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Research Article

Common Fixed Point Theorems for Conversely Commuting Mappings Using Implicit Relations

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The object of this paper is to utilize the notion of conversely commuting mappings due to Lü (2002) and prove some common fixed point theorems in Menger spaces via implicit relations. We give some examples which demonstrate the validity of the hypotheses and degree of generality of our main results.

1. Introduction

In 1986, Jungck [1] introduced the notion of compatible mappings in metric space. Most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides continuity of at least one of the mappings. Later on, Jungck and Rhoades [2] studied the notion of weakly compatible mappings and utilized it as a tool to improve commutativity conditions in common fixed point theorems. Many mathematicians proved several fixed point results in Menger spaces (see, e.g., [3–9]). In 2002, Lü [10] presented the concept of the converse commuting mappings as a reverse process of weakly compatible mappings and proved common fixed point theorems for single-valued mappings in metric spaces (also see [11]). Recently, Pathak and Verma [12, 13], Chugh et al. [14], and Chauhan et al. [15] proved some interesting common fixed point theorems for converse commuting mappings.

In this paper, we prove some unique common fixed point theorems for two pairs of converse commuting mappings in Menger spaces by using implicit relations.

2. Preliminaries

Definition 1 (see [16]). A t-norm is a function $\Delta : [0,1] \times [0,1] \to [0,1]$ satisfying

(T1) $\Delta(a,1) = a$, $\Delta(0,0) = 0$;

(T2) $\Delta(a,b) = \Delta(b,a)$;

(T3) $\Delta(c,d) \geq \Delta(a,b)$ for $c \geq a$, $d \geq b$;

(T4) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$ for all a, b, c in $[0,1]$.

Examples of t-norms are $\Delta(a,b) = \min\{a,b\}$, $\Delta(a,b) = ab$, and $\Delta(a,b) = \max\{a+b-1,0\}$.

Definition 2 (see [16]). A real valued function f on the set of real numbers is called a distribution function if it is nondecreasing, left continuous with $\inf_{u\in\mathbb{R}} f(u) = 0$ and $\sup_{u\in\mathbb{R}} f(u) = 1$.

We shall denote by $\mathfrak{I}$ the set of all distribution functions defined on $(-\infty,\infty)$, while $H(t)$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases} \quad (1)$$

If X is a nonempty set, $\mathcal{F} : X \times X \to \mathfrak{I}$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x,y) \in X \times X$ is represented by $F_{x,y}$.

Definition 3 (see [17]). A probabilistic metric space is an ordered pair $(X,\mathcal{F})$, where X is a nonempty set of elements and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;

(2) $F_{x,y}(t) = F_{y,x}(t)$;

(3) $F_{x,y}(0) = 0$;

(4) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t + s) = 1$ for all $x, y, z \in X$ and $t, s \geq 0$.

Every metric space (X, d) can always be realized as a probabilistic metric space by considering $\mathcal{F} : X \times X \to \mathfrak{I}$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$ and $t \in R$. So probabilistic metric spaces offer a wider framework than that of metric spaces and are better suited to cover even wider statistical situations; that is, every metric space can be regarded as a probabilistic metric space of a special kind.

Definition 4 (see [16]). A Menger space $(X, \mathcal{F}, \Delta)$ is a triplet, where $(X, \mathcal{F})$ is a probabilistic metric space and Δ is a t-norm satisfying the following condition:

$$F_{x,y}(t + s) \geq \Delta\left(F_{x,z}(t), F_{z,y}(s)\right), \tag{2}$$

for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 5 (see [2]). A pair (A, S) of self-mappings defined on a nonempty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points; that is, if $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

Definition 6 (see [10]). A pair (A, S) of self-mappings defined on a nonempty set X is called conversely commuting if, for all $x \in X$, $ASx = SAx$ implies $Ax = Sx$.

Definition 7 (see [10]). Let A and S be self-mappings of a nonempty set X. A point $x \in X$ is called commuting point of A and S if $ASx = SAx$.

Lemma 8 (see [18]). *Let $(X, \mathcal{F}, \Delta)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that*

$$F_{x,y}(kt) \geq F_{x,y}(t), \tag{3}$$

for all $t > 0$ with fixed $x, y \in X$, then $x = y$.

3. Implicit Relations

In 2005, Singh and Jain [19] studied an implicit function and obtained some fixed point results in framework of fuzzy metric spaces.

Let Φ be the set of all real continuous functions $\phi : [0, 1]^4 \to \mathbb{R}$, nondecreasing in first argument and satisfying the following conditions.

(ϕ-1) For $u, v \geq 0$, $\phi(u, v, u, v) \geq 0$ or $\phi(u, v, v, u) \geq 0$ implies that $u \geq v$.

(ϕ-2) $\phi(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

Example 9. Define $\phi : [0, 1]^4 \to \mathbb{R}$ as $\phi(t_1, t_2, t_3, t_4) = 18t_1 - 16t_2 + 8t_3 - 10t_4$. Then $\phi \in \Phi$.

Since then, Imdad and Ali [20] used the following class of implicit functions for the existence of a common fixed point due to Popa [21]. Many authors proved a number of common fixed point theorems using the notion of implicit relation on different spaces (see, e.g., [22–29]).

