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Research Article

Positive Periodic Solutions of Nicholson-Type Delay Systems with Nonlinear Density-Dependent Mortality Terms

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This paper is concerned with the periodic solutions for a class of Nicholson-type delay systems with nonlinear density-dependent mortality terms. By using coincidence degree theory, some criteria are obtained to guarantee the existence of positive periodic solutions of the model. Moreover, an example and a numerical simulation are given to illustrate our main results.

1. Introduction

In the last twenty years, the delay differential equations have been widely studied both in a theoretical context and in that of related applications [1–4]. As a famous and common delay dynamic system, Nicholson's blowflies model and its modifications have made remarkable progress that has been collected in [5] and the references cited there in. Recently, to describe the dynamics for the models of marine protected areas and B-cell chronic lymphocytic leukemia dynamics which belong to the Nicholson-type delay differential systems, Berezansky et al. [6], Wang et al. [7], and Liu [8] studied the problems on the permanence, stability, and periodic solution of the following Nicholson-type delay systems:

$$N_1'(t) = -\alpha_1(t)N_1(t) + \beta_1(t)N_2(t) + \sum_{j=1}^{m} c_{1j}(t)N_1\left(t - \tau_1 j(t)\right)e^{-\gamma_{1j}(t)N_1(t-\tau_{1j}(t))},$$

$$N_2'(t) = -\alpha_2(t)N_2(t) + \beta_2(t)N_1(t) + \sum_{j=1}^{m} c_{2j}(t)N_2\big(t-\tau_2 j(t)\big)e^{-\gamma_{2j}(t)N_2(t-\tau_{2j}(t))}, \tag{1.1}$$

where $\alpha_i, \beta_i, c_{ij}, \gamma_{ij}, \tau_{ij} \in C(R, (0+\infty))$, and $i = 1,2$, $j = 1,2,\ldots,m$.

In [5], Berezansky et al. also pointed out that a new study indicates that a linear model of density-dependent mortality will be most accurate for populations at low densities and marine ecologists are currently in the process of constructing new fishery models with nonlinear density-dependent mortality rates. Consequently, Berezansky et al. [5] presented an open problem: to reveal the dynamics of the following Nicholson's blowflies model with a nonlinear density-dependent mortality term:

$$N'(t) = -D(N(t)) + PN(t-\tau)e^{-aN(t-\tau)}, \tag{1.2}$$

where P is a positive constant and function D might have one of the following forms: $D(N) = aN/(N+b)$ or $D(N) = a - be^{-N}$ with positive constants $a, b > 0$.

Most recently, based upon the ideas in [5–8], Liu and Gong [9] established the results on the permanence for the Nicholson-type delay system with nonlinear density-dependent mortality terms. Consequently, the problem on periodic solutions of Nicholson-type system with $D(N) = a - be^{-N}$ has been studied extensively in [10–13]. However, to the best of our knowledge, there exist few results on the existence of the positive periodic solutions of Nicholson-type delay system with $D(N) = aN/(N+b)$. Motivated by this, the main purpose of this paper is to give the conditions to guarantee the existence of positive periodic solutions of the following Nicholson-type delay system with nonlinear density-dependent mortality terms:

$$\begin{aligned} N_1'(t) &= -D_{11}(t, N_1(t)) + D_{12}(t, N_2(t)) + \sum_{j=1}^{l} c_{1j}(t)N_1\big(t-\tau_{1j}(t)\big)e^{-\gamma_{1j}(t)N_1(t-\tau_{1j}(t))}, \\ N_2'(t) &= -D_{22}(t, N_2(t)) + D_{21}(t, N_1(t)) + \sum_{j=1}^{l} c_{2j}(t)N_2\big(t-\tau_{2j}(t)\big)e^{-\gamma_{2j}(t)N_2(t-\tau_{2j}(t))}, \end{aligned} \tag{1.3}$$

under the admissible initial conditions

$$x_{t_0} = \varphi, \qquad \varphi \in C_+ = C\Big([-r_1, 0], R_+^1\Big) \times C\Big([-r_2, 0], R_+^1\Big), \qquad \varphi_i(0) > 0, \tag{1.4}$$

where $D_{ij}(t, N) = a_{ij}(t)N/(b_{ij}(t) + N)$, $a_{ij}, b_{ij}, c_{ik}, \gamma_{ik} : R \to (0, +\infty)$ and $\tau_{ik} : R \to [0, +\infty)$ are all bounded continuous functions, and $r_i = \max_{1\le k\le l}\{\sup_{t\in R^1}\tau_{ik}(t)\}$, $i, j = 1,2$, $k = 1,\ldots,l$.

