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Research Article

Relational Demonic Fuzzy Refinement

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We use relational algebra to define a refinement fuzzy order called *demonic fuzzy refinement* and also the associated fuzzy operators which are fuzzy demonic join ($\sqcup_{\text{fuz}}$), fuzzy demonic meet ($\sqcap_{\text{fuz}}$), and fuzzy demonic composition ($\square_{\text{fuz}}$). Our definitions and properties are illustrated by some examples using mathematica software (fuzzy logic).

1. Introduction

1.1. Motivation. Fuzzy set theory provides a major newer paradigm in modeling and reasoning with uncertainty. Zadeh, a professor at University of California at Berkeley was the first to propose a theory of fuzzy sets and an associated logic, namely, fuzzy logic in [1]. Essentially, a fuzzy set is a set whose members may have degrees of membership between 0 and 1, as opposed to classical sets where each element must have either 0 or 1 as the membership degree; if 0, the element is completely outside the set; if 1, the element is completely in the set. As classical logic is based on classical set theory, fuzzy logic is based on fuzzy set theory. Since Zadeh's invention the concept of fuzzy sets has been extensively investigated in mathematics, science, and engineering. The notion of fuzzy relations is also a basic one in processing fuzzy information in relational structures; see, for example, Pedrycz [2]. Goguen [3] generalized the concepts of fuzzy sets and relations taking values on partially ordered sets. Fuzzy relations were initiated and applied to medical models of diagnosis by Sanchez [4]. Fuzzy set theory is now applied to problems in engineering, business, medical and related health sciences, and the natural sciences. Fuzzy relations play an important role in fuzzy modeling, fuzzy diagnosis, and fuzzy control. They also have applications in fields such as psychology, medicine, economics, and sociology.

There are countless applications for fuzzy logic. In fact, some claim that fuzzy logic is the encompassing theory over all types of logic. The items in this list are more common applications that one may encounter in everyday life.

(a) Temperature Control (Heating/Cooling) [5–7]. I do not think the university has figured this one out yet. The trick in temperature control is to keep the room at the same temperature consistently. Well, that seems pretty easy, right? But how much does a room have to cool off before the heat kicks in again? There must be some standard; so the heat (or air conditioning) is not in a constant state of turning on and off. Therein lies the fuzzy logic. The set is determined by what the temperature is actually set to. Membership in that set weakens as the room temperature varies from the set temperature. Once membership weakens to a certain point, temperature control kicks in to get the room back to the temperature it should be.

(b) Medical Diagnosis [8]. How many of what kinds of symptoms will yield a diagnosis? How often are doctors in error? Surely everyone has seen those lists of symptoms for a horrible disease that say "if you have at least 5 of these symptoms, you are at risk." It is a hypochondriac's haven. The question is as follows: how do doctors go from that list of symptoms to a diagnosis? Fuzzy logic. There is no guaranteed system to reach a diagnosis. If there was, we would not hear about cases of medical misdiagnosis. The diagnosis can only be some degree within the fuzzy set.

Fuzzy set theory appeared in 1965 [1]. Since then, it has received increasing attention by the scientific community and applied in almost all the general disciplines [9–11]. Fuzzy set theory provides a strict mathematical framework (there is nothing fuzzy about fuzzy set theory) in which vague conceptual phenomena can be precisely and rigorously studied. It can also be considered as a modeling language well suited for situations in which fuzzy relations, criteria, and phenomena exist. It will mean different things, depending on the application area and the way it is measured. In the meantime, numerous authors have contributed to this theory. In 1984 as many as 4000 publications may already exist. The first publications in fuzzy set theory by Zadeh [1] and Goguen [3, 12] show the intention of the authors to generalize the classical notion of a set. Zadeh [1] writes "The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of mathematics and computer science (pattern classification and information processing). Fuzzy logic is a superset of conventional logic that has been extended to handle the concept of partial truth-truth values between "completely true" and "completely false". As its name suggests, it is the logic underlying modes of reasoning which are approximate rather than exact. The importance of fuzzy theory derives from the fact that most modes of human reasoning and especially common sense reasoning are approximate in nature."

The calculus of relations has been an important component of the development of logic and algebra since the middle of the nineteenth century [13–15]. The main advantages of the relational formalization are uniformity and modularity. Actually, once problems in these fields are formalized in terms of relational calculus, these problems can be considered by using formulae of relations; that is, we need only calculus of relations in order to solve the problems. In the context of software development, one important approach is that of developing programs from specifications by stepwise refinement; see, for example, [16–20]. One point of view is that a specification is a relation constraining the input-output (resp., argument-result) behaviour of programs.

The demonic calculus of relations [21, 22] views any relation R from a set A to another set B as specifying those programs that terminate for all $a \in A$ wherever R associates any values from B with a, and then the program may only return values b for which $(a, b) \in R$. Consequently, a relation R refines another relation S if R specifies a larger domain of termination and fewer possibilities for return values. The demonic calculus of relations has the advantage that the demonic operations are defined on top of the conventional relation algebraic operations and can easily and usefully be mixed with the latter, allowing the application of numerous algebraic properties.

In Section 2, we present our mathematical tool, namely, relational algebra. First, we will give certain notions about elementary theory on relations and ordered structures. There, a concept of type can be defined that allows an abstract treatment of the domain (of definedness) of an element and also of assertions. After some auxiliary results (Section 3 on relational calculus and Section 4 on fuzzy calculus), we present, in Section 5, notions and properties of demonic fuzzy operators. Finally we conclude our work in Section 6.

2. Elementary Theory on Relations

Using Tarski [15] approach, we distinguish two levels of abstraction in the study of binary relations: *the elementary theory of relations* and *relational calculus*. First level defines the relations as sets of (pairs) and second level defines the relations as elementary objects on which the operations are defined and studied in an algebraic point of view.

In this paper, we need both levels of abstraction. The first one will give us our examples and the second one is useful for our proofs and formulas. So, in this section we will present both levels. A relation R from a set X to a set Y is a subset of pairs (x, y), where $x \in X$ and $y \in Y$. Formally,

$$R \subseteq X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}. \tag{1}$$

If $X = Y$, then R is *homogeneous* on X. The relations on finite sets can be represented by matrices. For example, the relation $R = \{(a, b), (a, c), (b, d), (c, d), (d, e)\}$ can be represented by

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{2}$$

The graphs and relations are closely linked.

Every finite relation can be interpreted as a representation graph and vice versa. Graph (1) which corresponds to the relation R is shown in Figure 1.

In matrix notation, an entry 0 corresponds to the absence of an edge between two vertices of the graph (the absence of the pair in the relation) and an entry 1 means the opposite.

As relations are sets, they are ordered by inclusion. The least relation between sets X and Y is the *empty* (also called zero), noted $\emptyset_{XY}$, and the greatest one, called *universal* relation, and noted L_{XY}. A particular relation defined for every set X is the *identity* relation, noted $I_X \overset{\text{def}}{=} \{(x, x) \mid x \in X\}$. The set of elements of X whose images by R is called *domain* of R, noted $\operatorname{dom}(R)$, and the set of images is noted $\operatorname{img}(R)$. Formally,

$$\begin{aligned} \operatorname{dom}(R) &\overset{\text{def}}{=} \{x \mid (\exists y : (x, y) \in R)\}, \\ \operatorname{img}(R) &\overset{\text{def}}{=} \{y \mid (\exists x : (x, y) \in R)\}. \end{aligned} \tag{3}$$

As relations are particular sets, we can apply the usual sets operations, which are union ($\cup$), intersection ($\cap$), and complementation ($\overline{\ }$). Relations are ordered by inclusion. More, their structure helps us to define other operations which are as follows.

(a) Inverse of a relation R, denoted $R^{\smile}$:

$$R^{\smile} = \{(x, y) \mid (y, x) \in R\}. \tag{4}$$

(b) For $R \subseteq X \times Z$ and $S \subseteq Z \times Y$, we define the composition of R and S, noted $R \circ S$, as follows:

$$R \circ S = \{(x, y) \mid (\exists z : z \in Z : (x, z) \in R \text{ and } (z, y) \in S)\}. \tag{5}$$

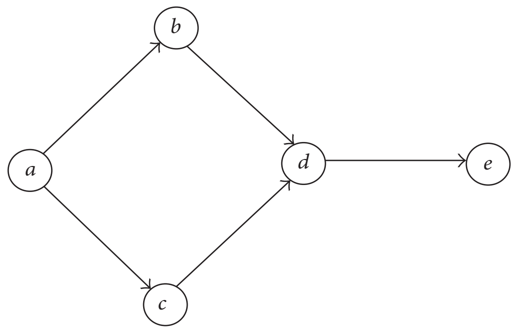

FIGURE 1: The graph associated to relation R.

We remark that $R \circ S \subseteq X \times Y$.

The composition operator symbol ($\circ$) will be omitted (i.e., we write (RS) for ($R \circ S$)).

We can now define the abstract algebraic structures having many properties of the relations. They are based on boolean algebras and other operators which are the composition ($\circ$) and the inverse ($\smile$) and also a particular element I (identity relation).

3. Relational Calculus

The origin of relational calculus goes back to last century with the work of de Morgan [23, 24] and Ströhlein [14], also in the beginning of present century with the work of Peirce [25, 26]. Their study has been revived by the work of Chin and Tarski [15, 27]. For more details on the algebra of relations see [27–33]. In what follows, we will give a definition of the algebra of relations and some important models of axioms characterizing this algebra. Most of our definitions are from [34].

Definition 1. An *abstract heterogeneous relational algebra* is an algebraic structure $(\mathscr{R}, \cup, \cap, ^{-}, ^{\smile}, \circ, \varnothing, L, I)$ on a nonempty set $\mathscr{R}$ of elements called *relations*, such that the next conditions are satisfied.

(a) Every structure $(\mathscr{R}, \cup, \cap, ^{-}, \varnothing, L)$ is a complete atomic boolean algebra, with zero element $\varnothing$, universal element L, and order $\subseteq$.

(b) Every relation R has an inverse $R^{\smile}$.

(c) $(\mathscr{R}, \circ)$ is a semigroup with precisely one unit element I.

(d) Schröder rule $P \circ Q \subseteq R \Leftrightarrow P^{\smile} \circ \overline{R} \subseteq \overline{Q} \Leftrightarrow \overline{R} \circ Q^{\smile} \subseteq \overline{P}$ is valid.

(e) Tarski rule is valid: $L \circ R \circ L = L$ if and only $R \neq \varnothing$.

The precedence of the relational operators, from highest to lowest, is the following: $^{-}, ^{\smile}$, bind equally, followed by ($\circ$) followed by $\cap$ and finally by $\cup$. The scope of $\bigcup_i$ and $\bigcap_i$ goes to the right as far as possible. The relation $R^{\smile}$ is called the *converse* of R. From Definition 1, the usual rules of the calculus of relations can be derived (see, e.g., [27, 34–36]). We assume these rules to be known and simply recall a few of them. We will present certain examples of models satisfying these axioms.

Example 2. (a) The algebra of binary relations on different sets is an important relation algebra, because it is very useful. Let $S_1, \dots, S_n$ be sets. Then

$$\mathscr{R} \stackrel{\text{def}}{=} \left\{ R \mid R \subseteq S_i \times S_j, 1 \leq i, j \leq n \right\}, \tag{6}$$

with relation operators, is a relation algebra. The operations $\cup$ and $\cap$ between relations Q and R are defined if and only Q and R have the same type. A relation is homogeneous if and if only $R : S_i \leftrightarrow S_i$ for a certain i. The composition $Q \circ R$ is defined if and if only $Q : S_i \leftrightarrow S_j$ and $R : S_j \leftrightarrow S_k$ for certain i, j, k.

(b) The set of all homogeneous binary relations on a set X, denoted $\mathrm{Rel}(X) \stackrel{\text{def}}{=} (\mathscr{P}(X \times X), \cup, \cap, ^{\smile}, ^{-}, \circ, \varnothing, X \times X, I_X)$, is a relation algebra.

(c) The algebra of boolean matrices is another important relation algebra.

