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On the Structure of a Distinguished-Limit Quasi-Isothermal Deflagration for the Generalized Reaction-Rate Model

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The structure of the quasi-isothermal deflagration is examined by means of an asymptotic analysis of the physical-plane boundary-value problem, with Lewis–Semenov number unity, in the limit of the activation-temperature ratio, $\beta = T_a/T_b$, greater than order unity, for the generalized reaction-rate-model case of: (1) the heat-addition–temperature ratio, $\alpha = (T_b - T_u)/T_u$, of order $\beta^{-1/2}$, less than order unity [where T_a, T_b, and T_u are the activation, adiabatic-flame (and/or burned-gas), and unburned-gas temperatures, respectively]; and (2) the exponent, a, which characterizes the pre-exponential thermal dependence of the reaction-rate term, unity. The examination indicates that, as in the order-unity heat-addition case, this deflagration has a four-region structure: the upstream diffusion-convection and downstream diffusion-reaction regions, and the far-upstream (or cold-boundary) and the far-downstream (or hot-boundary) regions.

Keywords: Activation-energy asymptotics; Cold-boundary difficulty; Laminar deflagration; Quasi-isothermal

Classification Categories: 80A32, 76D30, 76E30

1 STATEMENT OF THE PROBLEM

For the direct, first-order, one-step, irreversible, quasi-isothermal, unimolecular chemical reaction $R \rightarrow P$, the steady, one-dimensional,

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low-Mach-number, isobaric, laminar deflagration, for Lewis–Semenov number unity, is modeled by the following (nondimensional) boundary-value problem (cf. Bush and Krishnamurthy [1]) in the domain $(-\infty < \xi < \infty)$:

$$\frac{d\tau}{d\xi} = \tau - \varepsilon, \tag{1.1a}$$

$$\frac{d\varepsilon}{d\xi} = \Lambda(1-\tau)\tau^a \exp\{-\beta(1-\tau)/(\alpha^{-1}+\tau)\}; \tag{1.1b}$$

$$\tau, \varepsilon \to 0 \quad \text{as } \xi \to -\infty, \tag{1.2a}$$

$$\tau, \varepsilon \to 1 \quad \text{as } \xi \to \infty. \tag{1.2b}$$

In the above: (1) ξ is the (modified) spatial coordinate; (2) τ is the normalized temperature, and ε is the normalized stoichiometrically adjusted mass-flux fraction of the product; (3) β is the activation-temperature ratio, T_a/T_b, greater than order unity, and α is the heat-addition–temperature ratio, $(T_b-T_u)/T_u$, here, of less than order unity, where T_a, T_b, and T_u are the activation, adiabatic-flame (and/or burned-gas), and unburned-gas temperatures, respectively; (4) a is the exponent that characterizes the pre-exponential thermal dependence of the reaction-rate term, of order unity; and (5) Λ is the normalized Damköhler number, greater than order unity. It is worth noting that, in the present study, unlike in the earlier analysis (see [1]) of strong exothermic chemical reaction, the one-step reaction cannot be construed as *all* of R being converted to P. Rather, as appropriate in a cool flame, only a small fraction of the original fuel population is converted to product. Indeed, another plausible physical context for the present case arises for the deflagration of a very off-stoichiometric mixture of reactants, say, near the fuel-lean limit, for which the oxidant population is left essentially intact, the fuel is consumed completely, and the temperature rises just a little.

The previous paper had considered the generalized (reaction-rate) model boundary-value problem of (1.1) and (1.2), for $\beta \to \infty$, with $\alpha \sim O(1)$, and $a = 1$, by means of an asymptotic analysis in physical space. That analysis showed that a four-region structure consisting of two far (upstream and downstream) regions, in addition to the two *classical* near (upstream and downstream) regions, must be introduced for the deflagration in order to obtain uniformly valid solutions from the cold boundary to the hot boundary. In the present paper, by means of a parallel analysis, for $\beta \to \infty$, with $\alpha \sim O(\beta^{-1/2}) \to 0$, and $a = 1$, it is shown that, again, a four-region structure must be introduced for this quasi-isothermal deflagration in order to obtain uniformly valid solutions. Since, in general, β and α are independent parameters, this paper presents a (mathematical) distinguished-limit analysis for the understanding of this (physically interesting) quasi-isothermal case. For consideration of the (complementary) problem, defined by (1.1) and (1.2), for $\beta \leq O(1)$, $\alpha \ll O(1)$, and $a = 1$, say, see Zeldovich *et al.* [2].

