tag{10}$$

that is, $|\widetilde{k}(t) - k^*| < \epsilon$, $t > T_2$.

Let $u(t) = \begin{cases} k(t), & 0 \leq t \leq T_1, \\ \widetilde{k}(t - T_1), & T_1 \leq t < \infty, \end{cases}$ and then $u(t)$ is a solution of (8) with the initial $u(0) = k_0$ and $u(t) = k(t)$, $t > 0$, by the uniqueness of the solution of (8). Therefore, $|k(t) - k^*| < \epsilon$, $t > T_1 + T_2$, and the theorem holds. □

From the above theorem, we have following lemma.

Lemma 5. *If* $\lim_{t\to\infty} a(t) = a$, $\lim_{t\to\infty} b(t) = b$, $u_i(t)$, $i = 1, 2$, *are the solution of* (8) *with the initial values* $u_{10} > u_{20} > 0$, *then* $\lim_{t\to\infty} |u_1(t) - u_2(t)| = 0$.

Theorem 6. *If* $\lim_{t\to\infty} a(t) = a$, $\lim_{t\to\infty} b(t) = b$, *then the solution of* (8) *is Lyapunov asymptotically stable.*

Proof. For any given $\epsilon > 0$, choose $u_{10} = 3k_0/2$ and $u_{20} = k_0/2$; then, from Lemma 5, there exists $T > 0$ such that the solutions, $u_i(t)$, $i = 1, 2$, with the initial values u_{i0}, $i = 1, 2$, of (8) satisfy $|u_1(t) - u_2(t)| < \epsilon$, $t \geq T$.

Denote that

$$G(t, k) = a(t) f(k) - b(t) k; \tag{11}$$

then $\partial G/\partial k$ is continuous on the compact set

$$\mathscr{M} = \{(t, k) \mid 0 \leq t \leq T, u_1(t) \leq k \leq u_2(t)\}. \tag{12}$$

Let $M = \max_{(t,k)\in\mathscr{M}} (\partial G(k, t)/\partial k)$, $\delta_1 = \min\{\epsilon e^{-MT}, k_0/2\}$, and $\widetilde{k}(t)$ be the solution of (8) with the initial value $\widetilde{k}_0 \in (k_0 - \delta_1, k_0 + \delta_1)$; then, by the Gronwall inequality [13], we have

$$\left|\widetilde{k}(t) - k(t)\right| < \left|\widetilde{k}_0 - k_0\right| e^{MT} < \epsilon, \quad t \in [0, T]. \tag{13}$$

Since $u_2(t) < k(t) < u_1(t)$, $u_2(t) < \widetilde{k}(t) < u_1(t)$, $t > 0$, we have $|\widetilde{k}(t) - k(t)| < |u_1(t) - u_2(t)| < \epsilon$ for $t \geq T$. Therefore, the solution of (8) is Lyapunov asymptotically stable [14] by Lemma 5. This completes the proof of the theorem. □

2.5. Oscillation

Definition 7. The generalized exogenous variables $a(t)$ (or $b(t)$) are called persistent oscillation if, for any given $T > 0$, there exist c_T^i, d_T^i, $i = 1, 2$, $c_T^1 < d_T^1 < c_T^2 < d_T^2$, and $\epsilon > 0$, $\delta_1 > 0$ such that $d_T^i - c_T^i > \delta_1$, $i = 1, 2$, and one of the following inequalities holds:

$$\begin{aligned} \min_{c_T^1 \leq t \leq d_T^1} a(t) - \max_{c_T^2 \leq t \leq d_T^2} a(t) > \epsilon, \\ \min_{c_T^2 \leq t \leq d_T^2} a(t) - \max_{c_T^1 \leq t \leq d_T^1} a(t) > \epsilon. \end{aligned} \tag{14}$$

Lemma 8. *If the generalized variables* $a(t)$ *are persistent oscillation, then, for any given constants* $c, T, T > 0$, *there exist* $\delta_1 > 0$ *and* $t_T > T$ *such that one of the following inequalities holds:*

$$a(t) - c > \frac{\epsilon}{2}, \qquad c - a(t) > \frac{\epsilon}{2} \tag{15}$$

on the interval $[t_T - \delta_1/2, t_T + \delta_1/2]$.