We recall by the next examples how some of the operators are applied to boolean matrices. To respect the usual convention, we will use the boolean values $\{0, 1\}$ instead of the values $\{\varnothing, L\}$. Consider

$$I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \varnothing_{1\times 2} = \begin{pmatrix} 0 & 0 \end{pmatrix},$$
$$L_{2\times 3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}^{\smile} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{7}$$
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(d) We will give another example. The set of matrices whose entries are relations constitutes a relation algebra [34], with the operators defined as follows:

$$(R \cup S)_{ij} = R_{ij} \cup S_{ij}, \qquad \left(\overline{R}\right)_{ij} = \overline{R_{ij}},$$
$$(RS)_{ij} = \bigcup_k R_{ik} S_{kj}, \qquad (R \cap S)_{ij} = R_{ij} \cap S_{ij}, \tag{8}$$
$$(R^{\smile})_{ij} = \left(R_{ji}\right)^{\smile}.$$

The constant relations are defined as follows:

$$L_{ij} = L, \qquad \varnothing_{ij} = \varnothing, \qquad I_{ij} = \begin{cases} I & \text{if } i = j \\ \varnothing & \text{otherwise,} \end{cases} \tag{9}$$

where $R_{i,j}$ denotes entry i, j of matrix R. Of course, $R \cup S$ and $R \cap S$ exist only if matrices R and S have the same dimension; the composition RS exists only if the number of columns of R is the same as the number of rows of S. The entries of the identity matrix (which is square) are 0, except those of the diagonal which are 1. The entries of the zero matrix are $\varnothing$ and those of the universal matrix are L.

From Definition 1, the usual rules of the calculus of relations can be derived (see, e.g., [27, 34–38]). We assume these rules to be known and simply recall a few of them.

Theorem 3. *Let P, Q, R be relations and let X be an arbitrary index set. Consider the following:*

(1) $\overline{\bigcup_{i\in X} R_i} = \bigcap_{i\in X} \overline{R_i}$,

(2) $\overline{\bigcap_{i\in X} R_i} = \bigcup_{i\in X} \overline{R_i}$,

(3) $(Q \cap R) \cup \overline{R} = Q \cup \overline{R}$,

(4) $P \cap Q \subseteq R \Leftrightarrow P \subseteq \overline{Q} \cup R$,

(5) $Q \subseteq R \Leftrightarrow \overline{R} \subseteq \overline{Q}$,

(6) $Q(\bigcup_{i\in X} R_i) = \bigcup_{i\in X} QR_i$,

(7) $P(Q \cup R) = PQ \cup PR$,

(8) $(\bigcup_{i\in X} Q_i)R = \bigcup_{i\in X} Q_i R$,

(9) $(P \cup Q)R = PR \cup QR$,

(10) $Q(\bigcap_{i\in X} R_i) \subseteq \bigcap_{i\in X} QR_i$,

(11) $(\bigcap_{i\in X} Q_i)R \subseteq \bigcap_{i\in X} Q_i R$,

(12) $Q \subseteq R \Rightarrow PQ \subseteq PR$,

(13) $P \subseteq Q \Rightarrow PR \subseteq QR$,

(14) $(\bigcup_{i\in X} R_i)^{\smile} = \bigcup_{i\in X} R_i^{\smile}$,

(15) $(\bigcap_{i\in X} R_i)^{\smile} = \bigcap_{i\in X} R_i^{\smile}$,

(16) $(QR)^{\smile} = R^{\smile} Q^{\smile}$,

(17) $\overline{R}^{\smile} = \overline{R^{\smile}}$,

(18) $PQ \cap R \subseteq (P \cap RQ^{\smile})(Q \cap P^{\smile} R)$,

(19) $PQ \cap R \subseteq P(Q \cap P^{\smile} R)$,

(20) $PQ \cap R \subseteq (P \cap RQ^{\smile})Q$,

(21) $LL = L$,

(22) $(\bigcap_{i\in X} R_i L)L = \bigcap_{i\in X} R_i L$,

(23) $(\bigcup_{i\in X} R_i L)L = \bigcup_{i\in X} R_i L$,

(24) $(P \cap QL)R = PR \cap QL$,

(25) $(P \cap LQ^{\smile})R = P(R \cap QL)$,

(26) $QLR = QL \cap LR$,

(27) $\overline{RL}L = \overline{RL}$,

(28) $R = (I \cap RR^{\smile})R$.

Sometimes, instead of refering to laws 6, 7, 8, and 9, we refer to the operation $(\circ)$ as distributive. Then, we have $(\circ)$ which is monotonic instead of refering to laws 12 and 13.

In the following, we will give the definitions of certain properties.

Definition 4. A relation R is as follows:

(a) *deterministic* if and if only $R^{\smile} R \subseteq I$,

(b) *total* if and if only $L = RL$ (equivalent to $I \subseteq RR^{\smile}$),

(c) an *application* if and if only it is total and deterministic,

(d) *injective* if and if only $R^{\smile}$ is deterministic (i.e., $RR^{\smile} \subseteq I$),

(e) *surjective* if and if only $R^{\smile}$ is total (i.e., $LR = L$ or also $I \subseteq R^{\smile} R$),

(f) a *partial identity* if and if only $R \subseteq I$ (subidentity),

(g) a *vector* if and if only $R = RL$ (the vectors are usually denoted by the letter v),

(h) a *point* if and if only $R \neq \emptyset$, $R = RL$, and $RR^{\smile} \subseteq I$.

A *function* is a deterministic relation. The vectors RL and $R^{\smile} L$ are particular vectors characterizing, respectively, the domain and codomain of R. The set of vectors of a certain type is a complete boolean algebra [34, 36, 38].

In an algebra of boolean matrices, a vector is a matrix in which the rows are constant and a point is a vector with a nonzero row.

Example 5. Let $X = \{0, 1, 2\}$ and $V = \{0, 1\}$. Then

$$v \stackrel{\text{def}}{=} V \times X = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\} \quad (10)$$

is a vector that corresponds to a set of points V. The partial identity which corresponds to v is $a = \{(0,0), (1,1)\}$.

Let $R \stackrel{\text{def}}{=} \{(0,1), (0,2), (2,1)\}$ and let v and a be the vector and the partial identity given before.

(i) Prerestriction of R to v (or to a) is as follows:

$$v \cap R = aR = \{(0,1), (0,2)\}. \quad (11)$$

(ii) Postrestriction of R to v (or to a) is as follows:

$$v^{\smile} \cap R = Ra = \{(0,1), (2,1)\}. \quad (12)$$

(iii) Domain of R is

$$RL = \{(0,0), (0,1), (0,2), (2,0), (2,1), (2,2)\}. \quad (13)$$

The vector represents the subset $\{0, 2\}$.

(iv) The relation $R^{\smile} L = \{(1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$ is the vector characterizing the subset $\{1, 2\}$, which is the codomain of R.

4. Fuzzy Relational Calculus

Fuzzy set theory has been studied extensively over the past 30 years. Most of the early interest in fuzzy set theory pertained to representing uncertainty in human cognitive processes [1].

The concept of a fuzzy relation on a set was defined by Zadeh [1, 39] and other authors like Rosenfeld [40], Tamura et. al. [41], and Yeh and Bang [5] considered it further. Fuzzy relations are of fundamental importance in fuzzy logic and fuzzy set theory, including particularly fuzzy preference modeling, fuzzy mathematics, fuzzy inference, and many more.

Let $A, B \subseteq U$ be two sets; a *fuzzy relation* on $A \times B$ is defined by $\widetilde{R} = \{((x, y), \mu_{\widetilde{R}}(x, y)) \mid (x, y) \in A \times B\}$, where the map $\mu_{\widetilde{R}} : A \times B \to [0, 1]$ is called *membership function* and the value $\mu_{\widetilde{R}}(x, y) \in [0, 1]$ is called the degree of membership of (x, y) in $\widetilde{R}$ [40, 41].

As fuzzy relations are sets, they are ordered by inclusion. The least fuzzy relation between sets A and B is the *empty* (also called zero), noted $\widetilde{\varnothing}_{AB}$, and the greatest one, called *universal* fuzzy relation, and noted $\widetilde{L}_{AB}$. A particular fuzzy relation defined for every set A is the *identity* fuzzy relation, noted $\widetilde{I}_A \overset{\text{def}}{=} \{((x,y),\mu_{\widetilde{I}}(x,y)) \mid x \in A\}$, where $\mu_{\widetilde{I}}(x,y)=1$ if $x=y$ and 0 otherwise. The domain of $\widetilde{R}$, denoted by $\text{dom}(\widetilde{R})$, is defined as follows:

$$\text{dom}\left(\widetilde{R}\right) \overset{\text{def}}{=} \sup_{y\in B}\{((x,y),\mu_{\widetilde{R}}(x,y)) \mid \forall x \in A\}, \quad (14)$$

and the codomain of $\widetilde{R}$, denoted by $\text{codom}(\widetilde{R})$, is expressed by finding the maximal value of $\widetilde{R}$ a long A:

$$\text{codom}\left(\widetilde{R}\right) \overset{\text{def}}{=} \sup_{x\in A}\{((x,y),\mu_{\widetilde{R}}(x,y)) \mid \forall y \in B\}. \quad (15)$$

The domain and codomain are regarded as the height of row and columns of the fuzzy relation [42].

4.1. Operations on Fuzzy Relations. As fuzzy relations are particular fuzzy sets, we can apply the usual fuzzy sets operations, which are union (∪), intersection (∩), and complementation (‾), given in [43]. More, their structure helps us to define other operations which are as follows.

Definition 6 (see [1, 5, 39]). Let A, B, and C be sets and let $\widetilde{R}$ and $\widetilde{S}$ be fuzzy relations defined, respectively, on $A \times B$ and $B \times C$. One has

$$\begin{aligned}\widetilde{R} &= \{[(x,y),\mu_{\widetilde{R}}(x,y)] \mid (x,y) \in A\times B\},\\ \widetilde{S} &= \{((y,z),\mu_{\widetilde{S}}(y,z)) \mid (y,z) \in B\times C\}.\end{aligned} \quad (16)$$

(a) Inverse of a fuzzy relation $\widetilde{R}$, denoted $\widetilde{R}^{\smile}$:

$$\widetilde{R}^{\smile} = \{[(x,y),\mu_{\widetilde{R}}(x,y)] \mid (x,y) \in A\times B\}, \quad (17)$$

such that

$$\widetilde{R} = \{[(y,x),\mu_{\widetilde{R}}(y,x)] \mid (y,x) \in B\times A\}. \quad (18)$$

(b) The max-min composition $\widetilde{R} \circ \widetilde{S}$ is the fuzzy relation

$$\widetilde{R} \circ \widetilde{S} = \left\{\left[(x,z),\vee_{y\in B}\{\mu_{\widetilde{R}}(x,y)\wedge\mu_{\widetilde{S}}(y,z)\}\right] \mid x \in A,\ y \in B, z \in C\right\}. \quad (19)$$

(i) From now on, the composition operator symbol ($\circ$) will be omitted (i.e., we write ($\widetilde{R}\widetilde{S}$) for ($\widetilde{R}\circ\widetilde{S}$)).

(c) The semiscalars multiplication $k\widetilde{R}$ of a fuzzy relation $\widetilde{R}$ by a scalar $k(\in[0,1])$ is a fuzzy relation such that

$$k\widetilde{R} = \{[(x,y),k\mu_{\widetilde{R}}(x,y)] \mid (x,y) \in A\times B\}. \quad (20)$$

Example 7. (i) Let $\widetilde{R}$ and $\widetilde{S}$ be two fuzzy relations given as follows and $k=0.3$,

(ii) Let $A=\{x_1,x_2,x_3\}$, $B=\{y_1,y_2,y_3,y_4\}$, $\widetilde{R}$ = "x considerably larger than y," and $\widetilde{S}$ = "y very close to x"; then

$$\widetilde{R} = \begin{array}{c} \\ x_1\\ x_2\\ x_3\end{array}\!\!\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4\end{array}\\ \begin{pmatrix} 0.8 & 1 & 0.1 & 0.7\\ 0 & 0.8 & 0 & 0\\ 0.9 & 1 & 0.7 & 0.8\end{pmatrix}\end{array},$$
$$\widetilde{S} = \begin{array}{c} \\ x_1\\ x_2\\ x_3\end{array}\!\!\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4\end{array}\\ \begin{pmatrix} 0.4 & 0 & 0.9 & 0.6\\ 0.9 & 0.4 & 0.5 & 0.7\\ 0.3 & 0 & 0.8 & 0.5\end{pmatrix}\end{array}. \quad (21)$$

We have

$$\text{(i)}\quad \widetilde{R}\cup\widetilde{S} = \begin{array}{c} \\ x_1\\ x_2\\ x_3\end{array}\!\!\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4\end{array}\\ \begin{pmatrix} 0.8 & 1 & 0.9 & 0.7\\ 0.9 & 0.8 & 0.5 & 0.7\\ 0.9 & 1 & 0.8 & 0.8\end{pmatrix}\end{array}$$
$$\text{(ii)}\quad \widetilde{R}^{\smile} = \begin{array}{c} \\ y_1\\ y_2\\ y_3\\ y_4\end{array}\!\!\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3\end{array}\\ \begin{pmatrix} 0.8 & 0 & 0.9\\ 1 & 0.8 & 1\\ 0.1 & 0 & 0.7\\ 0.7 & 0 & 0.8\end{pmatrix}\end{array}$$
$$\text{(iii)}\quad \widetilde{R}\cap\widetilde{S} = \begin{array}{c} \\ x_1\\ x_2\\ x_3\end{array}\!\!\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4\end{array}\\ \begin{pmatrix} 0.4 & 0 & 0.1 & 0.6\\ 0 & 0.4 & 0 & 0\\ 0.3 & 0 & 0.7 & 0.5\end{pmatrix}\end{array}$$
$$\text{(iv)}\quad k\widetilde{S} = \begin{array}{c} \\ x_1\\ x_2\\ x_3\end{array}\!\!\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4\end{array}\\ \begin{pmatrix} 0.12 & 0 & 0.27 & 0.18\\ 0.27 & 0.12 & 0.15 & 0.21\\ 0.9 & 0 & 0.24 & 0.15\end{pmatrix}\end{array}. \quad (22)$$

These operations are illustrated, respectively, by Figures 2, 3, 4, 5, and 6.