For the previous case (see [1]) of $\beta \to \infty$, with $K = (1 + \alpha^{-1}) \sim O(1)$, the eigenvalue has been determined to have the representation

$$\Lambda \cong \frac{\beta^2}{K^2}\left\{\frac{1}{2} - \beta^{-1}[(I-1)K - 3] + \cdots\right\} \to \infty \tag{1.3}$$

with $(I-1) \doteq 0.344$. Now, for $\beta \to \infty$, with $\alpha \sim O(\beta^{-1/2}) \to 0$, and, in turn, with $K \sim O(\beta^{1/2}) \to \infty$, such that $\beta = k^2\hat{\beta} \to \infty$ and $K = k\beta^{1/2} = k^2\hat{\beta}^{1/2} \to \infty$, for $k \sim O(1)$, (1.3) takes the representation

$$\Lambda \cong \hat{\beta}\left\{\frac{1}{2} - \hat{\beta}^{-1/2}(I-1) + \cdots\right\} \to \infty. \tag{1.4}$$

From (1.3) and (1.4), for $\beta \to \infty$, it is clear that it is possible to go from a treatment with $K \sim O(1)$ to one with $K \sim O(\beta^{1/2})$; also, from (1.3) and (1.4), it is clear that it is not possible to go from one with $K \sim O(1)$ to one with $K \sim O(\beta)$.

2 ASYMPTOTIC ANALYSIS

The model deflagration boundary-value problem under consideration (i.e., $\hat{\beta} \to \infty, \alpha \sim O(\hat{\beta}^{-1/2}) \to 0$, and $a = 1$) requires the analysis of four principal regions: (1) a relatively thin downstream region, near the hot boundary, where $(1 - \tau) \sim O(\hat{\beta}^{-1/2})$; (2) a relatively thicker upstream region, near the cold boundary, where $\tau \sim O(1)$; (3) a far-upstream region (the thickness of which is comparable to that of the upstream region), nearer to the cold boundary, where $\tau \sim O(\hat{\beta}^{-1/2})$; and (4) a far-downstream region (the thickness of which is comparable to those of the upstream regions), nearer to the hot boundary, where $(1-\tau) \sim O(\hat{\beta}^{-1/2} \exp(-\hat{\beta}^{-1/2}))$.

2.1 The Downstream Region

The appropriate independent and dependent variables for the downstream region are:

$$\hat{\zeta}(\xi;\hat{\beta}) = \hat{\beta}^{1/2}\xi; \tag{2.1}$$

$$\tau(\xi;\hat{\beta}) \cong 1 - \hat{\beta}^{-1/2}\left[\hat{G}_0(\hat{\zeta}) + \hat{\beta}^{-1/2}\hat{G}_1(\hat{\zeta}) + \cdots\right], \tag{2.2a}$$

$$\varepsilon(\xi;\hat{\beta}) \cong \left[\hat{E}_0(\hat{\zeta}) + \hat{\beta}^{-1/2}\hat{E}_1(\hat{\zeta}) + \cdots\right]. \tag{2.2b}$$

Throughout the flow field, the eigenvalue, Λ, has the representation

$$\Lambda(\hat{\beta}) \cong \hat{\beta}\left[\hat{\Lambda}_0 + \hat{\beta}^{-1/2}\hat{\Lambda}_1 + \cdots\right]. \tag{2.3}$$

In terms of the downstream variables, (1.1) can be written as

$$\frac{d\hat{G}_0}{d\hat{\zeta}} = -(1-\hat{E}_0), \ldots; \tag{2.4a}$$

$$\frac{d\hat{E}_0}{d\hat{\zeta}} = \hat{\Lambda}_0\hat{G}_0\exp(-\hat{G}_0), \ldots. \tag{2.4b}$$

From (1.2b), the downstream boundary conditions for these equations are

$$\hat{G}_0 \to 0, \ldots, \quad \hat{E}_0 \to 1, \ldots \qquad \text{as } \hat{\zeta} \to \infty. \tag{2.5a}$$

In anticipation of the downstream-region/upstream-region matching, the upstream boundary conditions for these equations are taken to be

$$\hat{G}_0 \to \infty, \ldots, \quad \hat{E}_0 \to 0, \ldots \quad \text{as } \hat{\zeta} \to -\infty. \tag{2.5b}$$