Proof. Assume that the first inequality of (14) holds; then one of the following inequalities holds:

$$\min_{c_T^1 \leq t \leq d_T^1} a(t) - c > \frac{\epsilon}{2}, \qquad -\max_{c_T^2 \leq t \leq d_T^2} a(t) + c > \frac{\epsilon}{2}. \tag{16}$$

Without loss of the generality, we assume that the first inequality above holds and we have

$$a(t) - c > \frac{\epsilon}{2}, \qquad t \in \left[c_T^1, d_T^1\right]. \tag{17}$$

Taking $t_T = (d_T^1 + c_T^1)/2$, then (15) holds for $[t_T - \delta_1/2, t_T + \delta_1/2] \subset [c_T^1, d_T^1]$.

Similarly, we can prove the case that the second inequality holds. This completes the proof of the lemma. □

Theorem 9. *If one of the generalized variables is persistent oscillation and the other converges, then the solution of the differential equation* (8) *does not converge.*

Proof. Assume that $a(t)$ is persistent oscillation and $\lim_{t\to\infty} b(t) = b$.

If the solution $k(t)$ of the differential equation (8) converges, then there exists a $k^* > 0$ such that $\lim_{t\to\infty} k(t) = k^*$. Therefore, for given $\epsilon_1 > 0$, there exists $T_1 > 0$ such that $|f(k(t)) - f(k^*)| < \epsilon_1$, $|c - (k(t)/f(k(t)))b(t)| < \epsilon_1$, $t > T_1$, where $c = (k^*/f(k^*))b$.

Let $c_1 = \min\{f(k^*) - \epsilon_1, f(k^*) + \epsilon_1\}$; then, from (8) and Lemma 8, there exists $\epsilon > 0$, such that

$$\begin{aligned} &\left|k\left(t_T + \frac{\delta_1}{2}\right) - k\left(t_T - \frac{\delta_1}{2}\right)\right| \\ &= \left|\int_{t_T-\delta_1/2}^{t_T+\delta_1/2} f(k(s))[a(s) - c]\, ds \right. \\ &\quad \left. + \int_{t_T-\delta_1/2}^{t_T+\delta_1/2} f(k(s))\left[c - \frac{k(s)}{f(k(s))} b(s)\right] ds\right| \\ &\geq \frac{c_1 \epsilon \delta_1}{2} - \left(f(k^*) + \epsilon_1\right)\epsilon_1 \delta_1 \\ &\geq \frac{c_1 \epsilon \delta_1}{4} \end{aligned} \tag{18}$$

for $T > T_1$, provided $\epsilon_1 < \min\{c_1\epsilon/4(f(k^*) + 1), 1\}$.

Since $\lim_{t_T\to\infty}|k(t_T+\delta_1/2)-k(t_T-\delta_1/2)| = 0$, the inequality above does not hold for big enough T_1. This is a contradiction and the theorem holds. □

Lemma 10. *If generalized variable $a(t)$ (or $(b(t))$ which does not equal constant is a periodical function with the period $\omega > 0$, then it is persistent oscillation.*

Proof. Since $\omega > 0$, there exists $t_1, t_2 \in (0,\omega), t_1 < t_2$ such that $a(t_1) \neq a(t_2)$. Let $\epsilon = |a(t_1) - a_2(t_2)|/2$; then there exists $\delta > 0$ such that $0 < t_1-\delta/2 < t_1+\delta/2 < t_2-\delta/2 < t_2+\delta/2 < \omega$ and one of the following inequalities holds:

$$\begin{gathered}\min_{c_1\le t\le d_1} a(t) - \max_{c_2\le t\le d_2} a(t) > \epsilon,\\ \min_{c_2\le t\le d_2} a(t) - \max_{c_1\le t\le d_1} a(t) > \epsilon,\end{gathered} \tag{19}$$

where $c_i = t_i - \delta/2$, $d_i = t_i + \delta/2$, $i = 1, 2$.