Remark 8. Different versions of "composition" have been suggested which differ in their results and also with respect to their mathematical properties. The max-min composition has become the best known and the most frequently used. However, often the so-called *max-product* or *max-average* compositions lead to results that are more appealing; see [40].

(a) The max-prod composition ($\widetilde{R}\circ_{.}\widetilde{S}$) is defined as follows:

$$\widetilde{R}\circ_{.}\widetilde{S} = \left\{\left[(x,z),\vee_y\left\{\mu_{\widetilde{R}_1}(x,y)\cdot\mu_{\widetilde{R}_2}(y,z)\right\}\right] \mid x \in A,\ y \in B, z \in C\right\}. \quad (23)$$

(b) The max-av composition ($\widetilde{R}\circ_{\text{av}}\widetilde{S}$) is defined as follows:

$$\widetilde{R}\circ_{\text{av}}\widetilde{S} = \left\{\left[(x,z),\frac{1}{2}\vee_y\{\mu_{\widetilde{R}}(x,y)+\mu_{\widetilde{S}}(y,z)\}\right] \mid x \in A,\ y \in B, z \in C\right\}. \quad (24)$$

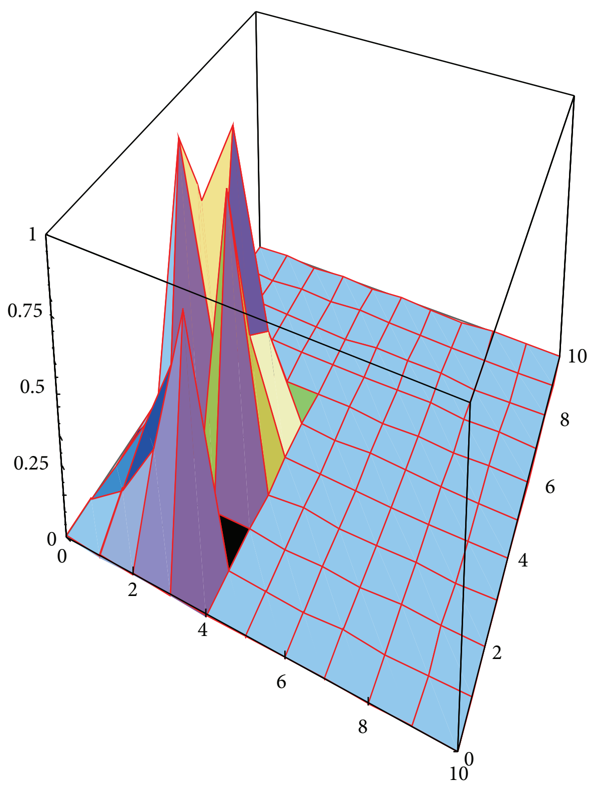

Figure 2: Fuzzy relation $\widetilde{R}$ in Example 9.

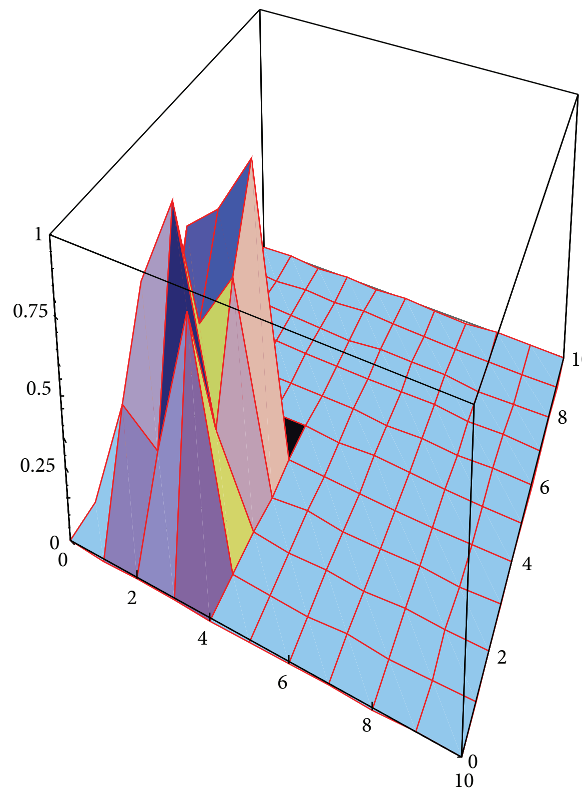

Figure 4: The max-min composition of fuzzy relations $\widetilde{R}$ and $\widetilde{S}$.

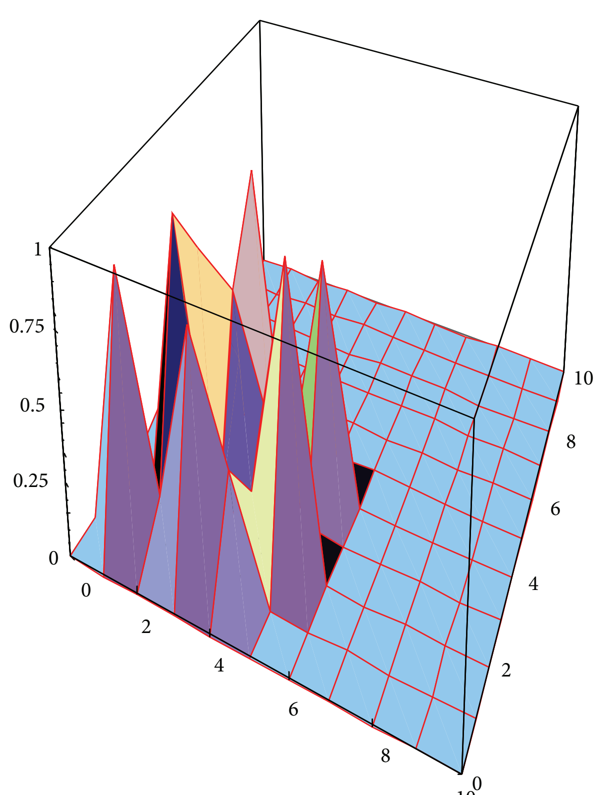

Figure 3: Fuzzy relations $\widetilde{S}$ in Example 9.

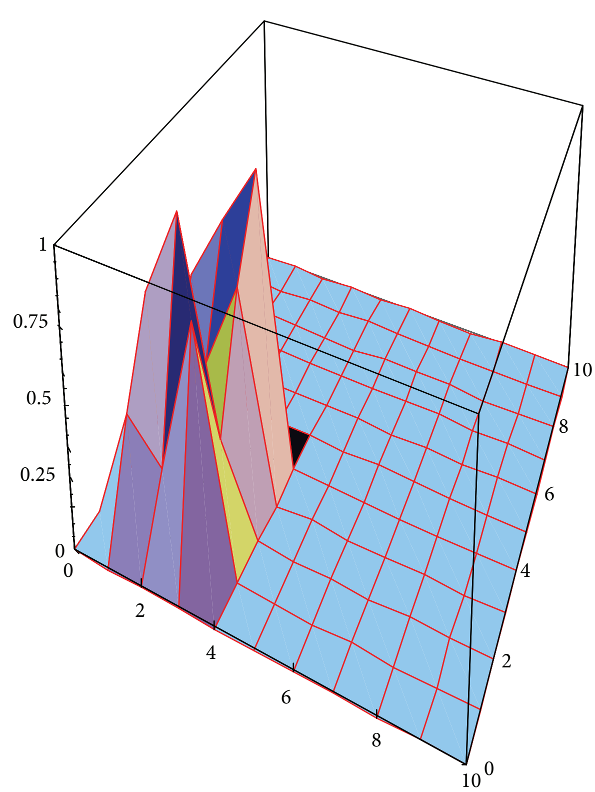

Figure 5: The max-product composition of fuzzy relations $\widetilde{R}$ and $\widetilde{S}$.

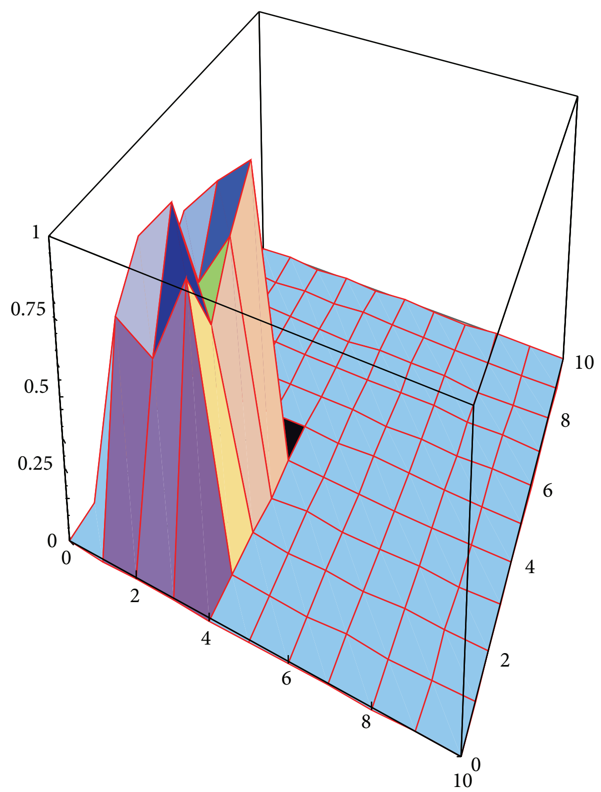

Figure 6: The max-av composition of fuzzy relations $\widetilde{R}$ and $\widetilde{S}$.

Example 9. Let $\widetilde{R}$ and $\widetilde{S}$ be defined by the following matrices [44]:

$$\widetilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{pmatrix} 0.1 & 0.2 & 0 & 1 & 0.7 \\ 0.3 & 0.5 & 0 & 0.2 & 1 \\ 0.8 & 0 & 1 & 0.4 & 0.3 \end{pmatrix} \end{array},$$

$$\widetilde{S} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.9 & 0 & 0.3 & 0.4 \\ 0.2 & 1 & 0.8 & 0 \\ 0.8 & 0 & 0.7 & 1 \\ 0.4 & 0.2 & 0.3 & 0 \\ 0 & 1 & 0 & 0.8 \end{pmatrix} \end{array}. \quad (25)$$

We will first compute the max-min composition $\widetilde{R}\widetilde{S}$. We will show in details the determination for $x = x_1$, $z = z_1$; Let $x = x_1$, $z = z_1$, and $y = y_i$, $1 \le i \le 5$:

(i) $\wedge\{\mu_{\widetilde{R}}(x_1, y_1), \mu_{\widetilde{S}}(y_1, z_1)\} = \wedge\{0.1, 0.9\} = 0.1$

(ii) $\wedge\{\mu_{\widetilde{R}}(x_1, y_2), \mu_{\widetilde{S}}(y_2, z_1)\} = \wedge\{0.2, 0.2\} = 0.2$

(iii) $\wedge\{\mu_{\widetilde{R}}(x_1, y_3), \mu_{\widetilde{S}}(y_3, z_1)\} = \wedge\{0, 0.8\} = 0$

(iv) $\wedge\{\mu_{\widetilde{R}}(x_1, y_4), \mu_{\widetilde{S}}(y_4, z_1)\} = \wedge\{1, 0.4\} = 0.4$

(v) $\wedge\{\mu_{\widetilde{R}}(x_1, y_5), \mu_{\widetilde{S}}(y_5, z_1)\} = \wedge\{0.7, 0\} = 0$

$$\begin{aligned} \widetilde{R}\widetilde{S}(x_1, z_1) &= ((x_1, z_1), \mu_{\widetilde{R}\widetilde{S}}(x_1, z_1)) \\ &= ((x_1, z_1), \vee\{0.1, 0.2, 0, 0.4, 0\}) \qquad (26) \\ &= ((x_1, z_1), 0.4). \end{aligned}$$

