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Research Article

On the Solutions Fractional Riccati Differential Equation with Modified Riemann-Liouville Derivative

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Fractional variational iteration method (FVIM) is performed to give an approximate analytical solution of nonlinear fractional Riccati differential equation. Fractional derivatives are described in the Riemann-Liouville derivative. A new application of fractional variational iteration method (FVIM) was extended to derive analytical solutions in the form of a series for these equations. The behavior of the solutions and the effects of different values of fractional order α are indicated graphically. The results obtained by the FVIM reveal that the method is very reliable, convenient, and effective method for nonlinear differential equations with modified Riemann-Liouville derivative

1. Introduction

In recent years, fractional calculus used in many areas such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, engineering, acoustics, viscoelasticity, material science and signal processing can be successfully modelled by linear or nonlinear fractional order differential equations [1–10]. As it is well known, Riccati differential equations concerned with applications in pattern formation in dynamic games, linear systems with Markovian jumps, river flows, econometric models, stochastic control, theory, diffusion problems, and invariant embedding [11–17]. Many studies have been conducted on solutions of the Riccati differential equations. Some of them, the approximate solution of ordinary Riccati differential equation obtained from homotopy perturbation method (HPM) [18–20], homotopy analysis method (HAM) [21], and variational iteration method proposed by He [22]. The He's homotopy perturbation method proposed by He [23–25] the variational iteration method [26] and

Adomian decomposition method (ADM) [27] to solve quadratic Riccati differential equation of fractional order.

The variational iteration method (VIM), which proposed by He [28, 29], was successfully applied to autonomous ordinary and partial differential equations and other fields. He [30] was the first to apply the variational iteration method to fractional differential equations. In recent years, a new modified Riemann-Liouville left derivative is suggested by Jumarie [31–35]. Recently, the fractional Riccati differential equation is solved with help of new homotopy perturbation method (HPM) [23].

In this paper, we extend the application of the VIM in order to derive analytical approximate solutions to nonlinear fractional Riccati differential equation:

$$D_*^{\alpha} y(x) = A(x) + B(x)y(x) + C(x)y^2(x), \quad x \in R,\ 0 < \alpha \le 1,\ t > 0, \tag{1.1}$$

subject to the initial conditions

$$y^{(k)}(0) = d_k, \quad k = 0, 1, 2, \dots, n-1, \tag{1.2}$$

where α is fractional derivative order, n is an integer, $A(x)$, $B(x)$, and $C(x)$ are known real functions, and d_k is a constant.

The goal of this paper is to extend the application of the variational iteration method to solve fractional nonlinear Riccati differential equations with modified Riemann-Liouville derivative.

The paper is organized as follows: In Section 2, we give definitions related to the fractional calculus theory briefly. In Section 3, we define the solution procedure of the fractional variational iteration method to show inefficiency of this method, we present the application of the FVIM for the fractional nonlinear Riccati differential equations with modified Riemann-Liouville derivative and numerical results in Section 4. The conclusions are then given in the final Section 5.

2. Basic Definitions

Here, some basic definitions and properties of the fractional calculus theory which can be found in [31–35].

Definition 2.1. Assume $f : R \to R$, $x \to f(x)$ denote a continuous (but not necessarily differentiable) function, and let the partition $h > 0$ in the interval $[0, 1]$. Jumarie's derivative is defined through the fractional difference [34]:

$$\Delta^{(\alpha)} = (FW - 1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h], \tag{2.1}$$

where $FWf(x) = f(x+h)$. Fractional derivative is defined as the following limit form [1, 7]:

$$f^{(\alpha)} = \lim_{h \to 0} \frac{\Delta^{\alpha}\left[f(x) - f(0)\right]}{h^{\alpha}}. \tag{2.2}$$

This definition is close to the standard definition of derivatives (calculus for beginners), and as a direct result, the αth derivative of a constant, $0 < \alpha < 1$, is zero.

Definition 2.2. The left-sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{\mu},\ \mu \geq -1$ is defined as

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \text{for } \alpha > 0,\ x > 0, \qquad J_a^0 f(x) = f(x). \tag{2.3}$$

The properties of the operator J^{α} can be found in [1, 7, 36].

Definition 2.3. The modified Riemann-Liouville derivative [33, 34] is defined as

$${}_0D_{\alpha}^{x} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^0 (x-\tau)^{n-\alpha} \big(f(\tau) - f(0)\big) d\tau, \tag{2.4}$$

where $x \in [0,1],\ n-1 \leq \alpha < n,$ and $n \geq 1$.

In addition, we want to give as in the following some properties of the fractional modified Riemann-Liouville derivative.