As an intermediate step, the leading-order boundary-value problem, from (2.4) and (2.5), may be written in phase-plane form, i.e.,

$$\frac{\mathrm{d}(1-\hat{E}_0)}{\mathrm{d}\hat{G}_0} = \hat{\Lambda}_0 \frac{\hat{G}_0 \exp(-\hat{G}_0)}{(1-\hat{E}_0)}; \tag{2.6a}$$

$$(1-\hat{E}_0) \to 0 \text{ as } \hat{G}_0 \to 0, \qquad (1-\hat{E}_0) \to 1 \text{ as } \hat{G}_0 \to \infty. \tag{2.6b}$$

The solution, $\hat{E}_0(\hat{G}_0)$, of (2.6) is

$$\hat{E}_0 = 1 - [1 - (1+\hat{G}_0)\exp(-\hat{G}_0)]^{1/2}, \tag{2.7}$$

for

$$\hat{\Lambda}_0 = \tfrac{1}{2}. \tag{2.8}$$

Once $\hat{E}_0(\hat{G}_0)$ is known, $\hat{\zeta} = \hat{\zeta}(\hat{G}_0)$ is determined from

$$\frac{\mathrm{d}\hat{\zeta}}{\mathrm{d}\hat{G}_0} = -\frac{1}{1-\hat{E}_0(\hat{G}_0)}: \quad \hat{\zeta} = -\int_{\hat{G}_0^{\mathrm{o}}}^{\hat{G}_0} \frac{\mathrm{d}\hat{t}}{1-\hat{E}_0(\hat{t})}, \tag{2.9}$$

for $\hat{\zeta} \to 0$ as $\hat{G}_0 \to \hat{G}_0^{\mathrm{o}} = \text{const.}$ (to be determined). From (2.9), at the "downstream edge" of this region,

$$\hat{\zeta} \sim \sqrt{2}\Big[-\log \hat{G}_0 + \log \hat{G}_0^{b} - \tfrac{1}{3}\hat{G}_0 + \cdots\Big] \to \infty \quad \text{as } \hat{G}_0 \to 0:$$

$$\hat{G}_0 \sim \hat{G}_0^{b} \exp(-\hat{\zeta}/\sqrt{2})(1+\cdots) \to 0 \quad \text{as } \hat{\zeta} \to \infty, \tag{2.10a}$$

where

$$\hat{G}_0^{b} = \hat{G}_0^{\mathrm{o}} \exp\left\{\frac{1}{\sqrt{2}}\int_0^{\hat{G}_0^{\mathrm{o}}} \left[\frac{1}{1-\hat{E}_0(\hat{t})} - \frac{\sqrt{2}}{\hat{t}}\right]\mathrm{d}\hat{t}\right\}. \tag{2.10b}$$

At the "upstream edge" of this region,

$$\hat{\zeta} \sim -(\hat{G}_0 - \hat{G}_0^u) + \tfrac{1}{2}(\hat{G}_0 + 2)\exp(-\hat{G}_0)(1 + \cdots) \to -\infty \quad \text{as } \hat{G}_0 \to \infty:$$

$$\hat{G}_0 \sim ((-\hat{\zeta}) + \hat{G}_0^u)(1 + \cdots) \to \infty \quad \text{as } \hat{\zeta} \to -\infty, \tag{2.11a}$$

where

$$\hat{G}_0^u = \hat{G}_0^{\circ} - \int_{\hat{G}_0^{\circ}}^{\infty} \left[\frac{1}{1 - \hat{E}_0(\hat{t})} - 1\right] \mathrm{d}\hat{t}. \tag{2.11b}$$

With the downstream and upstream behaviors for $\hat{G}_0(\hat{\zeta})$ determined, the corresponding behaviors for $\hat{E}_0(\hat{\zeta})$ are found to be

$$\hat{E}_0 \sim 1 - (\hat{G}_0^b/\sqrt{2})\exp(-\hat{\zeta}/\sqrt{2})(1 + \cdots) \to 1 \quad \text{as } \hat{\zeta} \to \infty; \tag{2.12}$$

$$\hat{E}_0 \sim \tfrac{1}{2}\exp(-\hat{G}_0^u)\exp(-(-\hat{\zeta}))[(-\hat{\zeta}) + (\hat{G}_0^u + 1)](1 + \cdots) \to 0 \quad \text{as } \hat{\zeta} \to -\infty. \tag{2.13}$$