In analogy to the above computation we now determine the grades of membership for all pairs (x_i, z_i), $1 \le i \le 3$, $1 \le j \le 4$ and finally we have

$$\widetilde{R}\widetilde{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.4 & 0.7 & 0.3 & 0.7 \\ 0.3 & 1 & 0.5 & 0.8 \\ 0.8 & 0.3 & 0.7 & 1 \end{pmatrix} \end{array}. \quad (27)$$

For the max-prod we obtain

$x = x_1$, $z = z_1$, and $y = y_i$, $1 \le i \le 5$:

(vi) $\mu_{\widetilde{R}}(x_1, y_1) \cdot \mu_{\widetilde{S}}(y_1, z_1) = 0.1 \cdot 0.9 = 0.09$

(vii) $\mu_{\widetilde{R}}(x_1, y_2) \cdot \mu_{\widetilde{S}}(y_2, z_1) = 0.2 \cdot 0.2 = 0.04$

(viii) $\mu_{\widetilde{R}}(x_1, y_3) \cdot \mu_{\widetilde{S}}(y_3, z_1) = 0 \cdot 0.8 = 0$

(ix) $\mu_{\widetilde{R}}(x_1, y_4) \cdot \mu_{\widetilde{S}}(y_4, z_1) = 1 \cdot 0.4 = 0.4$

(x) $\mu_{\widetilde{R}}(x_1, y_5) \cdot \mu_{\widetilde{S}}(y_5, z_1) = 0.7 \cdot 0 = 0$;

then

$$\begin{aligned} \widetilde{R}\widetilde{S}(x_1, z_1) &= ((x_1, z_1), \mu_{\widetilde{R}\widetilde{S}}(x_1, z_1)) \\ &= ((x_1, z_1), \{0.09 \vee 0.04 \vee 0 \vee 0.4 \vee 0\}) \qquad (28) \\ &= ((x_1, z_1), 0.4). \end{aligned}$$

After performing the remaining computations we obtain

$$\widetilde{R} \circ_{\cdot} \widetilde{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.4 & 0.7 & 0.3 & 0.56 \\ 0.27 & 1 & 0.4 & 0.8 \\ 0.8 & 0.3 & 0.7 & 1 \end{pmatrix} \end{array}. \quad (29)$$

The max-av composition finally yields:

i	$\mu(x_1, y_i) + \mu(y_i, z_1)$
1	1
2	0.4
3	0.8
4	1.4
5	0.7;

then

$$\frac{1}{2} \cdot \vee_y \{\mu_{\widetilde{R}}(x_1, y_i) + \mu_{\widetilde{S}}(y_i, z_1)\} = \frac{1}{2} \cdot (1.4) = 0.7$$

$$\widetilde{R} \circ_{\mathrm{av}} \widetilde{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.7 & 0.85 & 0.65 & 0.75 \\ 0.6 & 1 & 0.65 & 0.9 \\ 0.9 & 0.65 & 0.85 & 1 \end{pmatrix} \end{array}. \quad (30)$$

These operations are, respectively, illustrated by Figures 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, and 22.

Remark 10. The vectors $\widetilde{R}\widetilde{L}$ and $\widetilde{R}^{\smile}\widetilde{L}$ are particular vectors characterizing, respectively, the domain and codomain of $\widetilde{R}$, which are defined in (14) and (15).

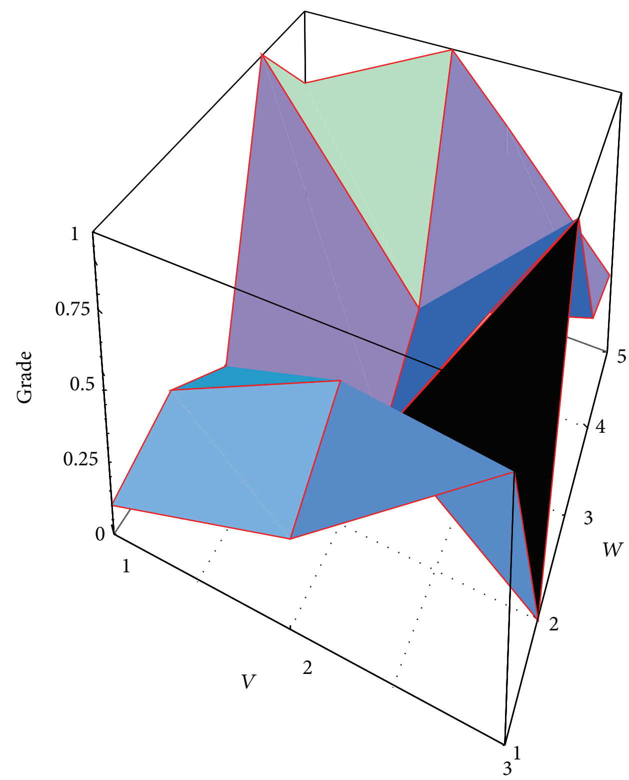

Figure 7: The fuzzy relation $\widetilde{R}$.

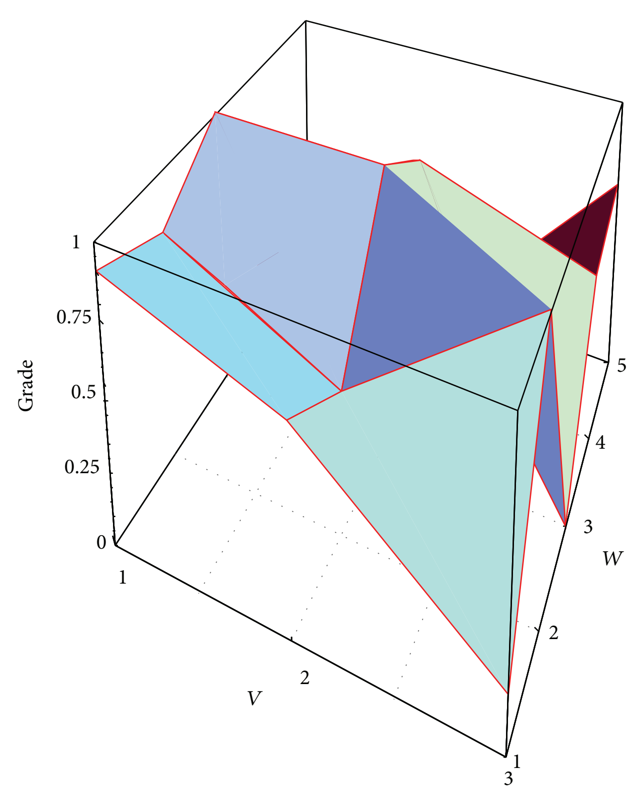

Figure 8: The complement of fuzzy relation $\widetilde{R}$.

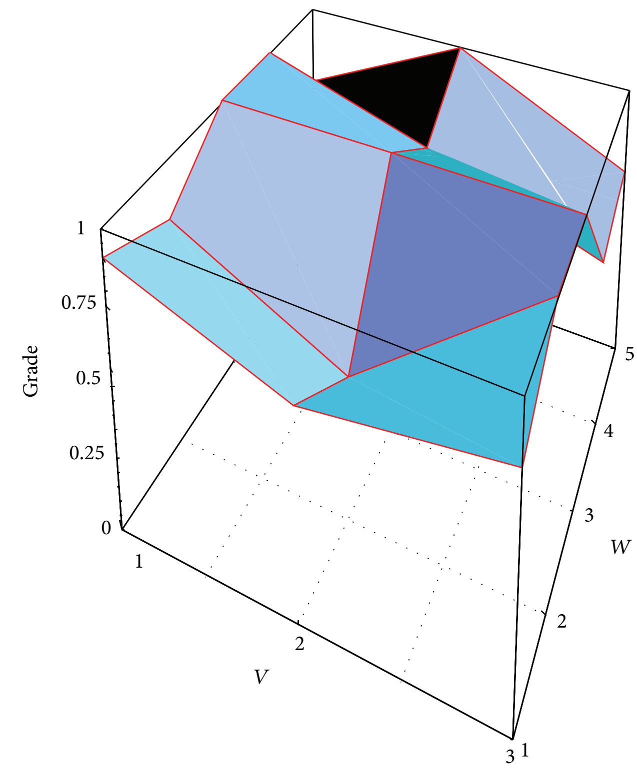

Figure 9: The fuzzy relation $\widetilde{R} \cup \overline{\widetilde{R}} \neq \widetilde{L}$.

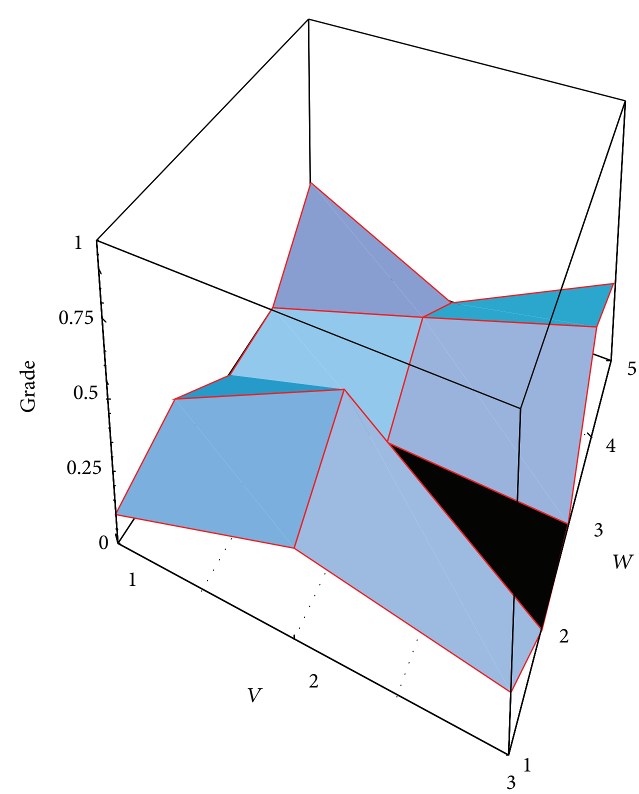

Figure 10: The fuzzy relation $\widetilde{R} \cap \overline{\widetilde{R}} \neq \widetilde{\emptyset}$.

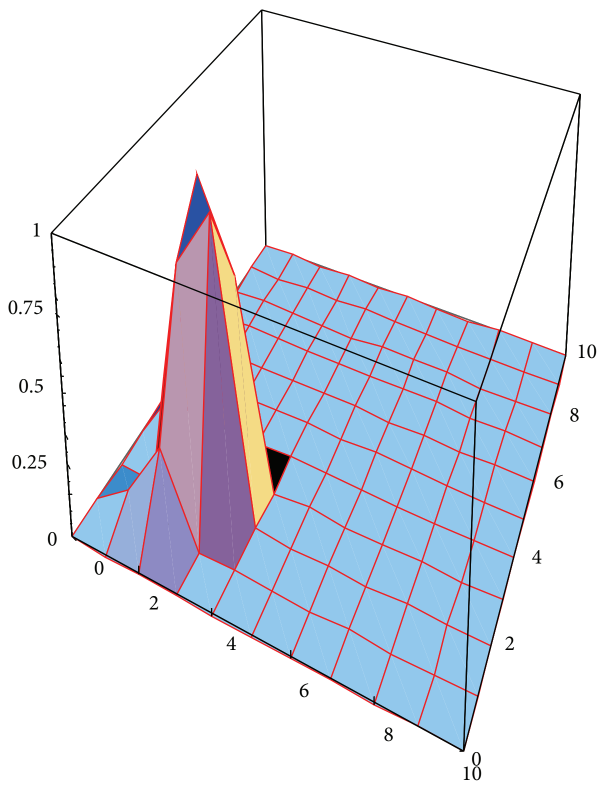

Figure 11: The demonic fuzzy relation $\widetilde{Q}$.

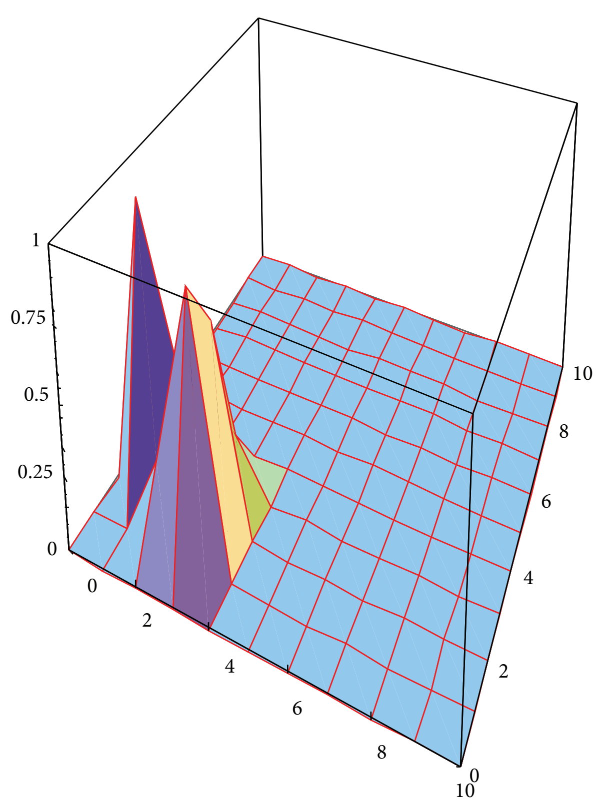

Figure 12: A demonic fuzzy relation $\widetilde{R}$.