For any given $T > 0$, there exists positive integer M_T such that $M_T\omega > T$. Let $c_T^i = M_T\omega + t_i - \delta/2$, $d_T^i = M_T\omega + t_i + \delta/2$, $i = 1, 2$; then, from (19), the conditions in Definition 7 hold for $a(t)$ is the period function with period ω. This completes the proof of the lemma. □

From Lemma 10, we have the following theorem.

Theorem 11. *If one of the generalized variables is periodical oscillation and the other converges, then the solution of the differential equation* (8) *does not converge.*

Example 12. If the intensive productive function is the Cobb-Douglas productive function, then the differential equation (3) becomes

$$\dot{k} = a(t)k^{\alpha} - b(t)k. \tag{20}$$

Let $z(t) = k^{1-\alpha}$; then the equation above is reduced by

$$\dot{z} = -b_1(t)z + a_1(t) \tag{21}$$

which is a linear differential equation, where $a_1(t) = (1-\alpha)a(t)$ and $b_1(t) = (1-\alpha)b(t)$, and its solution is given by

$$z(t) = e^{-\int_0^t b_1(s)ds}\left[z_0 + \int_0^t a_1(s)e^{\int_0^s b_1(\tau)d\tau}ds\right]. \tag{22}$$

Therefore, the solution of (20) is

$$k(t) = \left[e^{-\int_0^t b_1(s)ds}\left(k_0^{1-\alpha} + \int_0^t a_1(s)e^{\int_0^s b_1(\tau)d\tau}ds\right)\right]^{1/(1-\alpha)}. \tag{23}$$

When $a_1(t) = a_1 + a_2\sin x$, $b_1(t) = b$, the solution of the differential equation (20) is given by

$$k(t) = \left[\left(k_0^{1-\alpha} - \frac{a_1(1+b^2) - a_2 b}{b+b^2}\right)e^{-bt} + \frac{a_1 + a_1b^2 - a_2b\cos t + a_2b^2\sin t}{b+b^3}\right]^{1/(1-\alpha)}, \tag{24}$$

which approaches periodical oscillation.

3. Variable Population Growth

Suppose that $N(t)$ and $L(t)$ are the numbers of population and labor of an economy at time t and the labor force participation rate is $\lambda(t)$, $0 < \lambda(t) < 1$; then, $L(t) = \lambda(t)N(t)$. We further assume that the population growth rate, $n(t)$, is bounded; that is, $-n_1 \le n(t) \le n_2$, where $n_i \ge 0$, $i = 1, 2$.

Let $k = K/N$; then

$$\dot{k} = \left(\frac{\dot{K}}{K} - \frac{\dot{N}}{N}\right)k = \left(\frac{\dot{K}}{K} - n(t)\right)k = \frac{\dot{K}}{N} - n(t)k. \tag{25}$$

From $\dot{K} = sF(K,L) - \delta K$, we obtain the Solow-Swan model with the variable population growth rate and labor participation rate below:

$$\dot{k} = sF(k,\lambda(t)) - [\delta + n(t)]k. \tag{26}$$

3.1. Changeable Population Growth Rate. Assume that the labor participation is positive constant, that is, $\lambda(t) = \lambda_0$, and let $f(k) = F(k,\lambda_0)$; then, from (26), we obtain the Solow-Swan model with variable population growth rate

$$\begin{gathered}\dot{k} = sf(k) - [\delta + n(t)]k,\\ k(0) = k(0) > 0.\end{gathered} \tag{27}$$

Let $a(t) = s$, $b(t) = \delta + n(t)$; then, from Theorems 4 and 6, we have the following theorem.

Theorem 13. *If $n_1 + \delta > 0$, then, the extension interval of the solution of the differential equation* (27) *is $[0,+\infty)$; furthermore, if $\lim_{t\to+\infty} n(t) = n_0$, then the solution of the differential equation* (27) *is Lyapunov asymptotically stable and converges to the nonzero equilibrium of the differential equation*

$$\dot{k} = sf(k) - [\delta + n_0]k. \tag{28}$$

Remark 14. Theorem 19 provided by Guerrini [15] is a special case of the above theorem (there, $n(t)$ is required to be a monotone decreasing function).