Since, for this region, higher-order approximations are not pursued in this paper, from (2.10) and (2.12), τ and ε, as $\hat{\zeta} \to \infty$, are given by

$$\tau = 1 - \hat{\beta}^{-1/2}\left[\hat{G}_0^b \exp(-\hat{\zeta}/\sqrt{2})(1 + \cdots)\right]\left[1 + \mathrm{O}\left(\hat{\beta}^{-1/2}\right)\right]; \tag{2.14}$$

$$\varepsilon = \left[1 - \left(\hat{G}_0^b/\sqrt{2}\right)\exp(-\hat{\zeta}/\sqrt{2})(1 + \cdots)\right]\left[1 + \mathrm{O}\left(\hat{\beta}^{-1/2}\right)\right]. \tag{2.15}$$

When higher-order approximations for this region are pursued, it is found that the solutions for the downstream region, considered, are not uniformly valid as the hot boundary is approached ($\hat{\zeta} \to \infty$). This (near) downstream region must be supplemented by a far-downstream region. Details of this far-downstream region are presented in Section 2.4.

From (2.11) and (2.13), with $\hat{G}_0^u = -1$, τ and ε, as $\hat{\zeta} \to -\infty$, are given by

$$\tau = 1 - \hat{\beta}^{-1/2}[((-\hat{\zeta}) - 1)(1 + \cdots)]\left[1 + \mathrm{O}\left(\hat{\beta}^{-1/2}\right)\right]; \tag{2.16}$$

$$\varepsilon = [\tfrac{1}{2}\mathrm{e}(-\hat{\zeta})\exp(-(-\hat{\zeta}))(1 + \cdots)]\left[1 + \mathrm{O}\left(\hat{\beta}^{-1/2}\right)\right]. \tag{2.17}$$

2.2 The Upstream Region

For the upstream region, the independent variable is $\hat{\xi} = \xi$, with $-\infty < \hat{\xi} < 0$, and the appropriate dependent variables are

$$\tau(\xi;\hat{\beta}) \cong \left[\hat{F}_0(\hat{\xi}) + \hat{\beta}^{-1/2}\hat{F}_1(\hat{\xi}) + \cdots\right], \tag{2.18a}$$

$$\varepsilon(\xi;\hat{\beta}) \cong \hat{\beta}^{1/2}\left[\hat{J}_0(\hat{\xi}) + \hat{\beta}^{-1/2}\hat{J}_1(\hat{\xi}) + \cdots\right] \times \exp\left\{-\hat{\beta}^{1/2}\left[\hat{H}_0(\hat{\xi}) + \hat{\beta}^{-1/2}\hat{H}_1(\hat{\xi}) + \cdots\right]\right\},$$

$$\text{with } \hat{H}_0(\hat{\xi}) = (1 - \hat{F}_0(\hat{\xi})),\ \hat{H}_1(\hat{\xi}) = -\hat{F}_1(\hat{\xi}) + \frac{1}{k^2}(1 - \hat{F}_0(\hat{\xi}))^2, \ldots. \tag{2.18b}$$

The representation for the eigenvalue, Λ, is now

$$\Lambda(\hat{\beta}) \cong \hat{\beta}[\tfrac{1}{2} + \cdots]. \tag{2.19}$$

In terms of the upstream variables, (1.1) can be written as

$$\frac{d\hat{F}_0}{d\hat{\xi}} = \hat{F}_0,\ \frac{d\hat{F}_1}{d\hat{\xi}} = \hat{F}_1, \ldots; \tag{2.20a}$$

$$\hat{J}_0\frac{d\hat{F}_0}{d\hat{\xi}} = \frac{1}{2}(1 - \hat{F}_0)\hat{F}_0\text{:}\quad \hat{J}_0 = \frac{1}{2}(1 - \hat{F}_0), \ldots \tag{2.20b}$$

To leading orders of approximation, the boundary conditions are taken to be

$$\hat{F}_0 \to 1, \ldots,\quad \hat{J}_0 \to 0, \ldots \qquad \text{as } \hat{\xi} \to 0_- \tag{2.21a}$$

$$\hat{F}_0 \to 0, \ldots,\quad \hat{J}_0 \to \tfrac{1}{2}, \ldots \qquad \text{as } \hat{\xi} \to -\infty. \tag{2.21b}$$