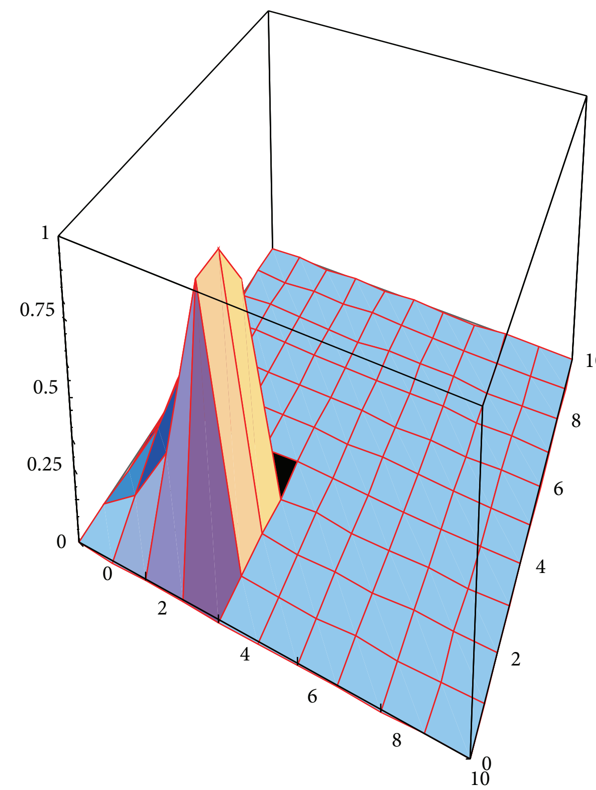

Figure 13: Demonic union of fuzzy relations $\widetilde{Q}$ and $\widetilde{R}$.

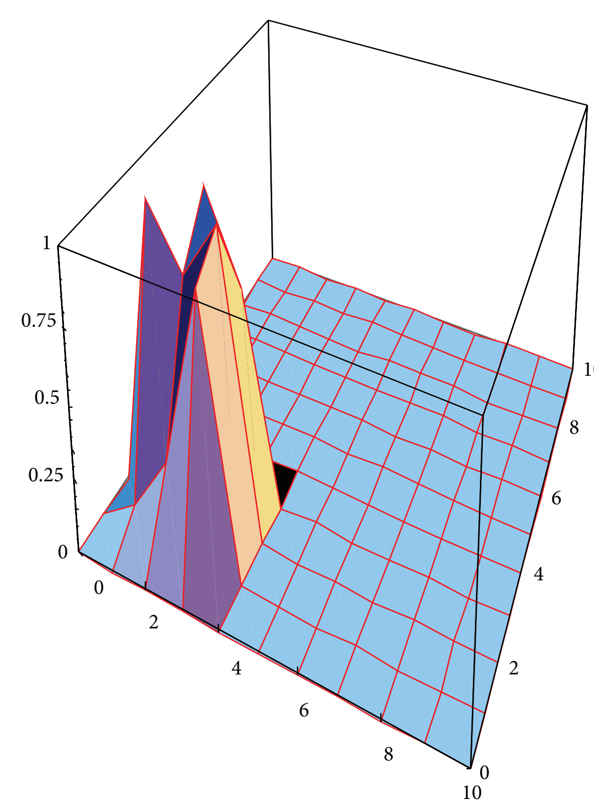

Figure 14: Angelic union of fuzzy relations $\widetilde{Q}$ and $\widetilde{R}$.

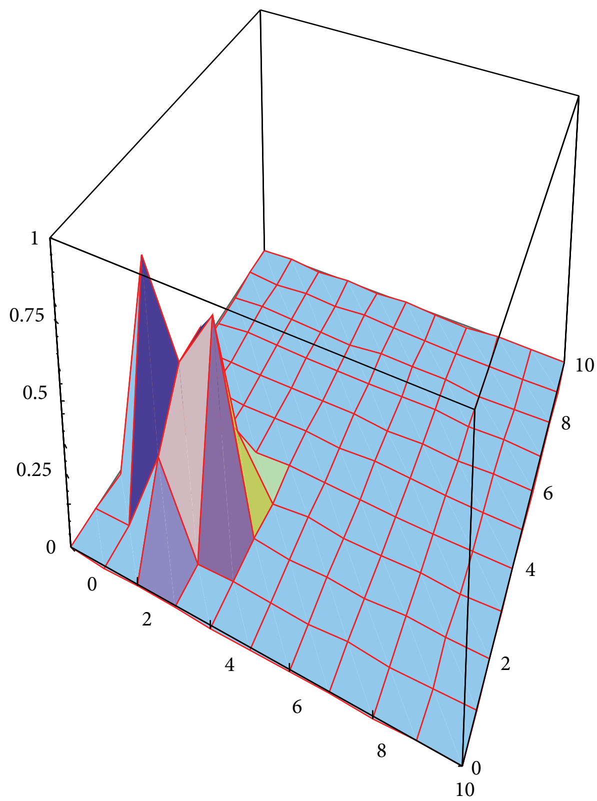

Figure 15: Demonic intersection of fuzzy relations $\widetilde{Q}$ and $\widetilde{R}$.

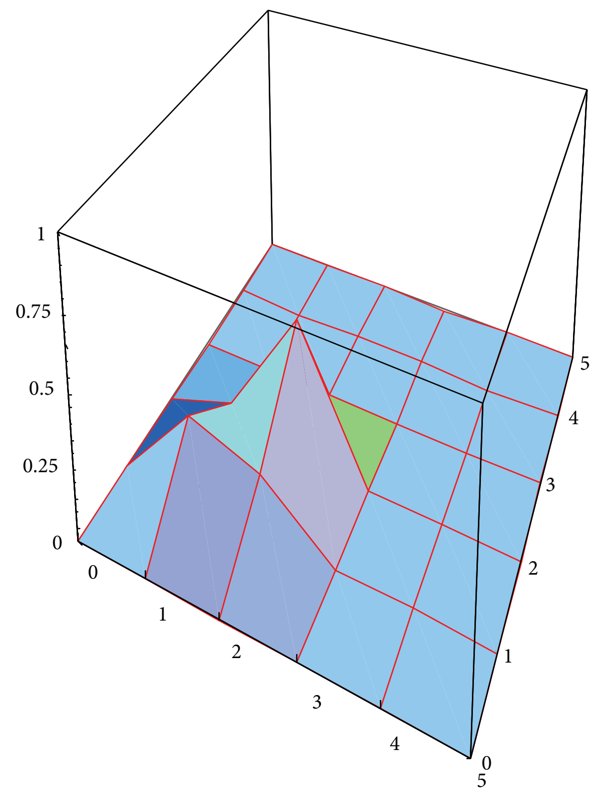

Figure 17: The demonic fuzzy relation $\widetilde{Q}$.

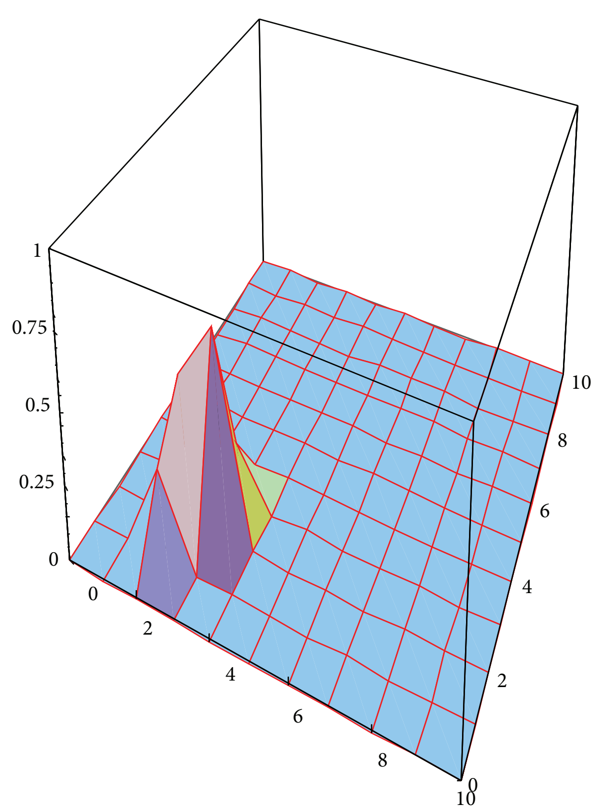

Figure 16: Angelic intersection of fuzzy relations $\widetilde{Q}$ and $\widetilde{R}$.

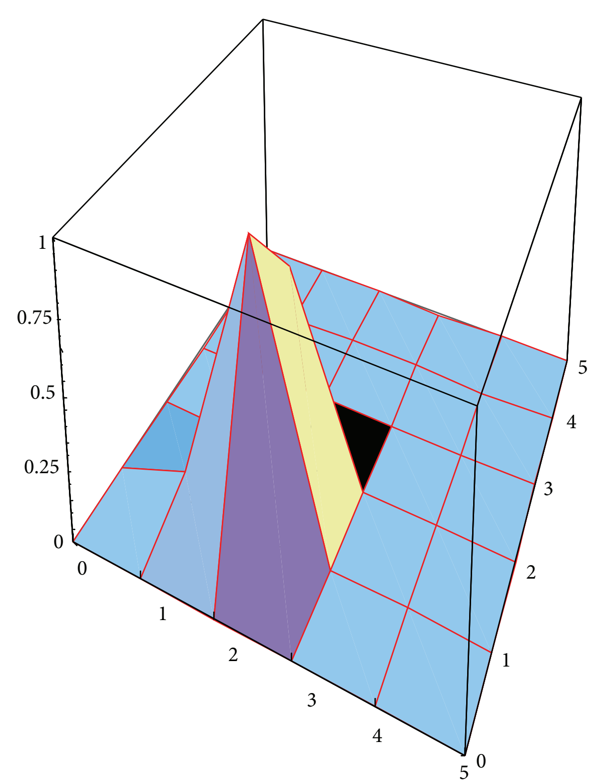

Figure 18: The demonic fuzzy relation $\widetilde{R}$.

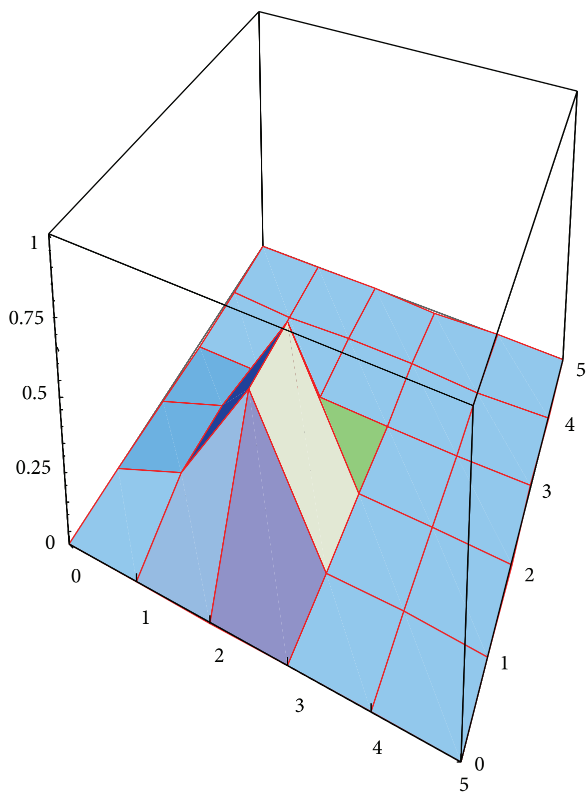

FIGURE 19: Demonic union of fuzzy relations $\widetilde{Q}$ and $\widetilde{R}$.