One of the most distinct characteristics in population change is the demographic transition, which has occurred in almost all developed countries and most developing countries [11]. The population growth rate in demographic transition appears in an inverted-U form and can be described by a function of time t, satisfying

$$\begin{gathered}n(0) = n_0 > 0, \qquad n'(t) > 0,\\ 0 < t < t_1, \quad n'(t) < 0, \quad t > t_1,\\ \lim_{t\to\infty} n(t) = 0.\end{gathered} \tag{29}$$

Corollary 15. *The growth of the economy which has undergone the demographic transition is stable and its per capita capital converges to the steady state of the economy with the zero population growth rate.*

From Theorems 1 and 9, we have the following.

Corollary 16. *If $k_i(t)$, $i = 1, 2$, are the solution of following equations:*

$$\dot{k} = sf(k) - [\delta + n_i(t)]k, \quad i = 1, 2, \tag{30}$$

with same initial value $k_1(0) = k_2(0) = k_0 > 0$ and $n_1(t) > n_2(t) > n_1$, then $k_2(t) > k_1(t)$, $t > 0$.

Corollary 16 implies that an economy with higher population growth rate has lower per capita capital and consumption than that with lower population growth rate.

Theorem 17. *If $n_1 + \delta > 0$ and $n(t)$ is persistent oscillation, then the solution of (27) does not converge.*

Corollary 18. *If the population growth is persistent oscillation of an economy, then its economic growth is not stable.*

3.2. Changeable Labor Force Participation. There are two major reasons that result in the change of the labor force participation: one is the changes in the age structure and the other is the population aging [16]. Here, we suppose that the aggregate population is stable; that is, $N(t)$ is a constant and $F(K, L)$ is the Cobb-Douglas production function $AK^{\alpha}L^{1-\alpha}$; then $n(t) = 0$ and (26) turns into

$$\dot{k} = \lambda^{1-\alpha}(t) f(k) - \delta k, \tag{31}$$

where $f(k) = sAk^{\alpha}$.

From Theorems 1, 3, 4, 6, and 9, we have the following.

Theorem 19. *The extension interval of the solution of the differential equation (31) is $[0, +\infty)$; furthermore, if $\lim_{t\to+\infty}\lambda(t) = \lambda_0 > 0$, then the solution of the differential equation (31) is Lyapunov asymptotically stable and converges to the nonzero equilibrium of the differential equation*

$$\dot{k} = \lambda_0^{1-\alpha} f(k) - \delta k. \tag{32}$$

Theorem 20. *If $k_i(t)$, $i = 1, 2$, are the solutions of the differential equations*

$$\dot{k} = \lambda_i^{1-\alpha}(t) f(k) - \delta k, \quad i = 1, 2 \tag{33}$$

with the same initial k_0 and $\lambda_2(t) > \lambda_1(t)$, $t \geq 0$, then $k_2(t) > k_1(t)$, $t > 0$.

Corollary 21. *The economy with higher labor force participation has higher per capita capital than that with lower labor force participation under the same other conditions; furthermore, the per capita capital of an economy with higher labor force participation tends stably to a higher steady state when the labor force participation tends to a stable level.*

One of the distinct characteristics in population aging is the decline of the labor force participation [16] and we have the following.

Corollary 22. *The economic growth of an economy is slowed down by population aging.*

Theorem 23. *If $\lambda(t)$ is persistent oscillation, then the solution of (31) does not converge.*

Corollary 24. *If the labor participation of an economy is persistent oscillation, then its economic growth is not stable.*

Remark 25. For the variable population growth rate, (31) becomes

$$\dot{k} = \lambda^{1-\alpha}(t) f(k) - [\delta + n(t)]k. \tag{34}$$

From Theorems 1, 3, 4, 6, and 9, we derive that the economic growth speeds up when the labor force participation rate increases and the population growth rate declines. Therefore, there exists "demographic dividend" in the late stage of demographic transition, in which the population growth rate declines and labor force participation rate increases.

4. Variable Saving Rate

Assume that the saving rate varies with time, the population growth rate is a constant n_0, and $0 < s_1 \leq s(t) \leq s_2 < 1$; then the classical Solow-Swan model changes into

$$\dot{k} = s(t) f(k) - (\delta + n_0)k. \tag{35}$$

From Theorems 1, 3, 4, 6, and 9, we have the following.