Directly, it is found that the leading-order temperature-function solutions are

$$\hat{F}_0 = \hat{B}_0 \exp(\hat{\xi}),\ \hat{F}_1 = \hat{B}_1 \exp(\hat{\xi}), \ldots,\quad \text{with } \hat{B}_0 = 1,\ \hat{B}_1 = 1, \ldots. \tag{2.22}$$

Thus,

$$\hat{F}_0 \sim [1 - (-\hat{\xi}) + \tfrac{1}{2}(-\hat{\xi})^2 - + \cdots] \to 1 \quad \text{as } \hat{\xi} \to 0_-; \tag{2.23a}$$

$$\hat{F}_0 \sim \exp(-(\hat{\xi})) \to 0 \quad \text{as } \hat{\xi} \to -\infty. \tag{2.23b}$$

From the solutions for the upstream region, τ and ε can be expressed as

$$\tau = \exp(\hat{\xi})\left[1 + \hat{\beta}^{-1/2} + \mathrm{O}(\beta^{-1})\right]; \tag{2.24}$$

$$\varepsilon = \frac{1}{2}\hat{\beta}^{1/2}[1 - \exp(\hat{\xi})]\exp\left\{-\hat{\beta}^{1/2}[1 - \exp(\hat{\xi})]\right\}$$
$$\times \exp\left\{-\frac{1}{k^2}[1 - \exp(\hat{\xi})]^2 + \exp(\hat{\xi})\right\}\left[1 + \mathrm{O}\left(\hat{\beta}^{-1/2}\right)\right]. \tag{2.25}$$

These upstream solutions for τ and ε, (2.24) and (2.25), as $\hat{\xi} \to 0_-$, match to the downstream solutions for τ and ε, (2.16) and (2.17), as $\hat{\zeta} \to -\infty$.

Upstream $(\hat{\xi} \to -\infty)$, for $\hat{F}_0 = \exp(\hat{\xi}) \to 0$, $\hat{\beta} \to \infty$, such that $\hat{\beta}^{1/2}\hat{F}_0 = \hat{\beta}^{1/2}\exp(\hat{\xi}) = \hat{f}_0 \sim \mathrm{O}(1)$, (2.24) and (2.25) yield

$$\tau = \hat{\beta}^{-1/2}\hat{f}_0\left[1 + \hat{\beta}^{-1/2} + \mathrm{O}\left(\hat{\beta}^{-1}\right)\right]; \tag{2.26}$$

$$\varepsilon = \hat{\beta}^{1/2}\exp\left(-\hat{\beta}^{1/2}\right)\frac{1}{2}\exp\left(-\frac{1}{k^2}\right)\exp(\hat{f}_0)\left[1 + \mathrm{O}\left(\hat{\beta}^{-1/2}\right)\right]. \tag{2.27}$$

Note that $\hat{\beta}^{-1/2} \to 0$ (algebraically), and that $\hat{\beta}^{1/2}\exp(-\hat{\beta}^{1/2}) \to 0$ (exponentially), as $\hat{\beta} \to \infty$. Further, with $\hat{F}_0 = \exp(\hat{\xi})$, it follows that

$$\hat{f}_0 = \hat{\beta}^{1/2}\exp(\hat{\xi}) = \exp(\hat{\xi}_u)\exp(\hat{\xi}) = \exp(\hat{\xi}_u + \hat{\xi}) \quad \text{with } \hat{\xi}_u = \tfrac{1}{2}\log\hat{\beta}. \tag{2.28}$$

Thus, $\hat{\beta}\hat{F}_0 = \hat{f}_0 \sim \mathrm{O}(1)$ for $(\hat{\xi}_u + \hat{\xi}) = \hat{\eta} \sim \mathrm{O}(1)$, i.e., $\hat{f}_0 = \exp(\hat{\eta}) \sim \mathrm{O}(1)$. In turn, (2.26) and (2.27) can be written as

$$\tau = \hat{\beta}^{-1/2}\exp(\hat{\eta})\left[1 + \hat{\beta}^{-1/2} + \mathrm{O}\left(\hat{\beta}^{-1}\right)\right]; \tag{2.29}$$

$$\varepsilon = \hat{\beta}^{1/2}\exp\left(-\hat{\beta}^{1/2}\right)\frac{1}{2}\exp\left(-\frac{1}{k^2}\right)\exp\{\exp(\hat{\eta})\}\left[1 + \mathrm{O}\left(\hat{\beta}^{-1/2}\right)\right]. \tag{2.30}$$