For convenience, we introduce some notations. Throughout this paper, given a bounded continuous function g defined on R^1, let g^+ and g^- be defined as

$$g^- = \inf_{t\in R^1} g(t), \qquad g^+ = \sup_{t\in R^1} g(t). \tag{1.5}$$

We also assume that $a_{ij}, b_{ij}, c_{ik}, \gamma_{ik} : R \to (0, +\infty)$ and $\tau_{ik} : R \to [0, +\infty)$ are all ω-periodic functions, $r_i = \max_{1 \le k \le l}\{\tau_{ik}^{+}\}$, and $i, j = 1, 2$, $k = 1, \ldots, l$.

Set

$$A_i = 2\int_0^{\omega} \frac{a_{ii}(t)}{b_{ii}(t)}dt, \qquad B_i = \sum_{j=1}^{l}\int_0^{\omega} c_{ij}(t)dt, \qquad \gamma_i^{+} = \max_{1\le j\le l}\left\{\gamma_{ij}^{+}\right\}, \qquad \gamma_i^{-} = \min_{1\le j\le l}\left\{\gamma_{ij}^{-}\right\},$$
$$D_1 = \int_0^{\omega} a_{12}(t)dt, \qquad D_2 = \int_0^{\omega} a_{21}(t)dt, \qquad C_i = \int_0^{\omega} a_{ii}(t)dt, \quad i = 1, 2. \tag{1.6}$$

Let $R^n(R^n_+)$ be the set of all (nonnegative) real vectors; we will use $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ to denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. We let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$ and define $\|x\| = \max_{1\le i\le n}|x_i|$. For matrix $A = (a_{ij})_{n\times n}$, A^T denotes the transpose of A. A matrix or vector $A \ge 0$ means that all entries of A are greater than or equal to zero. $A > 0$ can be defined similarly. For matrices or vectors A and B, $A \ge B$ (resp. $A > B$) means that $A - B \ge 0$ (resp. $A - B > 0$). We also define the derivative and integral of vector function $x(t) = (x_1(t), x_2(t))^T$ as $x' = (x_1'(t), x_2'(t))^T$ and $\int_0^{\omega} x(t)dt = (\int_0^{\omega} x_1(t)dt, \int_0^{\omega} x_2(t)dt)^T$.

The organization of this paper is as follows. In the next section, some sufficient conditions for the existence of the positive periodic solutions of model (1.3) are given by using the method of coincidence degree. In Section 3, an example and numerical simulation are given to illustrate our results obtained in the previous section.

2. Existence of Positive Periodic Solutions

In order to study the existence of positive periodic solutions, we first introduce the continuation theorem as follows.

Lemma 2.1 (continuation theorem [14]). *Let X and Z be two Banach spaces. Suppose that $L : D(L) \subset X \to Z$ is a Fredholm operator with index zero and $\widetilde{N} : X \to Z$ is L-compact on $\overline{\Omega}$, where Ω is an open subset of X. Moreover, assume that all the following conditions are satisfied:*

(1) *$Lx \ne \lambda\widetilde{N}x$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;*

(2) *$\widetilde{N}x \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \operatorname{Ker} L$;*

(3) *the Brouwer degree*

$$\deg\left\{Q\widetilde{N}, \Omega \cap \operatorname{Ker} L, 0\right\} \ne 0. \tag{2.1}$$

Then equation $Lx = \widetilde{N}x$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Our main result is given in the following theorem.

Theorem 2.2. *Suppose*

$$C_i > 2D_i, \qquad \ln \frac{2B_i}{A_i} > A_i, \quad i = 1,2, \tag{2.2}$$

$$\sum_{j=1}^{l} \frac{c_{1j}^{+}}{a_{11}^{-}\gamma_{1j}^{-}e} + \frac{a_{12}^{+}}{a_{11}^{-}} < 1, \qquad \sum_{j=1}^{l} \frac{c_{2j}^{+}}{a_{22}^{-}\gamma_{2j}^{-}e} + \frac{a_{21}^{+}}{a_{22}^{-}} < 1. \tag{2.3}$$