Theorem 26. *The extension interval of the solution of the differential equation (35) is $[0, +\infty)$; furthermore, if $\lim_{t\to+\infty}s(t) = s_0 > 0$, then the solution of the differential equation (35) is Lyapunov asymptotically stable and converges to the nonzero equilibrium of the differential equation*

$$\dot{k} = s_0 f(k) - (\delta + n_0)k. \tag{36}$$

Theorem 27. *If $k_i(t)$, $i = 1, 2$, are the solutions of the differential equations*

$$\dot{k} = s_i(t) f(k) - \delta k, \quad i = 1, 2, \tag{37}$$

with the same initial k_0 and $s_2(t) > s_1(t)$, $t \geq 0$, then $k_2(t) > k_1(t)$, $t > 0$.

Corollary 28. *The economy with higher saving rate has higher per capita capital than that with lower saving rate under the same other conditions; furthermore, the per capita capital of an economy tends stably to a higher steady state when the saving rate tends to a higher stable level.*

Theorem 29. *If $s(t)$ is persistent oscillation, then the solution of (35) does not converge.*

Corollary 30. *If the saving rate of an economy is persistent oscillation, then its economic growth is not stable.*

5. Exogenous Technological Change

5.1. The Hicks Neutral Technology. The Hicks neutral technological production function [1] is given by

$$Y = F(K, L, t) = T(t) F(K, L), \tag{38}$$

where $T(t)$ is an index of the state of the technology, and $\dot{T} \geq 0$.

Under this production function, the Solow-Swan model turns into

$$\dot{k} = sT(t) f(k) - (\delta + n) k, \tag{39}$$

which is a special case of the generalized Solow-Swan model (3). Therefore, from Theorems 1, 3, 4, 6, and 9, we have the following theorems and corollaries.

Theorem 31. *If $T(t)$ is bounded and $T(0) > 0$, then the extension interval of the solution of the differential equation (39) is $[0, +\infty)$; furthermore, if $\lim_{t \to +\infty} T(t) = \overline{T} > 0$, then the solution of the differential equation (39) is Lyapunov asymptotically stable and converges to the nonzero equilibrium of the differential equation*

$$\dot{k} = s\overline{T} f(k) - (\delta + n) k, \tag{40}$$

where $\overline{T} = \lim_{t \to \infty} T(t)$.

Theorem 32. *If $T_i(t)$, $i = 1, 2$, are bounded, $T_2(t) > T_1(t)$, $t \geq 0$, $T_1(0) > 0$, and $k_i(t)$, $i = 1, 2$, are the solutions of the differential equations*

$$\dot{k} = sT_i(t) f(k) - \delta k, \quad i = 1, 2, \tag{41}$$

with the same initial k_0, then $k_2(t) > k_1(t)$, $t > 0$.

Corollary 33. *The economy with higher technological level has higher per capita capital than that with lower technological level under the same other conditions; furthermore, the per capita capital of an economy tends stably to a higher steady state when its technological level tends to a higher stable level.*

Theorem 34. *If $T(t)$ is persistent oscillation, then the solution of (39) does not converge.*

Corollary 35. *If the technological level of an economy is persistent oscillation, then its economic growth is not stable.*

Remark 36. In the case of Solow neutral technology (capital augmenting technology) [1] and Cobb-Douglas production function, that is, $F(KB(t), L) = (KB(t))^{\alpha} L^{1-\alpha}$, the Solow-Swan model turns into

$$\dot{k} = sB^{\alpha}(t) k^{\alpha} - (\delta + n) k = a(t) f(k) - (\delta + n) k, \tag{42}$$

where $a(t) = sB^{\alpha}(t)$, $f(k) = k^{\alpha}$, and we have similar theorems and corollaries above.

5.2. The Harrod Neutral Technology. For the Harrod Neutral technology [1], the production function is given by $Y = F(K, LA(t))$, which is also called the labor-augmenting technological progress, and the Solow-Swan model becomes

$$\dot{k} = sf(k) - [\delta + g(t) + n] k, \tag{43}$$

where $k = K/LA(t)$ is the capital stock per unit of effective labor and $g(t) = \dot{A}/A$ is the technological growth rate.