From (2.29), it is seen that the temperature function, τ, of $O(\hat{\beta}^{-1/2})$, goes to zero (exponentially), as required from (1.2a), as the cold upstream boundary is approached (i.e., $\hat{\eta} \to -\infty$). However, from (2.30), it is seen that the mass-flux function, ε, of $O(\hat{\beta}^{1/2}\exp(-\hat{\beta}^{1/2}))$, goes to a finite value as the upstream boundary is approached – in contradiction of (1.2a). In Section 2.3, a far-upstream region, nearer to the upstream boundary, is introduced, the solutions of which resolve the "cold-boundary difficulty" suggested by (2.30).

2.3 The Far-Upstream Region

For the far-upstream region, based on (2.26)–(2.30), it is taken that the appropriate independent and dependent variables are

$$\hat{\eta}(\xi;\hat{\beta}) = \hat{\xi}_u(\hat{\beta}) + \hat{\xi} \quad \text{with } \hat{\xi}_u(\hat{\beta}) = \tfrac{1}{2}\log\hat{\beta}; \tag{2.31}$$

$$\tau(\xi;\hat{\beta}) \cong \hat{\beta}^{-1/2}\left[\hat{\Phi}_0(\hat{\eta}) + \hat{\beta}^{-1/2}\hat{\Phi}_1(\hat{\eta}) + \cdots\right], \tag{2.32a}$$

$$\varepsilon(\xi;\hat{\beta}) \cong \hat{\beta}^{1/2}\exp\left(-\hat{\beta}^{1/2}\right)\left[\hat{\Psi}_0(\hat{\eta}) + \hat{\beta}^{-1/2}\hat{\Psi}_1(\hat{\eta}) + \cdots\right]. \tag{2.32b}$$

The eigenvalue is still given by

$$\Lambda(\hat{\beta}) \cong \hat{\beta}\left[\tfrac{1}{2} + \cdots\right]. \tag{2.33}$$

In terms of the far-upstream variables, (1.1) can be written as

$$\frac{d\hat{\Phi}_0}{d\hat{\eta}} = \hat{\Phi}_0, \quad \frac{d\hat{\Phi}_1}{d\hat{\eta}} = \hat{\Phi}_1, \ldots; \tag{2.34a}$$

$$\frac{d\hat{\Psi}_0}{d\hat{\eta}} = \frac{1}{2}\exp\left(-\frac{1}{k^2}\right)\hat{\Phi}_0\exp(\hat{\Phi}_0), \ldots. \tag{2.34b}$$

The upstream boundary conditions for (2.34) are

$$\hat{\Phi}_0 \to 0, \ldots, \ \hat{\Psi}_0 \to 0, \ldots \qquad \text{as } \hat{\eta} \to -\infty. \tag{2.35}$$

In this region, the temperature-function solutions are

$$\hat{\Phi}_0 = \hat{A}_0\exp(\hat{\eta}), \quad \hat{\Phi}_1 = \hat{A}_1\exp(\hat{\eta}), \ldots, \qquad \text{with } \hat{A}_0 = 1, \quad \hat{A}_1 = 1, \ldots. \tag{2.36}$$

Here, it is noted that, upstream, $\hat{\Phi}_0(\hat{\eta}) = \exp(\hat{\eta}) \to 0$ as $\hat{\eta} \to -\infty$, and that, downstream, $\hat{\Phi}_0(\hat{\eta}) = \exp(\hat{\eta}) \to \infty$ as $\hat{\eta} \to \infty$. Consideration of (2.34a) and (2.34b) produces the following phase-plane problem:

$$\frac{d\hat{\Psi}_0}{d\hat{\Phi}_0} = \frac{1}{2}\exp\left(-\frac{1}{k^2}\right)\exp(\hat{\Phi}_0); \tag{2.37a}$$

$$\hat{\Psi}_0 \to 0 \quad \text{as } \hat{\Phi}_0 \to 0. \tag{2.37b}$$

The solution of (2.37) is determined to be

$$\hat{\Psi}_0 = \frac{1}{2}\exp\left(-\frac{1}{k^2}\right)[\exp(\hat{\Phi}_0) - 1], \tag{2.38a}$$

or, since $\hat{\Phi}_0 = \exp(\hat{\eta})$,

$$\hat{\Psi}_0 = \frac{1}{2}\exp\left(-\frac{1}{k^2}\right)[\exp\{\exp(\hat{\eta})\} - 1]. \tag{2.38b}$$

