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A GENERALIZATION OF A THEOREM BY CHEO AND YIEN CONCERNING DIGITAL SUMS

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ABSTRACT. For a non-negative integer n, let s(n) denote the digital sum of n. Cheo and Yien proved that for a positive integer x, the sum of the terms of the sequence

$$\{s(n) : n = 0, 1, 2, \ldots, (x-1)\}$$

is $(4.5)x\log x + O(x)$. In this paper we let k be a positive integer and determine that the sum of the sequence

$$\{s(kn) : n = 0, 1, 2, \ldots, (x-1)\}$$

is also $(4.5)x\log x + O(x)$. The constant implicit in the big-oh notation is dependent on k.

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1. INTRODUCTION.

In Cheo and Yien [1], it was proven that for a positive integer x,

$$\sum_{n=0}^{x-1} s(n) = (4.5)x\log x + O(x) \qquad (1.1)$$

where s(n) denotes the digital sum of n. Here, we will show that, in fact, for any positive integer k,

$$\sum_{n=0}^{x-1} s(kn) = (4.5)x\log x + O(x) \qquad (1.2)$$

where the constant implicit in the big-oh notation is dependent on k.

The following notation will be used to facilitate the proof of (1.2). For integers x and y,

$$x \bmod y \qquad (1.3)$$

will be the remainder when x is divided by y and, as usual, square brackets will denote the integral part operator. In addition, for non-negative integers m, i, and j we let

$$[m]^j = m \bmod 10^j, \qquad (1.4)$$

$$[m]_i = [m/10^i], \qquad (1.5)$$

and

$$[m]_i^j = \left[[m]^j\right]_i \tag{1.6}$$

for $i < j$.

Thus, the j right-most digits of m are given by (1.4) and the number determined by dropping the i right-most digits of m is given by (1.5). Therefore, the number determined from the jth right-most digit of m to the (i + 1)st right-most digit of m is given by (1.6).

2. A PROOF OF (1.2) WHEN k AND 10 ARE RELATIVE PRIME.

Let $(k,10) = 1$, x be a positive integer, and $L = [\log x]$. Then

$$\sum_{n=0}^{x-1} s(kn) = \sum_{n=0}^{x-1} s([kn]^L) + \sum_{n=0}^{x-} s([kn]_L) \tag{2.1}$$

$$= \sum_{n=0}^{x-1} s([kn]^L) + 0(x) . \tag{2.2}$$

This follows since for non-negative integers L and m,

$$m = [m]^L + 10^L[m]_L \tag{2.3}$$

and so

$$s(m) = s([m]^L) + s([m]_L). \tag{2.4}$$

Also, since each $s([kn]_L)$ is bounded by a constant (dependent on k), we have that the second term of (2.1) is $0(x)$.

Next, for $i = 0, 1, 2, \ldots, L$ define

$$x_i = [x]_{L+1-i}10^{L+1-i} . \tag{2.5}$$

Then,

$$\sum_{n=0}^{x-1} s([kn]^L) = \sum_{n=0}^{x_1-1} s([kn]^L) + \sum_{n=x_1}^{x-1} s([kn]^L) \tag{2.6}$$

$$= \sum_{n=0}^{x_1-1} s([kn]^L) + \sum_{n=x_1}^{x-1} s([kn]_{L-1}^L) + \sum_{n=x_1}^{x-1} s([kn]^{L-1}).$$

In the same way,

$$\sum_{n=x_1}^{x-1} s([kn]^{L-1}) = \sum_{n=x_1}^{x_2-1} s([kn]^{L-1}) + \sum_{n=x_2}^{x-1} s([kn]_{L-2}^{L-1})$$

$$+ \sum_{n=x_2}^{x-1} s([kn]^{L-2}) . \tag{2.7}$$

Continuing in this manner and combining terms, we have

$$\sum_{n=0}^{x-1} s([kn]^L) = \sum_{i=1}^{L} \sum_{n=x_{i-1}}^{x_i-1} s([kn]^{L+1-i})$$

$$+ \sum_{i=1}^{L} \sum_{n=x_i}^{x-1} s([kn]_{L-i}^{L+1-i}) . \tag{2.8}$$

Since

$$s([kn]_{L-i}^{L+1-i}) \tag{2.9}$$

is a decimal digit and

$$x - x_i = [x]^{L+1-i} \leqq 10^{L+1-i} \tag{2.10}$$

for each i, it follows that

$$\sum_{i=1}^{L} \sum_{n=x_i}^{x-1} s([kn]_{L-i}^{L+1-i}) = 0(x) . \tag{2.11}$$

To determine the value of the first term of (2.8), we need the following lemma. Its proof is straight forward and will not be given.

LEMMA 2. Let d and i be non-negative integers. Then for (k,10) = 1,

$$\{[kn]^i : n = d,d+1, \ldots,d+10^i-1\} = \{n: n = 0,1, \ldots,10^i-1\}. \tag{2.12}$$

By this lemma and the fact that

$$x_i - x_{i-1} = [x]_{L+1-i}^{L+2-i} 10^{L+1-i} \tag{2.13}$$

it follows that

$$\sum_{n=x_{i-1}}^{x_i - 1} s([kn]^{L+1-i}) = ([x]_{L+1-i}^{L+2-i}) \sum_{n=0}^{10^{L+1-i}} s(n) \tag{2.14}$$

for each i.

Now since

$$\sum_{n=0}^{10^{L+1-i}-1} s(n) = 4.5(L + 1 - i)10^{L+1-i} \tag{2.15}$$

by [2], we have that

$$\sum_{i=1}^{L} \sum_{n=x_{i-1}}^{x_i - 1} s([kn]^{L+1-i}) = (4.5)x\log x + 0(x) . \tag{2.16}$$

Using (2.16) and (2.11) in (2.8), by (2.2) we have the expression given in (1.2). The constant implicit in the big-oh notation is dependent on k with k and 10 relatively prime.

3. CONCLUSION.

For any positive integer k, there exists non-negative integers a, b, and r such that $k = 2^a5^br$ with (r,10) = 1. Note that if k = r, then we have (1.2). However, by use of the following generalization to Lemma 2, and some technical modifications, it can be shown that the restriction that k and 10 be relatively prime can be removed in the derivation of (2.1). That is,

$$\sum_{n=0}^{x-1} s(kn) = (4.5)x\log x + 0(x) \tag{3.1}$$

for any positive integer k.

LEMMA 3. Let $k = 2^a5^br$ with (r,10) = 1 and $i \geqq \max\{a,b\}$. Then for any non-negative integer d,

$$\{[kn]^i : n = d, d+1, d+2, \ldots , d + (10^i/2^a5^b) - 1\}$$
$$= \{2^a5^bn : n = 0, 1, 2, \ldots , (10^i/2^a5^b) - 1\} . \quad (3.2)$$

Finally, based on the above techniques, it is strongly conjectured that for any positive integers k_1 and k_2 , it again follows that

$$\sum_{n=0}^{x-1} s(k_1n + k_2) = (4.5)x\log x + O(x) . \quad (3.3)$$

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