From Theorems 1, 3, 4, 6, and 9, we have the following.

Theorem 37. *The extension interval of the solution of the differential equation (43) is $[0, +\infty)$; furthermore, if $\lim_{t \to +\infty} g(t) = \overline{g} > 0$, then the solution of the differential equation (43) is Lyapunov asymptotic stable and converges to equilibrium of the differential equation*

$$\dot{k} = sf(k) - (\delta + \overline{g} + n) k. \tag{44}$$

Remark 38. The special case of this theorem has been proved by Zhou et al. [17]; there the function $g(t)$ is given by $a + g - u(t)$, and $u(t)$ is the solution of the logistic equation $\dot{u} = u(a - bu)$.

Lemma 39. *If $g_i(0) \geq 0$, $\lim_{t \to \infty} g_i(t) = \overline{g}_i$, $i = 1, 2$, and $\overline{g}_2 > \overline{g}_1$, then, for any given positive constant B, there exists a time T such that $\int_0^t [g_2(\tau) - g_1(\tau)] d\tau > B$, $t > T$.*

Proof. Let $h(t) = \int_0^t [g_2(s) - g_1(s)] ds$; then, from $\lim_{t \to \infty} h(t) = \infty$, the lemma holds. □

Theorem 40. *Let $k_i(t)$, $i = 1, 2$, be the solutions of the differential equations*

$$\dot{k} = sf(k) - [\delta + g_i(t) + n] k, \quad i = 1, 2, \tag{45}$$

with the initial values k_{i0}, $i = 1, 2$.

(1) *If $k_{10} = k_{20}$ and $g_2(t) > g_1(t)$, $t \geq 0$, then $k_2(t) < k_1(t)$, $t > 0$.*

(2) *If $\overline{g}_2 = \lim_{t \to \infty} g_2(t) > \overline{g}_1 = \lim_{t \to \infty} g_1(t)$, then there exists a time T, such that $\tilde{k}_2(t) > \tilde{k}_1(t)$, where $\tilde{k}_i(t)$, $i = 1, 2$, are the per capita capital $K_i(t)/L_i(t)$, $i = 1, 2$.*

Proof. (1) It is directly deduced by Theorem 1.

(2) Since $\tilde{k}_i(t) = A_i(t) k_i(t) = A_{i0} e^{\int_0^t g_i(\tau) d\tau} k_i(t)$, $i = 1, 2$,

$$\frac{\tilde{k}_2(t)}{\tilde{k}_1(t)} = \frac{A_{20} k_2(t)}{A_{10} k_1(t)} e^{\int_0^t [g_2(\tau) - g_1(\tau)] d\tau} = \frac{A_{20} k_2(t)}{A_{10} k_1(t)} e^{h(t)}, \tag{46}$$

where A_{i0}, $i = 1, 2$, are initial technological levels.

Let k_i^*, $i = 1, 2$, be the positive solutions of the following equation:

$$sf(k) - [\delta + \overline{g}_i + n] k = 0, \quad i = 1, 2, \tag{47}$$

respectively; then, from $\lim_{t \to \infty} k_i(t) = k_i^*$, $i = 1, 2$, there exists T_1 such that $k_2(t) > k_2^*/2$, $k_1(t) < 3k_1^*/2$, $t > T_1$. Hence, $\tilde{k}_2(t)/\tilde{k}_1(t) > (A_{20} k_2^*/3A_{10} k_1^*) e^{h(t)}$.

By Lemma 39, there is a time T such that $h(t) > \ln(3A_{10} k_1^*/A_{20} k_2^*)$, $t > T$. Therefore, $\tilde{k}_2(t)/\tilde{k}_1(t) > 1$, $t > \max\{T, T_1\}$. This completes the proof of the theorem. □

Remark 41. This theorem implies that a developing economy can catch up and surpass a developed economy provided it maintains a higher technological growth rate than the latter one.

Corollary 42. *(1) An economy with lower technological growth rate has higher per capita capital of effective labor than* *that with higher technological growth rate under the same other conditions.*

(2) The per capita capital of an economy with higher final technological growth rate will excess that with lower final technological growth rate.