Let Ψ denote the family of all continuous functions $\psi : [0, 1]^4 \to \mathbb{R}$ satisfying the following conditions.

(ψ-1) For every $u > 0, v \geq 0$ with $\psi(u, v, u, v) \geq 0$ or $\psi(u, v, v, u) \geq 0$, we have $u > v$.

(ψ-2) $\psi(u, u, 1, 1) < 0$ for all $u > 0$.

Example 10. Define $\psi : [0, 1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - \varphi(\min\{t_2, t_3, t_4\})$, where $\varphi : [0, 1] \to [0, 1]$ is a continuous function such that $\varphi(s) > s$ for $0 < s < 1$.

Example 11. Define $\psi : [0, 1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - k\min\{t_2, t_3, t_4\}$, where $k > 1$.

Example 12. Define $\psi : [0, 1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - kt_2 - \min\{t_3, t_4\}$, where $k > 0$.

Example 13. Define $\psi : [0, 1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - at_2 - bt_3 - ct_4$, where $a > 1$ and $b, c \geq 0$ $(b, c \neq 1)$.

Example 14. Define $\psi : [0, 1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - at_2 - b(t_3 + t_4)$, where $a > 1$ and $0 \leq b < 1$.

Example 15. Define $\psi : [0, 1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1^3 - kt_2t_3t_4$, where $k > 1$.

In 2011, Gopal et al. [30] showed that the above-mentioned classes of functions Φ and Ψ are independent classes.

4. Main Results

First, we prove a unique common fixed point theorem for two pairs of self-mappings satisfying a class of implicit function Ψ.

Theorem 16. *Let A, B, S, and T be four self-mappings of a Menger space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm and the pairs (A, S) and (B, T) are conversely commuting, respectively, and satisfy the following conditions:*

$$\psi\left(F_{Ax,By}(t), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\right) \geq 0, \tag{4}$$

for all $x, y \in X$, $t > 0$, and $\psi \in \Psi$. If A and S have a commuting point and B and T have a commuting point, then A, B, S, and T have a unique common fixed point in X.

Proof. Let u be the commuting point of A and S. Then $ASu = SAu$. And let v be the commuting point of B and T. Then $BTv = TBv$. Since A and S are conversely commuting, we have $Au = Su$. Since B and T are conversely commuting, we have $Bv = Tv$. Hence $AAu = ASu = SAu = SSu$ and $BBv = BTv = TBv = TTv$.

(i) We claim that $Au = Bv$. On using (4) with $x = u$, $y = v$, we get

$$\psi\left(F_{Au,Bv}(t), F_{Su,Tv}(t), F_{Au,Su}(t), F_{Bv,Tv}(t)\right) \geq 0, \quad (5)$$

or, equivalently,

$$\psi\left(F_{Au,Bv}(t), F_{Au,Bv}(t), 1, 1\right) \geq 0. \quad (6)$$

Hence, for $F_{Au,Bv}(t) = 1$ for all $t > 0$, we have $Au = Bv$. Thus $Au = Su = Bv = Tv$.

(ii) Now, we show that Au is a fixed point of mapping A. In order to establish this, using (4) with $x = Au$, $y = v$, we have

$$\psi\left(F_{AAu,Bv}(t), F_{SAu,Tv}(t), F_{AAu,SAu}(t), F_{Bv,Tv}(t)\right) \geq 0, \quad (7)$$

and so

$$\psi\left(F_{AAu,Au}(t), F_{AAu,Au}(t), 1, 1\right) \geq 0. \quad (8)$$

Hence, for $F_{AAu,Au}(t) = 1$ for all $t > 0$, we get $AAu = Au$. Similarly we show that $Bv = BBv$. On using (4) with $x = u$, $y = Bv$, we have

$$\psi\left(F_{Au,BBv}(t), F_{Su,TBv}(t), F_{Au,Su}(t), F_{BBv,TBv}(t)\right) \geq 0, \quad (9)$$

or, equivalently,

$$\psi\left(F_{Bv,BBv}(t), F_{Bv,BBv}(t), 1, 1\right) \geq 0. \quad (10)$$

Thus, $F_{AAu,Au}(t) = 1$ for all $t > 0$ and we obtain $BBv = Bv$. Since $Au = Bv$, we have $Au = Bv = BBv = BAu$ which shows that Au is a fixed point of the mapping B. On the other hand, $Au = Bv = BBv = TBv = TAu$ and $Au = AAu = ASu = SAu$. Hence $Au(= w \in X)$ is a common fixed point of A, B, S, and T.

(iii) For the uniqueness of common fixed point, we use (4) with $x = w$ and $y = \hat{u}$, where $\hat{u}$ is another common fixed point of the mappings A, B, S, and T. Now we have

$$\psi\left(F_{Aw,B\hat{u}}(t), F_{Sw,T\hat{u}}(t), F_{Aw,Sw}(t), F_{B\hat{u},T\hat{u}}(t)\right) \geq 0, \quad (11)$$

and so

$$\psi\left(F_{w,\hat{u}}(t), F_{w,\hat{u}}(t), 1, 1\right) \geq 0. \quad (12)$$

Hence, we get $w = \hat{u}$. Therefore, w is a unique common fixed point of the mappings A, B, S, and T.