Then (1.3) *has a positive ω-periodic solution.*

Proof. Set $N(t) = (N_1(t), N_2(t))^T$ and $N_i(t) = e^{x_i(t)}$ $(i = 1,2)$. Then (1.3) can be rewritten as

$$\begin{aligned} x_1'(t) &= -\frac{a_{11}(t)}{b_{11}(t) + e^{x_1(t)}} + \frac{a_{12}(t)e^{x_2(t)-x_1(t)}}{b_{12}(t) + e^{x_2(t)}} \\ &\quad + \sum_{j=1}^{l} c_{1j}(t) e^{x_1(t-\tau_{1j}(t)) - x_1(t) - \gamma_{1j}(t) e^{x_1(t-\tau_{1j}(t))}} := \Delta_1(x,t), \\ x_2'(t) &= -\frac{a_{22}(t)}{b_{22}(t) + e^{x_2(t)}} + \frac{a_{21}(t)e^{x_1(t)-x_2(t)}}{b_{21}(t) + e^{x_1(t)}} \\ &\quad + \sum_{j=1}^{l} c_{2j}(t) e^{x_2(t-\tau_{2j}(t)) - x_2(t) - \gamma_{2j}(t) e^{x_2(t-\tau_{2j}(t))}} := \Delta_2(x,t). \end{aligned} \tag{2.4}$$

As usual, let $X = Z = \{x = (x_1(t), x_2(t))^T \in C(R, R^2) : x(t+\omega) = x(t) \text{ for all } t \in R\}$ be Banach spaces equipped with the supremum norm $\|\cdot\|$. For any $x \in X$, because of periodicity, it is easy to see that $\Delta(x,\cdot) = (\Delta_1(x,\cdot), \Delta_2(x,\cdot))^T \in C(R, R^2)$ is ω-periodic. Let

$$\begin{gathered} L : D(L) = \left\{ x \in X : x \in C^1\left(R, R^2\right) \right\} \ni x \longmapsto x' = \left(x_1', x_2'\right)^T \in Z, \\ P : X \ni x \longmapsto \left(\frac{1}{\omega} \int_0^{\omega} x_1(s)ds, \frac{1}{\omega} \int_0^{\omega} x_2(s)ds \right)^T \in X, \\ Q : Z \ni z \longmapsto \left(\frac{1}{\omega} \int_0^{\omega} z_1(s)ds, \frac{1}{\omega} \int_0^{\omega} z_2(s)ds \right)^T \in Z, \\ \widetilde{N} : X \ni x \longmapsto \Delta(x,\cdot) \in Z. \end{gathered} \tag{2.5}$$

It is easy to see that

$$\begin{gathered} \operatorname{Im} L = \left\{ x \mid x \in Z, \int_0^{\omega} x(s)ds = (0,0)^T \right\}, \qquad \operatorname{Ker} L = R^2, \\ \operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L. \end{gathered} \tag{2.6}$$

Thus, the operator L is a Fredholm operator with index zero. Furthermore, denoting by $L_P^{-1} : \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P$ the inverse of $L|_{D(L)\cap \operatorname{Ker} P}$, we have

$$\begin{aligned} L_P^{-1}y(t) &= -\frac{1}{\omega}\int_0^{\omega}\int_0^t y(s)ds\,dt + \int_0^t y(s)ds \\ &= \left(-\frac{1}{\omega}\int_0^{\omega}\int_0^t y_1(s)ds\,dt + \int_0^t y_1(s)ds, -\frac{1}{\omega}\int_0^{\omega}\int_0^t y_2(s)ds\,dt + \int_0^t y_2(s)ds\right)^T. \end{aligned} \tag{2.7}$$

It follows that

$$Q\widetilde{N}x = \frac{1}{\omega}\int_0^{\omega}\widetilde{N}x(t)dt = \left(\frac{1}{\omega}\int_0^{\omega}\Delta_1(x(t),t)dt, \frac{1}{\omega}\int_0^{\omega}\Delta_2(x(t),t)dt\right)^T, \tag{2.8}$$

$$\begin{aligned} L_P^{-1}(I-Q)\widetilde{N}x = \int_0^t \widetilde{N}x(s)ds - \frac{t}{\omega}\int_0^{\omega}\widetilde{N}x(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t \widetilde{N}x(s)ds\,dt \\ + \frac{1}{\omega}\int_0^{\omega}\int_0^t Q\widetilde{N}x(s)ds\,dt. \end{aligned} \tag{2.9}$$

Obviously, $Q\widetilde{N}$ and $L_P^{-1}(I-Q)\widetilde{N}$ are continuous. It is not difficult to show that $L_P^{-1}(I-Q)\widetilde{N}(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $Q\widetilde{N}(\overline{\Omega})$ is clearly bounded. Thus $\widetilde{N}$ is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Considering the operator equation $Lx = \lambda\widetilde{N}x, \lambda \in (0,1)$, we have

$$x'(t) = \left(x_1'(t), x_2'(t)\right)^T = \lambda\Delta(x,t) = (\lambda\Delta_1(x,t), \lambda\Delta_2(x,t))^T. \tag{2.10}$$

Suppose that $x = (x_1(t), x_2(t))^T \in X$ is a solution of (2.10) for some $\lambda \in (0,1)$.