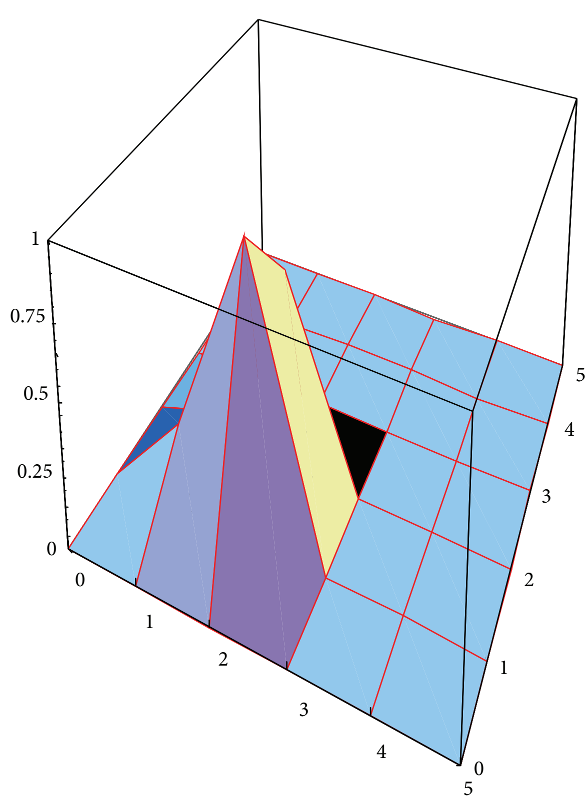

FIGURE 20: Angelic union of fuzzy relations $\widetilde{Q}$ and $\widetilde{R}$.

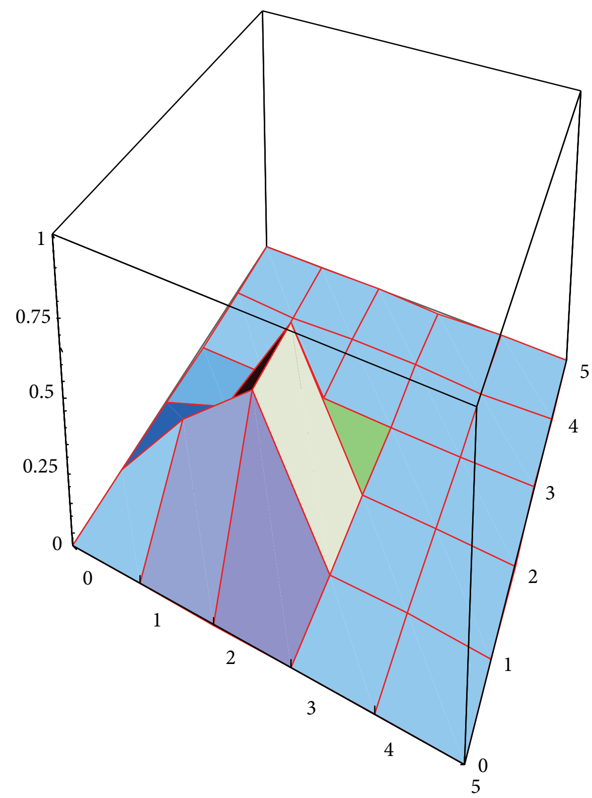

FIGURE 21: Demonic intersection of fuzzy relations $\widetilde{Q}$ and $\widetilde{R}$.

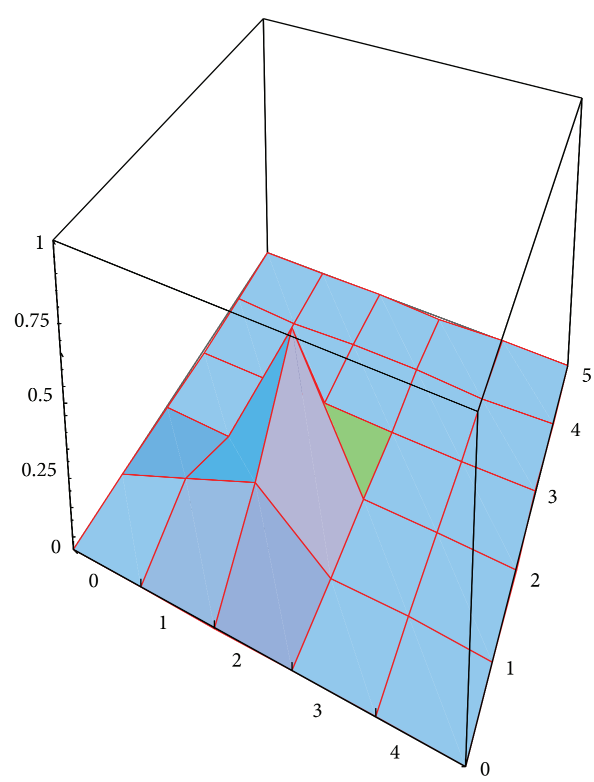

FIGURE 22: Angelic intersection of fuzzy relations $\widetilde{Q}$ and $\widetilde{R}$.

4.2. Properties of Fuzzy Relations. Just as for relations, the properties of commutativity, associativity, distributivity, involution, and idempotency all hold for fuzzy relations. Moreover, De Morgan's principles hold for fuzzy relations just as they do for relations, and the empty relation $\widetilde{\emptyset}$ and the universal relation $\widetilde{L}$ are analogous to the empty set and the whole set in set-theoretic form, respectively. Fuzzy relations are not constrained, as is the case for fuzzy sets in general, by the excluded middle axioms. Since a fuzzy relation $\widetilde{R}$ is also a fuzzy set, there is overlap between a relation and its complement [45]; hence,

$$\widetilde{R} \cup \overline{\widetilde{R}} \neq \widetilde{L}, \qquad \widetilde{R} \cap \overline{\widetilde{R}} \neq \widetilde{\emptyset}. \tag{31}$$

Example 11. Let

$$\widetilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{pmatrix} 0.1 & 0.2 & 0 & 1 & 0.7 \\ 0.3 & 0.5 & 0 & 0.2 & 1 \\ 0.8 & 0 & 1 & 0.4 & 0.3 \end{pmatrix} \end{array}. \tag{32}$$

Then

(i) $$\overline{\widetilde{R}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{pmatrix} 0.9 & 0.8 & 1 & 0 & 0.3 \\ 0.7 & 0.5 & 1 & 0.8 & 0 \\ 0.2 & 1 & 0 & 0.6 & 0.7 \end{pmatrix} \end{array},$$

(ii) $$\widetilde{R} \cap \overline{\widetilde{R}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{pmatrix} 0.1 & 0.2 & 0 & 0 & 0.3 \\ 0.3 & 0.5 & 0 & 0.2 & 0 \\ 0.2 & 0 & 0 & 0.4 & 0.3 \end{pmatrix} \end{array} \neq \widetilde{\emptyset}, \tag{33}$$

(iii) $$\widetilde{R} \cup \overline{\widetilde{R}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{pmatrix} 0.9 & 0.8 & 1 & 1 & 0.7 \\ 0.7 & 0.5 & 1 & 0.8 & 1 \\ 0.8 & 1 & 1 & 0.6 & 0.7 \end{pmatrix} \end{array} \neq \widetilde{L}.$$

In the following, we will give some properties of fuzzy relations.

Definition 12. Let $\widetilde{R}_1$ and $\widetilde{R}_2$ be fuzzy relations on $A \times B$, one have $\widetilde{R}_1 = \{((x,y), \mu_{\widetilde{R}_1}(x,y))\}$, $\widetilde{R}_2 = \{((x,y), \mu_{\widetilde{R}_2}(x,y))\}$. Then, one has the following.

(a) Equality

$$\widetilde{R}_1 = \widetilde{R}_2 \text{ if and only if } \mu_{\widetilde{R}_1}(x,y) = \mu_{\widetilde{R}_2}(x,y). \tag{34}$$

(b) Inclusion

(i) If $\mu_{\widetilde{R}_1}(x,y) \leq \mu_{\widetilde{R}_2}(x,y)$, the relation $\widetilde{R}_1$ is included in $\widetilde{R}_2$ or $\widetilde{R}_2$ is larger $\widetilde{R}_1$, denoted by $\widetilde{R}_1 \subseteq \widetilde{R}_2$.

(ii) If $\widetilde{R}_1 \subseteq \widetilde{R}_2$ and in addition if for at least one pair (x,y),

$$\mu_{\widetilde{R}_1}(x,y) < \mu_{\widetilde{R}_2}(x,y). \tag{35}$$

Then one has the proper inclusion $\widetilde{R}_1 \subseteq \widetilde{R}_2$.

The following properties in Theorem 13 and Proposition 14 have been proved to hold for fuzzy relations (see [11, 46]).

Theorem 13. *Let $\widetilde{R}$, $\widetilde{S}$, and $\widetilde{T}$ be fuzzy relations. Then*

(a) *associativity of composition:* $\widetilde{R}(\widetilde{S}\widetilde{T}) = (\widetilde{R}\widetilde{S})\widetilde{T}$;

(b) *distributivity over union:* $\widetilde{R}(\widetilde{S} \cup \widetilde{T}) = (\widetilde{R}\widetilde{S}) \cup (\widetilde{R}\widetilde{T})$;

(c) *weak distributivity over intersection:* $\widetilde{R}(\widetilde{S} \cap \widetilde{T}) \subseteq (\widetilde{R}\widetilde{S}) \cap (\widetilde{R}\widetilde{T})$;

(d) *commutativity:* $\widetilde{R} \cap \widetilde{S} = \widetilde{S} \cap \widetilde{R}$, $\widetilde{R} \cup \widetilde{S} = \widetilde{S} \cup \widetilde{R}$;

(e) *associativity:* $\widetilde{R} \cap (\widetilde{S} \cap \widetilde{T}) = (\widetilde{R} \cap \widetilde{S}) \cap \widetilde{T}$, $\widetilde{R} \cup (\widetilde{S} \cup \widetilde{T}) = (\widetilde{R} \cup \widetilde{S}) \cup \widetilde{T}$;

(f) *distributivity:* $\widetilde{R} \cap (\widetilde{S} \cup \widetilde{T}) = (\widetilde{R} \cap \widetilde{S}) \cup (\widetilde{R} \cap \widetilde{T})$, $\widetilde{R} \cup (\widetilde{S} \cap \widetilde{T}) = (\widetilde{R} \cup \widetilde{S}) \cap (\widetilde{R} \cup \widetilde{T})$;

(g) *idempotency:* $\widetilde{R} \cup \widetilde{R} = \widetilde{R}$, $\widetilde{R} \cap \widetilde{R} = \widetilde{R}$;

(h) *identity:* $\widetilde{R} \cap \widetilde{\phi} = \widetilde{\phi}$, $\widetilde{R} \cup \widetilde{\phi} = \widetilde{R}$, $\widetilde{R} \cap \widetilde{L} = \widetilde{R}$, $\widetilde{R} \cup \widetilde{L} = \widetilde{L}$;

(i) *involution:* $\overline{\overline{\widetilde{R}}} = \widetilde{R}$;

(j) *De Morgans law:* $\overline{\widetilde{R} \cup \widetilde{S}} = \overline{\widetilde{R}} \cap \overline{\widetilde{S}}$, $\overline{\widetilde{R} \cap \widetilde{S}} = \overline{\widetilde{R}} \cup \overline{\widetilde{S}}$.

Proposition 14. *A fuzzy relation $\widetilde{R} \subseteq A \times A$ is as follows:*

(a) *reflexive if and only $\widetilde{I} \subseteq \widetilde{R}$; that is,* $(\forall x : \mu_{\widetilde{R}}(x,x) = 1)$;

(b) *transitive if and only $\widetilde{R}\widetilde{R} \subseteq \widetilde{R}$; that is,* $(\forall x, y, z : \mu_{\widetilde{R}}(x,z) \geq \wedge\{\mu_{\widetilde{R}}(x,y), \mu_{\widetilde{R}}(y,z)\})$;

(c) *symmetric if and only $\widetilde{R} \subseteq \widetilde{R}^{\smile}$; that is,* $(\forall x, y : \mu_{\widetilde{R}}(x,y) = \mu_{\widetilde{R}}(y,x))$;

(d) *antisymmetric if and only $\widetilde{R}^{\smile} \subseteq \overline{\widetilde{R}} \cup \widetilde{I}$; that is,*

$$\begin{aligned} &\forall x, y : \mu_{\widetilde{R}}(x,y) \neq \mu_{\widetilde{R}}(y,x) \\ &\textit{or} \quad \mu_{\widetilde{R}}(x,y) = \mu_{\widetilde{R}}(y,x) = 0; \end{aligned} \tag{36}$$

(e) *equivalence if and if only $\widetilde{R}$ verifies properties (a), (b), and (c);*

(f) *order if and if only $\widetilde{R}$ verifies properties (a), (b), and (d);*

(g) *preorder if and if only $\widetilde{R}$ verifies properties (a) and (b).*

Example 15. Let $\widetilde{R}, \widetilde{S}, \widetilde{Q}, \widetilde{T}$ and $\widetilde{P}$ be fuzzy relations:

$$\text{(i)}\quad \widetilde{R} = \begin{array}{cc} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{pmatrix} 1 & 0 & 0.2 & 0.3 \\ 0 & 1 & 0.1 & 1 \\ 0.2 & 0.7 & 1 & 0.4 \\ 0 & 1 & 0.4 & 1 \end{pmatrix} \end{array}$$

$$\text{(ii)}\quad \widetilde{S} = \begin{array}{cc} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{pmatrix} 0.2 & 1 & 0.4 & 0.4 \\ 0 & 0.6 & 0.3 & 0 \\ 0 & 1 & 0.3 & 0 \\ 0.1 & 1 & 1 & 0.1 \end{pmatrix} \end{array}$$

$$\text{(iii)}\quad \widetilde{P} = \begin{array}{cc} & \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} & \begin{pmatrix} 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 & 0.8 & 0.2 \\ 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.2 \\ 0.6 & 0.2 & 0.6 & 0.8 & 0.2 & 1 \end{pmatrix} \end{array} \qquad (37)$$

$$\text{(iv)}\quad \widetilde{T} = \begin{array}{cc} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{pmatrix} 0.4 & 0 & 0.7 & 0 \\ 0 & 1 & 0.9 & 0.6 \\ 0.8 & 0.4 & 0.7 & 0.4 \\ 0 & 0.1 & 0 & 0 \end{pmatrix} \end{array}$$

$$\text{(v)}\quad \widetilde{Q} = \begin{array}{cc} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{pmatrix} 0 & 0.1 & 0 & 0.1 \\ 0.1 & 1 & 0.2 & 0.3 \\ 0 & 0.2 & 0.8 & 0.8 \\ 0.1 & 0.3 & 0.8 & 1 \end{pmatrix} \end{array}.$$

$\widetilde{R}$ is a reflexive fuzzy relation, $\widetilde{S}$ is a transitive fuzzy relation, $\widetilde{Q}$ is a symmetric fuzzy relation, $\widetilde{T}$ is an antisymmetric fuzzy relation, and $\widetilde{P}$ is an equivalence fuzzy relation.