The solution of (2.38b) for $\hat{\Psi}_0(\hat{\eta})$, is one for which $\hat{\Psi}_0(\hat{\eta}) \to 0$ as $\hat{\eta} \to -\infty$ and for which $\hat{\Psi}_0(\hat{\eta}) \to \infty$ as $\hat{\eta} \to \infty$.

Hence, for this far-upstream region, τ and ε can be expressed as

$$\tau = \hat{\beta}^{-1/2}\exp(\hat{\eta})\left[1 + \hat{\beta}^{-1/2} + O\left(\hat{\beta}^{-1}\right)\right]; \tag{2.39}$$

$$\varepsilon = \hat{\beta}^{1/2}\exp\left(-\hat{\beta}^{1/2}\right)\frac{1}{2}\exp\left(-\frac{1}{k^2}\right)[\exp\{\exp(\hat{\eta})\} - 1]\left[1 + O\left(\hat{\beta}^{-1/2}\right)\right]. \tag{2.40}$$

The function τ of (2.39) is that of (2.29), and, as such, satisfies its upstream boundary condition ($\hat{\eta} \to -\infty$). The function ε of (2.40) now satisfies its upstream boundary condition ($\hat{\eta} \to -\infty$) – and the cold-boundary difficulty is resolved. Further, these far-upstream solutions for τ and ε, (2.39) and (2.40), as $\hat{\eta} \to \infty$, are seen to match to the upstream ones, (2.24) and (2.25), as $\hat{\xi} \to -\infty$.

2.4 The Far-Downstream Region

When higher-order approximations for the downstream region (of Section 2.1) are pursued, it is determined that, for this region, the solutions for τ and ε are not uniformly valid as $\hat{\zeta} \to \infty$ and $\hat{\beta} \to \infty$, such that $\hat{\beta}^{-1/2}\hat{\zeta} = \hat{\xi}$ is of order unity. This nonuniformity indicates that, for the boundary-value problem under consideration, the downstream region should be supplemented by a far-downstream region.

Based on (2.14) and (2.15), as well as the above, the appropriate independent and dependent variables for this far-downstream region are

$$\hat{\lambda}(\xi;\hat{\beta}) = \hat{\beta}^{-1/2}\hat{\zeta} = \hat{\xi}; \tag{2.41}$$

$$\tau(\xi;\hat{\beta}) \cong 1 - \hat{\beta}^{-1/2}\exp\left(-\hat{\beta}^{1/2}\hat{\lambda}/\sqrt{2}\right)\left[\hat{u}_0(\hat{\lambda}) + \hat{\beta}^{-1/2}\hat{u}_1(\hat{\lambda}) + \cdots\right], \tag{2.42a}$$

$$\varepsilon(\xi;\hat{\beta}) \cong 1 - \exp\left(-\hat{\beta}^{1/2}\hat{\lambda}/\sqrt{2}\right)\left[\hat{v}_0(\hat{\lambda}) + \hat{\beta}^{-1/2}\hat{v}_1(\hat{\lambda}) + \cdots\right]. \tag{2.42b}$$

Further, the eigenvalue is

$$\Lambda(\hat{\beta}) \cong \hat{\beta}\left[\tfrac{1}{2} - \hat{\beta}^{-1/2}(I-1) + \cdots\right]. \tag{2.43}$$

This eigenvalue representation is the one derived in (1.4).

In terms of the far-downstream variables, (1.1) can be written as

$$\left(\frac{1}{\sqrt{2}}\hat{u}_0 - \hat{v}_0\right) = 0, \quad \left(\frac{1}{\sqrt{2}}\hat{u}_1 - \hat{v}_1\right) = \left(\frac{d\hat{u}_0}{d\hat{\lambda}} - \hat{u}_0\right), \ldots; \tag{2.44a}$$

$$\left(\frac{1}{\sqrt{2}}\hat{u}_0 - \hat{v}_0\right) = 0, \quad \left(\frac{1}{\sqrt{2}}\hat{u}_1 - \hat{v}_1\right) = \sqrt{2}\left((I-1)\hat{u}_0 - \frac{d\hat{v}_0}{d\hat{\lambda}}\right). \tag{2.44b}$$