Firstly, we claim that there exists a positive number H such that $\|x\| < H$. Integrating the first equation of (2.10) and in view of $x \in X$, it results that

$$0 = \int_0^{\omega} x_1'(t)dt = \lambda\int_0^{\omega}\Delta_1(x,t)dt, \tag{2.11}$$

which together with (2.4) implies that

$$\begin{aligned} &\int_0^{\omega}\left|\sum_{j=1}^{l} c_{1j}(t)e^{x_1(t-\tau_{1j}(t))-x_1(t)-\gamma_{1j}(t)e^{x_1(t-\tau_{1j}(t))}} + \frac{a_{12}(t)e^{x_2(t)-x_1(t)}}{b_{12}(t)+e^{x_2(t)}}\right|dt \\ &\quad = \int_0^{\omega}\frac{a_{11}(t)}{b_{11}(t)+e^{x_1(t)}}dt \\ &\quad < \int_0^{\omega}\frac{a_{11}(t)}{b_{11}(t)}dt. \end{aligned} \tag{2.12}$$

Similarly, we have

$$\begin{aligned}
&\int_0^{\omega}\left|\sum_{j=1}^{l} c_{2j}(t)e^{x_2(t-\tau_{2j}(t))-x_2(t)-\gamma_{2j}(t)e^{x_2(t-\tau_{2j}(t))}}+\frac{a_{21}(t)e^{x_1(t)-x_2(t)}}{b_{21}(t)+e^{x_1(t)}}\right|dt \\
&\quad=\int_0^{\omega}\frac{a_{22}(t)}{b_{22}(t)+e^{x_2(t)}}dt \\
&\quad<\int_0^{\omega}\frac{a_{22}(t)}{b_{22}(t)}dt.
\end{aligned}\tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\begin{aligned}
\int_0^{\omega}\left|x_1'(t)\right|dt &\le \lambda\int_0^{\omega}\left|\sum_{j=1}^{l} c_{1j}(t)e^{x_1(t-\tau_{1j}(t))-x_1(t)-\gamma_{1j}(t)e^{x_1(t-\tau_{1j}(t))}}+\frac{a_{12}(t)e^{x_2(t)-x_1(t)}}{b_{12}(t)+e^{x_2(t)}}\right|dt \\
&\quad+\lambda\int_0^{\omega}\left|\frac{a_{11}(t)}{b_{11}(t)+e^{x_1(t)}}\right|dt \\
&< 2\int_0^{\omega}\frac{a_{11}(t)}{b_{11}(t)}dt = A_1,
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
\int_0^{\omega}\left|x_2'(t)\right|dt &\le \lambda\int_0^{\omega}\left|\sum_{j=1}^{l} c_{2j}(t)e^{x_2(t-\tau_{2j}(t))-x_2(t)-\gamma_{2j}(t)e^{x_2(t-\tau_{2j}(t))}}+\frac{a_{21}(t)e^{x_1(t)-x_2(t)}}{b_{21}(t)+e^{x_1(t)}}\right|dt \\
&\quad+\lambda\int_0^{\omega}\left|\frac{a_{22}(t)}{b_{22}(t)+e^{x_2(t)}}\right|dt \\
&< 2\int_0^{\omega}\frac{a_{22}(t)}{b_{22}(t)}dt = A_2.
\end{aligned}\tag{2.15}$$

Since $x \in X$, there exist $\xi_1, \xi_2, \eta_1, \eta_2 \in [0, \omega]$ such that

$$x_i(\xi_i) = \min_{t\in[0,\omega]} x_i(t), \quad x_i(\eta_i) = \max_{t\in[0,\omega]} x_i(t), \quad x_i'(\xi_i) = x_i'(\eta_i) = 0, \quad i = 1, 2. \tag{2.16}$$