5. Demonic Fuzzy Order and Fuzzy Demonic Operators

First, we will give the rationale behind the definition of refinement called the *refinement ordering*. If we consider a relation R as a specification of the input-output behavior of a program p, then p *refines* R (or that p is correct with respect to R) if,

(i) for any input i in the domain of R, i' is a possible output of p only if $(i, i') \in R$;

(ii) p always terminates for any input belonging to the domain of R [47]. For an input that does not belong to the domain of specification R, program p may return any result or return no result; that is, the specifier does not care what happens following the submission of such an input.

In the following, we will define the refinement fuzzy ordering (*demonic fuzzy inclusion*). The associated fuzzy operators are fuzzy demonic join ($\widetilde{\sqcup}$), fuzzy demonic meet ($\widetilde{\sqcap}$), and fuzzy demonic composition ($\widetilde{\boxdot}$). We will give the definitions and needed properties of these operators. We will illustrate them with simple examples using mathematica (*fuzzy logic*).

Definition 16. One says that a fuzzy relation $\widetilde{Q}$ *fuzzy refines* a fuzzy relation $\widetilde{R}$, denoted by $\widetilde{Q} \mathrel{\widetilde{\sqsubseteq}} \widetilde{R}$, if and if only

$$\begin{gathered} \vee_{y\in B}\left\{\mu_{\widetilde{R}}(x,y)\right\} \subseteq \vee_{y\in B}\left\{\mu_{\widetilde{Q}}(x,y)\right\}, \\ \wedge\left\{\mu_{\widetilde{Q}}(x,y), \vee_{y\in B}\left\{\mu_{\widetilde{R}}(x,y)\right\}\right\} \subseteq \mu_{\widetilde{R}}(x,y), \end{gathered} \qquad (38)$$

where $\mu_{\widetilde{R}}$ and $\mu_{\widetilde{Q}}$ are, respectively, the membership functions of $\widetilde{R}$ and $\widetilde{Q}$.

In other words, $\widetilde{Q}$ fuzzy refines $\widetilde{R}$ if and only if the prerestriction of $\widetilde{Q}$ to the domain of $\widetilde{R}$ is included in $\widetilde{R}$: this means that $\widetilde{Q}$ must not produce results not allowed by $\widetilde{R}$ for those states that are in the domain of $\widetilde{R}$.

It is easy to show that this definition is equivalent to definition (given [43]). In other words,

$$\widetilde{Q}\widetilde{L} \subseteq \widetilde{R}\widetilde{L} \wedge \widetilde{Q} \cap \widetilde{R}\widetilde{L} \subseteq \widetilde{R} \qquad (39)$$

if and only if

$$\begin{gathered} \vee_{y\in B}\left\{\mu_{\widetilde{R}}(x,y)\right\} \subseteq \vee_{y\in B}\left\{\mu_{\widetilde{Q}}(x,y)\right\}, \\ \wedge\left\{\mu_{\widetilde{Q}}(x,y), \vee_{y\in B}\left\{\mu_{\widetilde{R}}(x,y)\right\}\right\} \subseteq \mu_{\widetilde{R}}(x,y). \end{gathered} \qquad (40)$$

Example 17. Let

$$\widetilde{Q} = \begin{pmatrix} 0.1 & 0 \\ 1 & 0.2 \end{pmatrix}, \qquad \widetilde{R} = \begin{pmatrix} 0.1 & 0.1 \\ 0.4 & 0.4 \end{pmatrix}. \qquad (41)$$

We have

$$\begin{aligned} \vee_{y\in B}\mu_{\widetilde{R}} = \begin{pmatrix} 0.1 \\ 0.4 \end{pmatrix} \subseteq \begin{pmatrix} 0.1 \\ 1 \end{pmatrix} = \vee_{y\in B}\mu_{\widetilde{Q}}. \\ \wedge\left\{\mu_{\widetilde{Q}}(x,y), \vee_{y\in B}\left\{\mu_{\widetilde{R}}(x,y)\right\}\right\} &= \begin{pmatrix} 0.1 & 0 \\ 0.4 & 0.2 \end{pmatrix} \\ &\subseteq \begin{pmatrix} 0.1 & 0.1 \\ 0.4 & 0.4 \end{pmatrix} \\ &= \mu_{\widetilde{R}} \end{aligned} \qquad (42)$$

Then

$$\widetilde{Q} \mathrel{\widetilde{\sqsubseteq}} \widetilde{R}. \qquad (43)$$

Let

$$\widetilde{P} = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.4 & 0.5 & 0.9 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}, \qquad \widetilde{S} = \begin{pmatrix} 0.3 & 0.2 & 0.4 \\ 0.7 & 0.8 & 0.8 \\ 0.3 & 0.5 & 0.6 \end{pmatrix}. \qquad (44)$$

We have

$$\vee_{y\in B}\mu_{\widetilde{S}} = \begin{pmatrix}0.4\\0.8\\0.6\end{pmatrix} \subseteq \begin{pmatrix}0.5\\0.9\\0.7\end{pmatrix} = \vee_{y\in B}\mu_{\widetilde{P}},$$

$$\begin{aligned}\wedge\left\{\mu_{\widetilde{P}}(x,y), \vee_{y\in B}\left\{\mu_{\widetilde{S}}(x,y)\right\}\right\} &= \begin{pmatrix}0.3&0.2&0.4\\0.4&0.5&0.8\\0.1&0.2&0.6\end{pmatrix}\\ &\subseteq \begin{pmatrix}0.3&0.2&0.4\\0.7&0.8&0.8\\0.3&0.5&0.6\end{pmatrix}\\ &= \mu_{\widetilde{S}}.\end{aligned} \tag{45}$$

Then

$$\widetilde{P} \mathrel{\widetilde{\sqsubseteq}} \widetilde{S}. \tag{46}$$

Let

$$\widetilde{Q} = \begin{pmatrix}0.1&0.2&0.4\\0.5&0.7&0.9\end{pmatrix}, \qquad \widetilde{R} = \begin{pmatrix}0.2&0.2&0.3\\0.4&0.5&0.8\end{pmatrix}$$
$$\widetilde{Q} \mathrel{\widetilde{\not\sqsubseteq}} \widetilde{R} \tag{47}$$

because

$$\begin{aligned}\wedge\left\{\mu_{\widetilde{Q}}(x,y), \vee_{y\in B}\left\{\mu_{\widetilde{R}}(x,y)\right\}\right\} &= \begin{pmatrix}0.1&0.2&0.3\\0.5&0.7&0.8\end{pmatrix}\\ &\not\subseteq \begin{pmatrix}0.2&0.2&0.3\\0.4&0.5&0.8\end{pmatrix}\\ &= \mu_{\widetilde{R}}.\end{aligned} \tag{48}$$

Let

$$\widetilde{P} = \begin{pmatrix}0.5&0.2&0.7\\0.7&0&0.3\end{pmatrix}, \qquad \widetilde{S} = \begin{pmatrix}0.1&0.3&0.4\\0.9&1&0.5\end{pmatrix}$$
$$\widetilde{P} \mathrel{\widetilde{\not\sqsubseteq}} \widetilde{S} \tag{49}$$

because

$$\vee_{y\in B}\mu_{\widetilde{S}} = \begin{pmatrix}0.4\\1\end{pmatrix} \not\subseteq \begin{pmatrix}0.7\\0.7\end{pmatrix} = \vee_{y\in B} \not\subseteq \mu_{\widetilde{P}}. \tag{50}$$

Theorem 18. *The fuzzy relation $\widetilde{\sqsubseteq}$ is a partial order.*

5.1. Fuzzy Demonic Operators and Illustration with Mathematica. In this subsection, we will present fuzzy demonic operators and some of their properties.

Definition 19. Let $\widetilde{Q}$ and $\widetilde{R}$ be fuzzy relations.

(a) Their supremum is $\widetilde{Q} \mathbin{\widetilde{\sqcup}} \widetilde{R}$ and their membership is

$$\begin{aligned}\mu_{(\widetilde{Q}\widetilde{\sqcup}\widetilde{R})}(x,y) = {}&\left[\mu_{\widetilde{Q}}(x,y) \vee \mu_{\widetilde{R}}(x,y)\right] \wedge \left[\vee_{y\in B}\left(\mu_{\widetilde{Q}}(x,y)\right)\right]\\ &\wedge \left[\vee_{y\in B}\left(\mu_{\widetilde{R}}(x,y)\right)\right]\end{aligned} \tag{51}$$

and satisfies

$$\begin{aligned}&\vee_{y\in B}\left\{\mu_{(\widetilde{Q}\widetilde{\sqcup}\widetilde{R})}(x,y)\right\}\\ &\quad = \left[\vee_{y\in B}\left(\mu_{\widetilde{Q}}(x,y)\right)\right] \wedge \left[\vee_{y\in B}\left(\mu_{\widetilde{R}}(x,y)\right)\right].\end{aligned} \tag{52}$$

The operator ($\widetilde{\sqcup}$) is called *fuzzy demonic union*. This definition is equivalent to the definition ([43]). In other words,

$$Q \sqcup R = (Q \cup R) \cap QL \cap RL \tag{53}$$

if and only if

$$\begin{aligned}\mu_{(\widetilde{Q}\widetilde{\sqcup}\widetilde{R})}(x,y) = {}&\left[\mu_{\widetilde{Q}}(x,y) \vee \mu_{\widetilde{R}}(x,y)\right]\\ &\wedge \left[\vee_{y\in B}\left(\mu_{\widetilde{Q}}(x,y)\right)\right] \wedge \left[\vee_{y\in B}\left(\mu_{\widetilde{R}}(x,y)\right)\right].\end{aligned} \tag{54}$$

(b) Their infimum, if it exists, is $\widetilde{Q} \mathbin{\widetilde{\sqcap}} \widetilde{R}$ and their membership is

$$\begin{aligned}\mu_{(\widetilde{Q}\widetilde{\sqcap}\widetilde{R})}(x,y) = {}&\left[\mu_{\widetilde{Q}}(x,y) \wedge \mu_{\widetilde{R}}(x,y)\right]\\ &\vee \left[\mu_{\widetilde{Q}}(x,y) \wedge \left(1 - \vee_{y\in B}\left(\mu_{\widetilde{R}}(x,y)\right)\right)\right]\\ &\vee \left[\mu_{\widetilde{R}}(x,y) \wedge \left(1 - \vee_{y\in B}\left(\mu_{\widetilde{Q}}(x,y)\right)\right)\right]\end{aligned} \tag{55}$$

and it satisfies

$$\begin{aligned}&\vee_{y\in B}\left\{\mu_{(\widetilde{Q}\widetilde{\sqcap}\widetilde{R})}(x,y)\right\}\\ &\quad = \left[\vee_{y\in B}\left(\mu_{\widetilde{Q}}(x,y)\right)\right] \vee \left[\vee_{y\in B}\left(\mu_{\widetilde{R}}(x,y)\right)\right].\end{aligned} \tag{56}$$