Theorem 43. *If $g(t)$ is persistent oscillation, then the solution of* (43) *does not converge.*

Corollary 44. *If the technological growth rate of an economy with the labor-augmenting technological progress is persistent oscillation, then its economic growth is not stable.*

6. Summary

From the analysis in Section 2, we see that the stability of the nonautomatic differential equation depends on the generalized exogenous variables. If they converge, then the solution of the equation is Lyapunov asymptotically stable and does not converge if one of the generalized exogenous variables is persistent oscillation. Therefore, the economy described by the model stably grows when the generalized exogenous variables tend to a stable level and presents fluctuation when one of the generalized exogenous variables is persistent oscillation.

In Section 3, we analyze the effect of the demographic factors on the economic growth. One demographic factor is the change of population growth rate in the period of demographic transition. There does not exist substantial effect on the economic growth in long term for the population growth rate tends to zero after the demographic transition. However, the economic growth speeds up in the later period of the demographic transition in which the labor force participation rate rises and the population growth rate decreases. This implies that the "demographic dividend" appears in this period and theoretically confirms the evidence provided by Bloom et al. [11] through empirical analysis.

The other demographic factor affecting economic growth that we inquired is population aging. The distinct characteristic of population aging is that the labor force participation rate declines when the total population is stable. From the analysis in Section 3.2, we see that population aging slows down the economic growth.

The third demographic factor is unstable population growth rate and unstable labor force participation rate. If one of them is persistent oscillation, then the economy presents long-term fluctuation.

The effect of the saving rate change on economic growth is discussed in Section 4. The per capita capital of the economy with higher final saving rate will excess that with lower final saving rate. Under the same initial per capita capital, the per capita capital of the economy with higher saving rate is bigger than that with lower saving rate in whole period of economic growth. If the saving rate is persistent oscillation, then the economy presents long-term fluctuation.

Three types of variable neutral technology with time are put into the model to analyze their effects on the economic growth. It is obtained that the economy with higher technological level (Hicks neutral technology or Solow neutral technology) grows faster than that with lower technological level. The per capita capital of an economy tends to a stable level if the technological level tends to a stable level and the economy presents long-term fluctuation if the technologic level is persistent oscillation.

For the Harrod neutral technology, we show that the per capita of an economy with higher technological growth rate will excess that with lower technological growth whatever how high initial per capita capital the latter had and how high technological growth rate in the early stage the latter had. This result implies that a developing economy can catch up a developed economy provided it maintains a higher technological growth rate than the latter in long term.

If the technological growth rate is persistent oscillation, the economic growth is not stable and the economy presents long-term fluctuation.

Appendix

Some Cited Definitions, Theorems, and the Details of the Proof

Theorem A.1 (differential inequality [13]). *Let $U(t,u)$, $u(t_0) = u_0$ on an open (t,u)-set E and $u = u^0(t)$ the maximal solution of $\dot{u} = U(t,u), u(t_0) = u_0$. Let $v(t)$ be a continuous function on $[t_0, t_0 + a]$ satisfying the conditions $v(t_0) \le u_0, (t, v(t)) \in E$, and $v(t)$ has a right derivative $D_R v(t)$ on $t_0 \le t < t_0 + a$. Then, on a common interval of existence of $u_0(t)$ and $v(t)$, $v(t) \le u^0(t)$.*

Theorem A.2 (Extension Theorem [13]). *Let $f(t,y)$ be continuous on an open (t,y)-set E and let $y(t)$ be a solution of $\dot{y} = f(t,y)$ on some interval. Then $y(t)$ can be extended (as a solution) over a maximal interval of existence (ω_-, ω_+). Also, if (ω_-, ω_+) is a maximal interval of existence, then $y(t)$ tends to the boundary ∂E of E as $t \to \omega_-$ and $t \to \omega_+$.*

Theorem A.3 (Gronwall's inequality [13]). *Let $u(t), v(t)$ be nonnegative, continuous functions on $[a,b]$; $C \ge 0$ a constant; and*

$$v(t) \le C + \int_a^t v(s)\, u(s)\, \mathrm{d}s, \quad a \le t \le b. \tag{A.1}$$

Then

$$v(t) \le C \exp\left(\int_a^t u(s)\, \mathrm{d}s\right), \quad a \le t \le b; \tag{A.2}$$

in particular, if $C = 0$, then $v(t) \equiv 0$.