It follows from (2.12) and (2.14) that

$$\begin{aligned}
\frac{A_1}{2} &= \int_0^{\omega}\frac{a_{11}(t)}{b_{11}(t)}dt > \int_0^{\omega}\frac{a_{11}(t)}{b_{11}(t)+e^{x_1(t)}}dt \\
&= \int_0^{\omega}\sum_{j=1}^{l} c_{1j}(t)e^{x_1(t-\tau_{1j}(t))-x_1(t)-\gamma_{1j}(t)e^{x_1(t-\tau_{1j}(t))}}dt + \int_0^{\omega}\frac{a_{12}(t)e^{x_2(t)-x_1(t)}}{b_{12}(t)+e^{x_2(t)}}dt
\end{aligned}$$

$$\begin{aligned} &> e^{x_1(\xi_1)-x_1(\eta_1)-\gamma_1^+ e^{x_1(\eta_1)}} \sum_{j=1}^{l} \int_0^{\omega} c_{1j}(t)dt \\ &= B_1 e^{x_1(\xi_1)-x_1(\eta_1)-\gamma_1^+ e^{x_1(\eta_1)}}, \end{aligned} \tag{2.17}$$

which implies that

$$x_1(\xi_1) < \ln \frac{A_1}{2B_1} + x_1(\eta_1) + \gamma_1^+ e^{x_1(\eta_1)}. \tag{2.18}$$

Using (2.14) yields

$$x_1(t) \leq x_1(\xi_1) + \int_0^{\omega} \left|x_1'(t)\right| dt < \ln \frac{A_1}{2B_1} + x_1(\eta_1) + \gamma_1^+ e^{x_1(\eta_1)} + A_1, \quad t \in [0, \omega]. \tag{2.19}$$

In particular,

$$x_1(\eta_1) < x_1(\xi_1) + \int_0^{\omega} \left|x_1'(t)\right| dt < \ln \frac{A_1}{2B_1} + x_1(\eta_1) + \gamma_1^+ e^{x_1(\eta_1)} + A_1. \tag{2.20}$$

It follows that

$$x_1(\eta_1) > \ln\left(\frac{1}{\gamma_1^+} \left(\ln \frac{2B_1}{A_1} - A_1 \right) \right). \tag{2.21}$$

Again from (2.14), we have

$$x_1(t) \geq x_1(\eta_1) - \int_0^{\omega} \left|x_1'(t)\right| dt > \ln\left(\frac{1}{\gamma_1^+} \left(\ln \frac{2B_1}{A_1} - A_1 \right) \right) - A_1 := H_{11}, \quad t \in [0, \omega]. \tag{2.22}$$

Similarly, we can obtain

$$x_2(t) \geq x_2(\eta_2) - \int_0^{\omega} \left|x_2'(t)\right| dt > \ln\left(\frac{1}{\gamma_2^+} \left(\ln \frac{2B_2}{A_2} - A_2 \right) \right) - A_2 := H_{21}, \quad t \in [0, \omega]. \tag{2.23}$$

Since $x_1'(\xi_1) = 0$, from (2.10), we have

$$\begin{aligned} \frac{a_{11}(\xi_1)}{b_{11}(\xi_1) + e^{x_1(\xi_1)}} &= \sum_{j=1}^{l} c_{1j}(\xi_1) e^{x_1(\xi_1-\tau_{1j}(\xi_1))-x_1(\xi_1)-\gamma_{1j}(\xi_1)e^{x_1(\xi_1-\tau_{1j}(\xi_1))}} \\ &\quad + \frac{a_{12}(\xi_1) e^{x_2(\xi_1)-x_1(\xi_1)}}{b_{12}(\xi_1) + e^{x_2(\xi_1)}}. \end{aligned} \tag{2.24}$$

Hence, from (2.24) and the fact that $\sup_{u\geq 0} ue^{-u} = 1/e$, we have

$$\begin{aligned}
\frac{e^{x_1(\xi_1)}}{b_{11}^{+} + e^{x_1(\xi_1)}} &\leq \frac{e^{x_1(\xi_1)}}{b_{11}(\xi_1) + e^{x_1(\xi_1)}} \\
&= \sum_{j=1}^{l} \frac{c_{1j}(\xi_1)}{a_{11}(\xi_1)\gamma_{1j}(\xi_1)} \gamma_{1j}(\xi_1) e^{x_1(\xi_1-\tau_{1j}(\xi_1))} e^{-\gamma_{1j}(\xi_1) e^{x_1(\xi_1-\tau_{1j}(\xi_1))}} \\
&\quad + \frac{a_{12}(\xi_1)}{a_{11}(\xi_1)\left(1 + b_{12}(\xi_1) e^{-x_2(\xi_1)}\right)} \\
&< \sum_{j=1}^{l} \frac{c_{1j}^{+}}{a_{11}^{-}\gamma_{1j}^{-} e} + \frac{a_{12}^{+}}{a_{11}^{-}}.
\end{aligned} \tag{2.25}$$