□

Now, we give an example which illustrates Theorem 16.

Example 17. Let $X = [1, \infty)$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$, define

$$F_{x,y}(t) = \begin{cases} \dfrac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases} \quad (13)$$

for all $x, y \in X$. Define $\Delta(a, b) = \min\{a, b\}$. Clearly $(X, \mathscr{F}, \Delta)$ is a Menger space. Let $\psi : [0, 1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - \varphi(\min\{t_2, t_3, t_4\})$ with $\varphi(s) = \sqrt{s}$ for $0 < s < 1$. Define the self-mappings A, B, S, and T by

$$A(x) = \begin{cases} 2x - 1, & \text{if } x < 2; \\ 1, & \text{if } x \geq 2, \end{cases}$$

$$S(x) = \begin{cases} x^2, & \text{if } x < 2; \\ x + 3, & \text{if } x \geq 2, \end{cases}$$

$$B(x) = \begin{cases} 2x - 1, & \text{if } x < 2; \\ 2, & \text{if } x \geq 2, \end{cases} \quad (14)$$

$$T(x) = \begin{cases} 3x^2 - 2, & \text{if } x < 2; \\ x^2 + 1, & \text{if } x \geq 2. \end{cases}$$

Hence the pairs (A, S) and (B, T) are conversely commuting and 1 is a unique common fixed point of the mappings A, B, S, and T.

Corollary 18. *The conclusions of Theorem 16 remain true if condition* (4) *is replaced by one of the following conditions: for all* $x, y \in X$

$$F_{Ax,By}(t) \geq \varphi\left(\min\left\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\right\}\right), \quad (15)$$

where $\varphi : [0, 1] \to [0, 1]$ *is a continuous function such that* $\varphi(s) > s$ *for all* $0 < s < 1$;

$$F_{Ax,By}(t) \geq k\left(\min\left\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\right\}\right), \quad (16)$$

where $k > 1$;

$$F_{Ax,By}(t) \geq kF_{Sx,Ty}(t) + \min\left\{F_{Ax,Sx}(t), F_{By,Ty}(t)\right\}, \quad (17)$$

where $k > 0$;

$$F_{Ax,By}(t) \geq aF_{Sx,Ty}(t) + bF_{Ax,Sx}(t) + cF_{By,Ty}(t), \quad (18)$$

where $a > 1$ *and* $b, c \geq 0$ $(b, c \neq 1)$;

$$F_{Ax,By}(t) \geq aF_{Sx,Ty}(t) + b\left[F_{Ax,Sx}(t) + F_{By,Ty}(t)\right], \quad (19)$$

where $a > 1$ *and* $0 \leq b < 1$;

$$F_{Ax,By}(t) \geq kF_{Sx,Ty}(t)\, F_{Ax,Sx}(t)\, F_{By,Ty}(t), \quad (20)$$

where $k > 1$.

Proof. The proof of each inequality (15)–(20) easily follows from Theorem 16 in view of Examples 10–15. □

Now we state a unique common fixed point theorem satisfying a class of implicit function Φ.

Theorem 19. *Let A, B, S, and T be four self-mappings on a Menger space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm, the pairs (A, S) and (B, T) are conversely commuting, respectively, and satisfying*

$$\phi\left(F_{Ax,By}(kt), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(kt)\right) \geq 0,$$
$$\phi\left(F_{Ax,By}(kt), F_{Sx,Ty}(t), F_{Ax,Sx}(kt), F_{By,Ty}(t)\right) \geq 0, \tag{21}$$

for all $x, y \in X$, $t > 0$, $k \in (0,1)$ and $\phi \in \Phi$. If A and S have a commuting point and B and T have a commuting point, then A, B, S, and T have a unique common fixed point in X.

Proof. The proof of this theorem can be completed on the lines of the proof of Theorem 16 (in view of Lemma 8); hence details are omitted. □

Example 20. In the setting of Example 17, define $\phi : [0,1]^4 \to \mathbb{R}$ as $\phi(t_1, t_2, t_3, t_4) = 18t_1 - 16t_2 - 8t_3 + 10t_4$ besides retaining the rest. Therefore, all the conditions of Theorem 19 are satisfied for some fixed $k \in (0,1)$ and 1 is a unique common fixed point of the mappings A, B, S, and T.

By choosing A, B, S, and T suitably, we can deduce corollaries involving two as well as three self-mappings. For the sake of naturality, we only derive the following corollary (due to Theorem 16) involving a pair of self-mappings.

Corollary 21. *Let A and S be two self-mappings of a Menger space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm and the mappings A and S are conversely commuting satisfying*

$$\psi\left(F_{Ax,Ay}(t), F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t)\right) \geq 0, \tag{22}$$

for all $x, y \in X$, $t > 0$, and $\psi \in \Psi$. If A and S have a commuting point, then A and S have a unique common fixed point in X.

Conflict of Interests

The authors declare that they have no conflict of interests.

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