Definition A.4 (Lyapunov Stability [14]). Let $x^*(t)$ be a given real or complex solution vector of the n-dimensional system $\dot{x} = X(x,t)$. Then we have the following.

(i) $x^*(t)$ is Lyapunov stable for $t \ge t_0$ if and only if to each value of $\epsilon > 0$, however small, there corresponds a value of $\delta > 0$ (where δ may depend only on ϵ and t_0) such that

$$\left\| x(t_0) - x^*(t_0) \right\| < \delta \Longrightarrow \left\| x(t) - x^*(t) \right\| < \epsilon \tag{A.3}$$

for all $t > t_0$, where $x(t)$ represents any other neighbouring solution.

(ii) If the given system is autonomous, the reference to t_0 in (i) may be disregarded; the solution $x^*(t)$ is either Lyapunov stable or not, for all t_0.

(iii) Otherwise the solution $x^*(t)$ is unstable in the sense of Lyapunov.

Definition A.5 (uniform stability [14]). If a solution is stable for $t \geq t_0$, and the δ of Definition A.4 is independent of t_0, the solution is uniformly stable on $t \geq t_0$.

Definition A.6 (asymptotic stability [14]). Let x^* be a stable (or uniformly stable) solution for $t \geq t_0$. If additionally there exists $\eta(t_0) > 0$ such that

$$\left\| x(t_0) - x^*(t_0) \right\| < \eta \Longrightarrow \lim_{t \to \infty} \left\| x(t) - x^*(t) \right\| = 0, \quad \text{(A.4)}$$

then the solution is said to be asymptotically stable (or uniformly and asymptotically stable).

Derivation of Formula of (13). Since the solutions $k(t), \widetilde{k}(t)$ satisfy

$$\begin{aligned} k(t) &= k_0 + \int_0^t G(\tau, k(\tau))\, d\tau, \quad t > 0, \\ \widetilde{k}(t) &= \widetilde{k}_0 + \int_0^t G\left(\tau, \widetilde{k}(\tau)\right) d\tau, \quad t > 0, \end{aligned} \quad \text{(A.5)}$$

we have

$$\begin{aligned} &\left| \widetilde{k}(t) - k(t) \right| \\ &\leq \left| \widetilde{k}_0 - k_0 \right| + \int_0^t \left| G\left(\tau, \widetilde{k}(\tau)\right) - G(\tau, k(\tau)) \right| d\tau \\ &\leq \left| \widetilde{k}_0 - k_0 \right| + M \int_0^t \left| \widetilde{k}(\tau) - k(\tau) \right| d\tau. \end{aligned} \quad \text{(A.6)}$$

By Gronwall's inequality, we obtain

$$\left| \widetilde{k}(t) - k(t) \right| < \left| \widetilde{k}_0 - k_0 \right| e^{MT} < \epsilon, \quad t \in [0, T]. \quad \text{(A.7)}$$

Conflict of Interests

The authors declare there is no conflict of interests regarding the publication of this paper.

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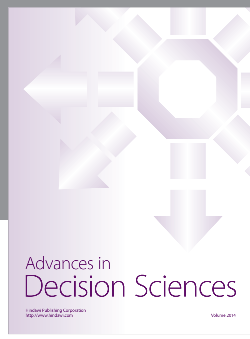

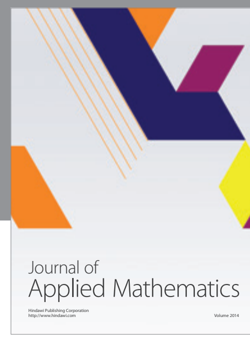

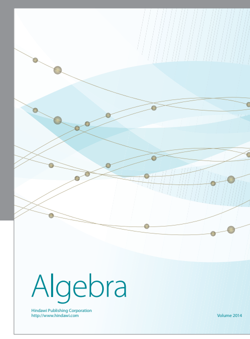

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