Noting that $u/(b_{11}^{+} + u)$ is strictly monotone increasing on $[0, +\infty)$ and

$$\sup_{u\geq 0} \frac{u}{b_{11}^{+} + u} = 1 > \sum_{j=1}^{l} \frac{c_{1j}^{+}}{a_{11}^{-}\gamma_{1j}^{-} e} + \frac{a_{12}^{+}}{a_{11}^{-}}, \tag{2.26}$$

it is clear that there exists a constant $k_1 > 0$ such that

$$\frac{u}{b_{11}^{+} + u} > \sum_{j=1}^{l} \frac{c_{1j}^{+}}{a_{11}^{-}\gamma_{1j}^{-} e} + \frac{a_{12}^{+}}{a_{11}^{-}} \quad \forall u \in [k_1, +\infty). \tag{2.27}$$

In view of (2.25) and (2.27), we get

$$e^{x_1(\xi_1)} \leq k_1, \qquad x_1(\xi_1) \leq \ln k_1. \tag{2.28}$$

In the same way, there exists a constant $k_2 > 0$ such that

$$x_2(\xi_2) \leq \ln k_2. \tag{2.29}$$

Again from (2.14), (2.15), (2.28), and (2.29), we get

$$\begin{aligned}
x_1(t) &\leq x_1(\xi_1) + \int_0^{\omega} \left| x_1'(t) \right| dt < \ln k_1 + A_1, \\
x_2(t) &\leq x_2(\xi_1) + \int_0^{\omega} \left| x_2'(t) \right| dt < \ln k_1 + A_2.
\end{aligned} \tag{2.30}$$

Then, we can choose two sufficiently large positive constants $H_{12} > \ln k_1 + A_1$ and $H_{22} > \ln k_2 + A_2$ such that

$$x_1(t) < H_{12}, \qquad x_2(t) < H_{22}, \qquad \ln b_{11}^{+} < H_{12}, \qquad \ln b_{22}^{+} < H_{22}. \tag{2.31}$$

Let $H > \max\{|H_{11}|, |H_{21}|, H_{12}, H_{22}\}$ be a fix constant such that

$$e^H > \frac{1}{\gamma_1^-}\left(H - \ln\frac{C_1 - 2D_1}{2B_1}\right), \qquad e^H > \frac{1}{\gamma_2^-}\left(H - \ln\frac{C_2 - 2D_2}{2B_2}\right). \tag{2.32}$$

Then (2.22), (2.23), and (2.31) imply that $\|x\| < H$, if $x \in X$ is solution of (2.10). So we can define an open bounded set as $\Omega = \{x \in X : \|x\| < H\}$ such that there is no $\lambda \in (0,1)$ and $x \in \partial\Omega$ such that $Lx = \lambda\widetilde{N}x$. That is to say $Lx \neq \lambda\widetilde{N}x$ for all $x \in \partial\Omega \cap D(L), \lambda \in (0,1)$.

Secondly, we prove that $\widetilde{N}x \notin \operatorname{Im} L$ for all $x \in \partial\Omega \cap \operatorname{Ker} L$. That is $((Q\widetilde{N}(x))_1, (Q\widetilde{N}(x))_2)^T \neq (0,0)^T$ for all $x \in \partial\Omega \cap \operatorname{Ker} L$.

If $x(t) = (x_1(t), x_2(t))^T \in \partial\Omega \cap \operatorname{Ker} L$, then $x(t)$ is a constant vector in R^2, and there exists some $i \in \{1,2\}$ such that $|x_i| = H$. Assume $|x_1| = H$, so that $x_1 = \pm H$. Then, we claim

$$\left(Q\widetilde{N}(x)\right)_1 > 0 \quad \text{for } x_1 = -H, \qquad \left(Q\widetilde{N}(x)\right)_1 < 0 \quad \text{for } x_1 = H. \tag{2.33}$$

If $(Q\widetilde{N}(x))_1 \leq 0$ for $x_1 = -H$, it follows from (2.2) and (2.8) that

$$\int_0^\omega \Delta_1(x,t)dt \leq 0, \quad \text{for } x_1 = -H. \tag{2.34}$$

Hence,

$$\begin{aligned}
\frac{A_1}{2} &= \int_0^\omega \frac{a_{11}(t)}{b_{11}(t)}dt \\
&> \int_0^\omega \frac{a_{11}(t)}{b_{11}(t) + e^{-H}}dt \\
&\geq \int_0^\omega \left[\sum_{j=1}^{l} c_{1j}(t)e^{-\gamma_{1j}(t)e^{-H}} + \frac{a_{12}(t)e^{x_2+H}}{b_{12}(t) + e^{x_2}}\right]dt \\
&> \int_0^\omega \sum_{j=1}^{l} c_{1j}(t)e^{-\gamma_{1j}^+e^{-H}}dt \\
&\geq e^{-\gamma_1^+e^{-H}} \sum_{j=1}^{l} \int_0^\omega c_{1j}(t)dt \\
&= B_1 e^{-\gamma_1^+e^{-H}},
\end{aligned} \tag{2.35}$$

which implies

$$-H > \ln\left(\frac{1}{\gamma_1^+}\ln\frac{2B_1}{A_1}\right) > \ln\left(\frac{1}{\gamma_1^+}\left(\ln\frac{2B_1}{A_1} - A_1\right)\right) - A_1 = H_{11}. \tag{2.36}$$