The operator ($\widetilde{\sqcap}$) is called *fuzzy demonic intersection*. This definition is equivalent to the definition given in ([18, 19, 36, 43, 48–50]). In other words,

$$Q \sqcap R = Q \cap R \cup Q \cap \overline{RL} \cup R \cap \overline{QL} \tag{57}$$

if and only if

$$\begin{aligned}\mu_{(\widetilde{Q}\widetilde{\sqcap}\widetilde{R})}(x,y) = {}&\left[\mu_{\widetilde{Q}}(x,y) \wedge \mu_{\widetilde{R}}(x,y)\right]\\ &\vee \left[\mu_{\widetilde{Q}}(x,y) \wedge \left(1 - \vee_{y\in B}\left(\mu_{\widetilde{R}}(x,y)\right)\right)\right]\\ &\vee \left[\mu_{\widetilde{R}}(x,y) \wedge \left(1 - \vee_{y\in B}\left(\mu_{\widetilde{Q}}(x,y)\right)\right)\right].\end{aligned} \tag{58}$$

For $\widetilde{Q} \mathbin{\widetilde{\sqcap}} \widetilde{R}$ to exist, one has to verify

$$\begin{aligned}\mu_{\widetilde{L}}(x,y) \subseteq \vee_{y\in B}&\left[\left(\mu_{\widetilde{Q}}(x,y) \vee \left(1 - \vee_{y\in B}\mu_{\widetilde{Q}}(x,y)\right)\right)\right.\\ &\left.\wedge \left(\mu_{\widetilde{R}}(x,y) \vee \left(1 - \vee_{y\in B}\mu_{\widetilde{R}}(x,y)\right)\right)\right].\end{aligned} \tag{59}$$

This condition is equivalent to

$$\left(\vee_{y\in B}\mu_{\widetilde{Q}}(x,y)\right)\wedge\left(\vee_{y\in B}\mu_{\widetilde{R}}(x,y)\right)\subseteq\vee_{y\in B}\left(\mu_{\widetilde{Q}}(x,y)\wedge\mu_{\widetilde{R}}(x,y)\right),\tag{60}$$

which can be interpreted as follows: the existence condition simply means that, on the intersection of their domains, $\widetilde{Q}$ and $\widetilde{R}$ have to agree for at least one value.

Example 20. We know that $\widetilde{Q}\,\widetilde{\sqcup}\,\widetilde{R}\neq\widetilde{Q}\cup\widetilde{R}$ and $\widetilde{Q}\,\widetilde{\sqcap}\,\widetilde{R}\neq\widetilde{Q}\cap\widetilde{R}$.

(a) Let $\widetilde{Q}=\begin{pmatrix}0.1&0&0.2\\0.3&0.8&1\\0&1&0.7\end{pmatrix}$ and $\widetilde{R}=\begin{pmatrix}0&1&0\\0.3&0.5&0.4\\0.9&0.7&0.2\end{pmatrix}$

(i) $\widetilde{Q}\,\widetilde{\sqcup}\,\widetilde{R}=\begin{pmatrix}0.1&0.2&0.2\\0.3&0.5&0.5\\0.9&0.9&0.7\end{pmatrix}$ but $\widetilde{Q}\cup\widetilde{R}=\begin{pmatrix}0.1&1&0.2\\0.3&0.8&1\\0.9&1&0.7\end{pmatrix}$

(ii) $\widetilde{Q}\,\widetilde{\sqcap}\,\widetilde{R}=\begin{pmatrix}0&0.8&0\\0.3&0.5&0.5\\0&0.7&0.2\end{pmatrix}$ but $\widetilde{Q}\cap\widetilde{R}=\begin{pmatrix}0&0&0\\0.3&0.5&0.4\\0&0.7&0.2\end{pmatrix}$.

(b) Let $\widetilde{Q}=\begin{pmatrix}0.3&0.1\\0.2&0.5\end{pmatrix}$ and $\widetilde{R}=\begin{pmatrix}0.1&0\\1&0.7\end{pmatrix}$

(i) $\widetilde{Q}\,\widetilde{\sqcup}\,\widetilde{R}=\begin{pmatrix}0.1&0.1\\0.5&0.5\end{pmatrix}$ but $\widetilde{Q}\cup\widetilde{R}=\begin{pmatrix}0.3&0.1\\1&0.7\end{pmatrix}$

(ii) $\widetilde{Q}\,\widetilde{\sqcap}\,\widetilde{R}=\begin{pmatrix}0.3&0.1\\0.5&0.5\end{pmatrix}$ but $\widetilde{Q}\cap\widetilde{R}=\begin{pmatrix}0.1&0\\0.2&0.5\end{pmatrix}$.

These demonic operations are illustrated, respectively, by Figures 23 and 24.

Now we need to define the relative fuzzy implication.

In what follows, we will give our definition of the relative fuzzy implication and some examples.

Definition 21. The binary operator ($\widetilde{\triangleright}$) is called *relative fuzzy implication*, and its a membership function is defined as follows:

$$\mu_{\widetilde{Q}\widetilde{\triangleright}\widetilde{R}}(x,z)=1-\vee_{y\in B}\left\{\wedge\left\{\mu_{\widetilde{Q}}(x,y),1-\mu_{\widetilde{R}}(y,z)\right\}\right\}.\tag{61}$$

This definition is equivalent to definition ([18, 19, 36, 48–50]) which is

$$Q\triangleright R\overset{\text{def}}{=}\overline{Q\overline{R}}\tag{62}$$

if and only if

$$\mu_{\widetilde{Q}\widetilde{\triangleright}\widetilde{R}}(x,z)=1-\vee_{y\in B}\left\{\wedge\left\{\mu_{\widetilde{Q}}(x,y),1-\mu_{\widetilde{R}}(y,z)\right\}\right\}.\tag{63}$$

Example 22. (a) Let $\widetilde{Q}=\begin{pmatrix}0&0.1\\0.3&0.5\end{pmatrix}$ and $\widetilde{R}=\begin{pmatrix}0.9&0.2\\0.7&1\end{pmatrix}$

(i) $\widetilde{Q}\,\widetilde{\triangleright}\,\widetilde{R}=\begin{pmatrix}0&0.1\\0.3&0.5\end{pmatrix}$.

(b) Let $\widetilde{Q}=\begin{pmatrix}0.1&0.2&0.4\\0.5&0.5&1\\0&0&0.3\end{pmatrix}$ and $\widetilde{R}=\begin{pmatrix}0.7&0.8&0.9\\1&0&0.1\\0.6&0.2&0.3\end{pmatrix}$

(i) $\widetilde{Q}\,\widetilde{\triangleright}\,\widetilde{R}=\begin{pmatrix}0.6&0.6&0.6\\0.6&0.2&0.3\\0.7&0.7&0.7\end{pmatrix}$.

In what follows, we will give the definition of the fuzzy demonic composition.

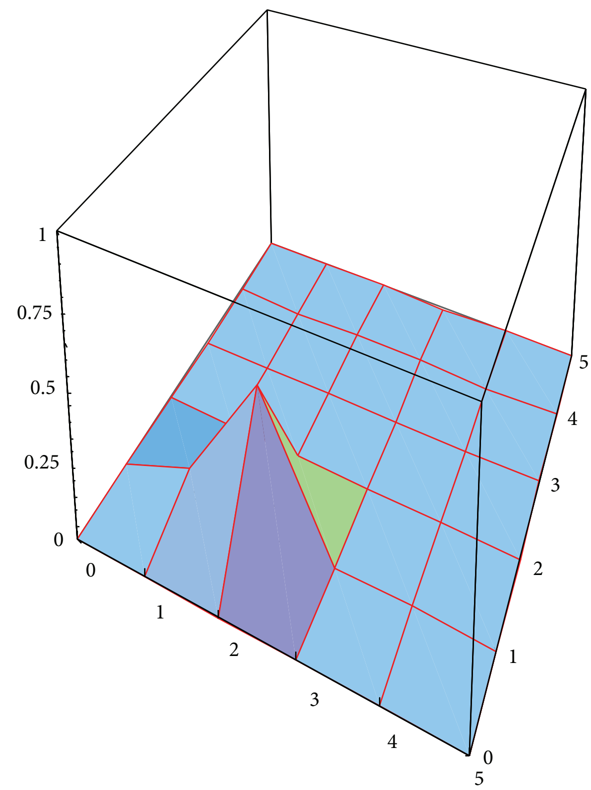

Figure 23: Demonic composition in Example 24 (a).

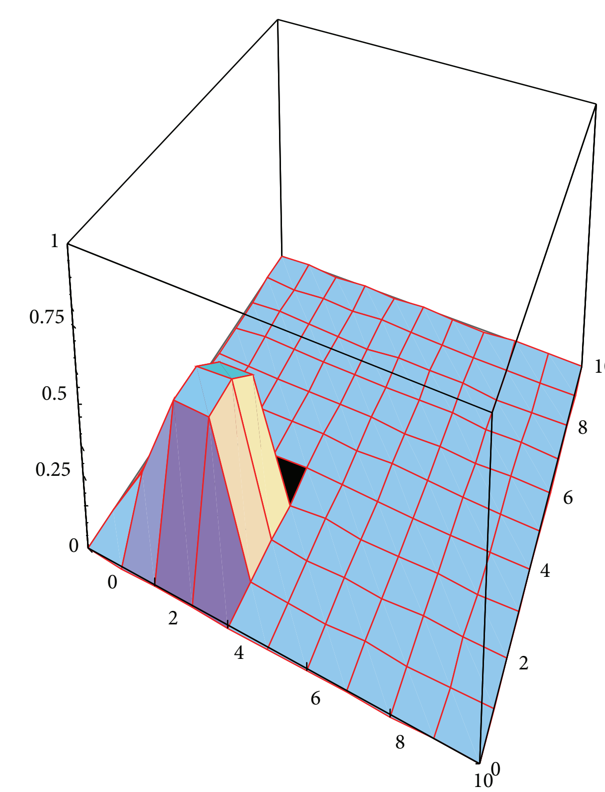

Figure 24: Demonic composition in Example 24 (b).

Definition 23. The *fuzzy demonic composition* of relations $\widetilde{Q}$ and $\widetilde{R}$ is $(\widetilde{Q} \,\widetilde{\square}\, \widetilde{R})$, and its membership function is given by

$$\begin{aligned} &\mu_{\widetilde{Q} \widetilde{\square} \widetilde{R}}(x, y) \\ &\quad = \wedge \Big[\vee_{y \in B} \left\{ \wedge \left\{ \mu_{\widetilde{Q}}(x, y), \mu_{\widetilde{R}}(y, z) \right\} \right\}, \\ &\qquad 1 - \vee_{y \in B} \left\{ \wedge \left\{ \mu_{\widetilde{Q}}(x, y), 1 - \vee_{y \in B} \left(\mu_{\widetilde{R}}(x, y) \right) \right\} \right\} \Big]. \end{aligned} \tag{64}$$

In other words,

$$Q \square R = QR \cap Q \triangleright RL \tag{65}$$

if and only if

$$\begin{aligned} &\wedge \Big[\vee_{y \in B} \left\{ \wedge \left\{ \mu_{\widetilde{Q}}(x, y), \mu_{\widetilde{R}}(y, z) \right\} \right\}, \\ &\quad 1 - \vee_{y \in B} \left\{ \wedge \left\{ \mu_{\widetilde{Q}}(x, y), 1 - \vee_{y \in B} \left(\mu_{\widetilde{R}}(x, y) \right) \right\} \right\} \Big]. \end{aligned} \tag{66}$$

Example 24. Consider the following:

(a) $\begin{pmatrix} 0.3 & 0.1 \\ 0.2 & 0.5 \end{pmatrix} \widetilde{\square} \begin{pmatrix} 0.1 & 0 \\ 1 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.1 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}$

(b) $\begin{pmatrix} 0.1 & 0 & 0.2 \\ 0.3 & 0.8 & 1 \\ 0 & 1 & 0.7 \end{pmatrix} \widetilde{\square} \begin{pmatrix} 0 & 1 & 0 \\ 0.3 & 0.5 & 0.4 \\ 0.9 & 0.7 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.5 & 0.5 & 0.4 \end{pmatrix}$.

Figures 23 and 24 represent fuzzy demonic composition of two relations.

6. Conclusion

In this paper, we have presented the notion of relational fuzzy calculus specially a fuzzy refinement order ($\widetilde{\sqsubseteq}$) also the definitions of the operators associated to this order which are ($\widetilde{\triangleright}$), fuzzy demonic operators ($\widetilde{\sqcap}, \widetilde{\sqcup}$), and fuzzy composition ($\widetilde{\square}$) and give some of their properties. These operators have been illustrated by mathematica (*fuzzy logic*).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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