The zeroth-order equations of (2.44a) and (2.44b) both yield

$$\hat{v}_0 = \frac{1}{\sqrt{2}}\hat{u}_0. \tag{2.45}$$

In turn, the first-order equations of (2.44a) and (2.44b), in combination with (2.45), yield

$$\frac{d\hat{u}_0}{d\hat{\lambda}} - \frac{1}{2}\left[1 + \sqrt{2}(I-1)\right]\hat{u}_0 = 0. \qquad (2.46)$$

Thus, from (2.45) and (2.46), the zeroth-order far-upstream-region solutions are determined to be

$$\hat{u}_0 = \sqrt{2}\hat{v}_0 = \hat{G}_0^b \exp\{\tfrac{1}{2}[1 + \sqrt{2}(I-1)]\,\hat{\lambda}\}, \qquad (2.47)$$

in order that the solutions for τ and ε for this region, as $\hat{\lambda} \to 0$, match to those, (2.14) and (2.15), for the (near-) downstream region, as $\hat{\zeta} \to \infty$.

Hence, for this far-downstream region, τ and ε can be expressed as

$$\tau = 1 - \hat{\beta}^{-1/2}\hat{G}_0^b \exp\left\{-\hat{\beta}^{1/2}\frac{\hat{\lambda}}{\sqrt{2}}\left[1 - \hat{\beta}^{-1/2}\left(\frac{1}{\sqrt{2}} + (I-1)\right)\right]\right\}\left[1 + O\left(\hat{\beta}^{-1/2}\right)\right]; \qquad (2.48)$$

$$\varepsilon = 1 - \frac{\hat{G}_0^b}{\sqrt{2}} \exp\left\{-\hat{\beta}^{1/2}\frac{\hat{\lambda}}{\sqrt{2}}\left[1 - \hat{\beta}^{-1/2}\left(\frac{1}{\sqrt{2}} + (I-1)\right)\right]\right\}\left[1 + O\left(\hat{\beta}^{-1/2}\right)\right]. \qquad (2.49)$$

The functions τ and ε, of (2.48) and (2.49), respectively, satisfy (1.2b), in that they both go to unity (exponentially) as the hot boundary is approached (i.e., $\hat{\lambda} \to \infty$).

3 RESULTS AND DISCUSSION

The foregoing asymptotic analysis for the generalized reaction-rate model boundary-value problem for the quasi-isothermal deflagration has revealed that a four-region flame structure is required in order to obtain uniformly valid solutions from the cold boundary to the hot boundary, as it is for the order-unity heat-addition deflagration. The details of this structure are shown in Fig. 1. The (near-) downstream and (near-) upstream regions, of Sections 2.1 and 2.2,

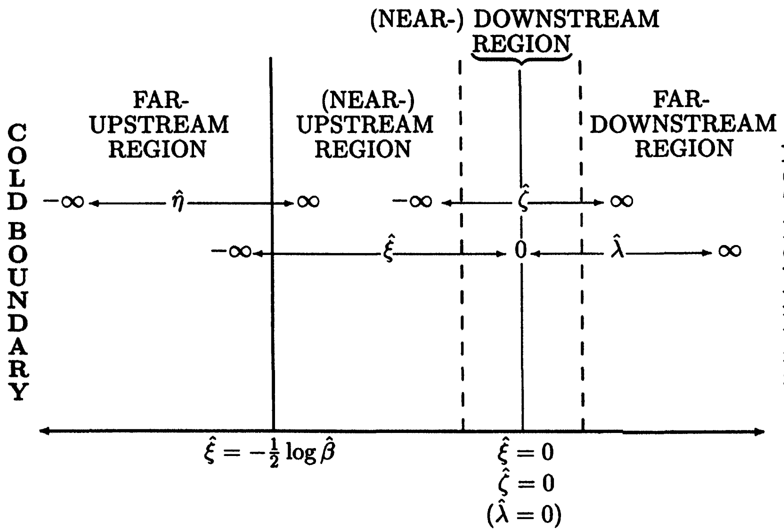

FIGURE 1 Schematic diagram of the four-region structure of deflagration.

respectively, must be complemented by the far-upstream and far-downstream regions, of Sections 2.3 and 2.4, respectively.

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