This is a contradiction and implies that $(Q\widetilde{N}(x))_1 > 0$ for $x_1 = -H$.

If $(Q\widetilde{N}(x))_1 \geq 0$ for $x_1 = H$, it follows from (2.2) and (2.8) that

$$\begin{aligned}
&\int_0^\omega \Delta_1(x,t)dt \geq 0, \quad \text{for } x_1 = H, \\
\frac{C_1}{2}e^{-H} &= \int_0^\omega \frac{a_{11}(t)}{2e^H}dt \\
&< \int_0^\omega \frac{a_{11}(t)}{b_{11}(t)+e^H}dt \\
&\leq \int_0^\omega \sum_{j=1}^{l} c_{1j}(t)e^{-\gamma_{1j}(t)e^H}dt + \int_0^\omega \frac{a_{12}(t)e^{x_2-H}}{b_{12}(t)+e^{x_2}}dt \\
&\leq \int_0^\omega \sum_{j=1}^{l} c_{1j}(t)e^{-\gamma_1^- e^H}dt + \int_0^\omega \frac{a_{12}(t)}{b_{12}(t)e^{H-x_2}+e^H}dt \\
&< e^{-\gamma_1^- e^H}\sum_{j=1}^{l}\int_0^\omega c_{1j}(t)dt + \int_0^\omega \frac{a_{12}(t)}{e^H}dt \\
&= B_1 e^{-\gamma_1^- e^H} + D_1 e^{-H}.
\end{aligned} \tag{2.37}$$

Consequently,

$$e^H < \frac{1}{\gamma_1^-}\left(H - \ln\frac{C_1 - 2D_1}{2B_1}\right), \tag{2.38}$$

a contradiction to the choice of H. Thus, $(Q\widetilde{N}(x))_1 < 0$ for $x_1 = H$.

Similarly, if $|x_2| = H$, we obtain

$$\left(Q\widetilde{N}(x)\right)_2 > 0 \quad \text{for } x_2 = -H, \qquad \left(Q\widetilde{N}(x)\right)_2 < 0 \quad \text{for } x_2 = H. \tag{2.39}$$

Consequently, (2.33) and (2.39) imply that $\widetilde{N}x \notin \operatorname{Im} L$ for all $x \in \partial\Omega \cap \operatorname{Ker} L$.

Furthermore, let $0 \leq \mu \leq 1$ and define continuous function $H(x,\mu)$ by setting

$$H(x,\mu) = -(1-\mu)x + \mu Q\widetilde{N}x. \tag{2.40}$$

For all $x(t) = (x_1(t), x_2(t))^T \in \partial\Omega \cap \operatorname{Ker} L$, then there exists some $i \in \{1,2\}$ such that $|x_i| = H$. There are two cases: $x_1 = \pm H$ or $x_2 = \pm H$. When $x_1 = H$ or $x_2 = H$, from (2.33) and (2.39), it is obvious that $(H(x,\mu))_1 < 0$ or $(H(x,\mu))_2 < 0$. Similarly, if $x_1 = -H$ or $x_2 = -H$, it results that $(H(x,\mu))_1 > 0$ or $(H(x,\mu))_2 > 0$. Hence $H(x,\mu) \neq (0,0)^T$ for all $x \in \partial\Omega \cap \ker L$.

Finally, using the homotopy invariance theorem, we obtain

$$\deg\left\{Q\widetilde{N}, \Omega \cap \ker L, (0,0)^T\right\} = \deg\left\{-x, \Omega \cap \ker L, (0,0)^T\right\} \neq 0. \tag{2.41}$$

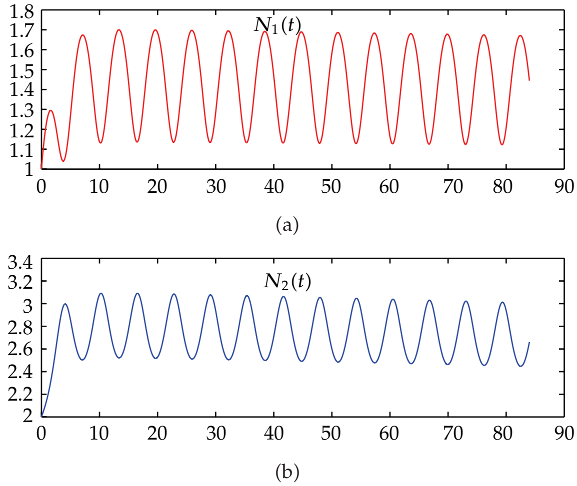

Figure 1: Numerical solution $N(t) = (N_1(t), N_2(t))^T$ of systems (3.1) for initial value $\varphi(t) \equiv (1,2)^T$.

It then follows from the continuation theorem that $Lx = \widetilde{N}x$ has a solution

$$x^*(t) = \left(x_1^*(t), x_2^*(t)\right)^T \in \operatorname{Dom} L \bigcap \overline{\Omega}, \tag{2.42}$$

which is an ω-periodic solution to (2.4). Therefore $N^*(t) = (N_1^*(t), N_2^*(t))^T = (e^{x_1^*(t)}, e^{x_2^*(t)})^T$ is a positive ω-periodic solution of (1.3) and the proof is complete. □

3. An Example

In this section, we give an example to demonstrate the results obtained in the previous section.

Example 3.1. Consider the following Nicholson-type delay system with nonlinear density-dependent mortality terms:

$$\begin{aligned}
N_1'(t) = &-\frac{(5+\sin t)N_1(t)}{5+\sin t+N_1(t)} + \frac{(2+\cos t)N_2(t)}{2+\cos t+N_2(t)} \\
&+ e^{4\pi}\left(1+\frac{\cos t}{4}\right)N_1(t-|2+\cos t|)e^{-e^{4\pi+|\sin t|}N_1(t-|2+\cos t|)} \\
&+ e^{4\pi}\left(1+\frac{\sin t}{4}\right)N_1(t-|2+\sin t|)e^{-e^{4\pi+|\cos t|}N_1(t-|2+\sin t|)}, \\
N_2'(t) = &-\frac{(5+\cos t)N_2(t)}{5+\cos t+N_2(t)} + \frac{(2+\sin t)N_1(t)}{2+\sin t+N_1(t)}
\end{aligned}$$

$$+ e^{4\pi}\left(1+\frac{\sin t}{4}\right)N_2(t-|2+\sin t|)e^{-e^{4\pi+|\cos t|}N_2(t-|2+\sin t|)}$$
$$+ e^{4\pi}\left(1+\frac{\cos t}{4}\right)N_2(t-|2+\cos t|)e^{-e^{4\pi+|\sin t|}N_2(t-|2+\cos t|)}. \tag{3.1}$$

Obviously, $A_i = 4\pi$, $B_i = 4\pi e^{4\pi}$, $C_i = 10\pi$, $D_i = 4\pi$ $(i = 1,2)$, $c_{ij}^+ = (5/4)e^{4\pi}$, $\gamma_{ij}^- = e^{4\pi}$ $(i, j = 1,2)$, $a_{12}^+ = a_{21}^+ = 3$, $a_{11}^- = a_{22}^- = 4$, then

$$\begin{gathered}\ln\frac{2B_i}{A_i} = \ln 2 + 4\pi > 4\pi = A_i, \qquad C_i = 10\pi > 8\pi = 2D_i, \quad i = 1,2,\\ \sum_{j=1}^{l}\frac{c_{1j}^+}{a_{11}^-\gamma_{1j}^- e} + \frac{a_{12}^+}{a_{11}^-} = \frac{5}{8e} + \frac{3}{4} \approx 0.9799 < 1,\\ \sum_{j=1}^{l}\frac{c_{2j}^+}{a_{22}^-\gamma_{2j}^- e} + \frac{a_{21}^+}{a_{22}^-} = \frac{5}{8e} + \frac{3}{4} \approx 0.9799 < 1,\end{gathered} \tag{3.2}$$

which means the conditions in Theorem 2.2 hold. Hence, the model (3.1) has a positive 2π-periodic solution in $\overline{\Omega}$, where $\Omega = \{x \in X : \|x\| < 10000\}$. The fact is verified by the numerical simulation in Figure 1.

Remark 3.2. Equation (3.1) is a form of Nicholson's blowflies delayed systems with nonlinear density-dependent mortality terms, but as far as we know there are no that results can be applicable to (3.1) to obtain the existence of positive 2π-periodic solutions. This implies the results of this paper are essentially new.

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