Fractional Leibniz product law:

$${}_0D_{\alpha}^{x}(uv) = u^{(\alpha)}v + uv^{(\alpha)}. \tag{2.5}$$

Fractional Leibniz Formulation:

$${}_0I_{\alpha}^{x}\, D_{\alpha}^{x}(uv) \ = f(x) - f(0), \quad 0 < \alpha \leq 1. \tag{2.6}$$

Fractional the integration of part:

$${}_aI_{\alpha}^{b}\left(u^{(\alpha)}v\right) = (uv)|_a^b - {}_aI_{\alpha}^{b}\left(uv^{(\alpha)}\right). \tag{2.7}$$

Definition 2.4. Fractional derivative of compounded functions [33, 34] is defined as

$$d^{\alpha} f \cong \Gamma(1+\alpha) df, \quad 0 < \alpha < 1. \tag{2.8}$$

Definition 2.5. The integral with respect to $(dx)^{\alpha}$ [33, 34] is defined as the solution of the fractional differential equation:

$$dy \cong f(x)(dx)^{\alpha}, \quad x \geq 0,\ y(0) = 0,\ 0 < \alpha < 1. \tag{2.9}$$

Lemma 2.6. *Let $f(x)$ denote a continuous function [33, 34] then the solution of* (2.5) *is defined as*

$$y = \int_0^x f(\tau)(d\tau)^{\alpha} = \alpha \int_0^x (x-\tau)^{\alpha} f(\tau) d\tau, \quad 0 < \alpha < 1. \tag{2.10}$$

For example, $f(x) = x^{\beta}$ *in* (2.10) *one obtains*

$$\int_0^x \tau^{\beta}(d\tau)^{\alpha} = \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} x^{\beta+\alpha}, \quad 0 < \alpha < 1. \tag{2.11}$$

Definition 2.7. Assume that the continuous function $f : R \to R$, $x \to f(x)$ has a fractional derivative of order $k\alpha$, for any positive integer k and any α, $0 < \alpha \leq 1$; then the following equality holds, which is

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\alpha k!} f^{\alpha k}(x), \quad 0 < \alpha \leq 1. \tag{2.12}$$

On making the substitution $h \to x$ and $x \to 0$, we obtain the fractional Mc-Laurin series:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\alpha k!} f^{\alpha k}(0), \quad 0 < \alpha \leq 1. \tag{2.13}$$

3. Fractional Variational Iteration Method

To describe the solution procedure of the fractional variational iteration method [31–35], we consider the following fractional Riccati differential equation:

$$D_*^{\alpha} y(x) = A(x) + B(x)y(x) + C(x)y^2(x), \quad x \in R,\ 0 < \alpha \leq 1,\ t > 0. \tag{3.1}$$

According to the VIM, we can build a correct functional for (3.1) as follows:

$$\begin{aligned} y_{n+1}(x) &= y_n(x) + I^{\alpha}\left[\lambda(\tau)\left(\frac{d^{\alpha}y_n(\tau)}{d\tau^{\alpha}} - \left(A(\tau) + B(\tau)y(\tau) + C(\tau)y^2(\tau)\right)\right)\right] \\ y_{n+1}(x) &= y_n(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\lambda(\tau)\left(\frac{d^{\alpha}y_n(\tau)}{d\tau^{\alpha}} - \left(A(\tau) + B(\tau)y(\tau) + C(\tau)y^2(\tau)\right)\right)d\tau. \end{aligned} \tag{3.2}$$

Using (2.3), we obtain a new correction functional:

$$y_{n+1}(x) = y_n(x) + \frac{1}{\Gamma(\alpha+1)}\int_0^x \lambda(\tau)\left(\frac{d^{\alpha}y_n(\tau)}{d\tau^{\alpha}} - \left(A(\tau) + B(\tau)y(\tau) + C(\tau)y^2(\tau)\right)\right)(d\tau)^{\alpha}. \tag{3.3}$$

It is obvious that the sequential approximations y_k, $k \geq 0$ can be established by determining λ, a general Lagrange's multiplier, which can be identified optimally with the variational theory. The function $\tilde{y}_n$ is a restricted variation which means $\delta\tilde{y}_n = 0$. Therefore, we first designate the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $y_{n+1}(x)$, $n \geq 0$ of the solution $y(x)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function y_0. The initial values are usually used for choosing the zeroth approximation y_0. With λ determined, then several approximations y_k, $k \geq 0$ follow immediately [37]. Consequently, the exact solution may be procured by using

$$y(x) = \lim_{n \to \infty} y_n(x). \tag{3.4}$$

4. Applications

In this section, we present the solution of two examples of the Riccati differential equations as the applicability of FVIM.

Example 4.1. Let usconsider the fractional Riccati differential equation, we get

$$\frac{d^\alpha y}{dx^\alpha} = -y^2(x) + 1, \qquad 0 < \alpha \leq 1, \tag{4.1}$$

with initial conditions:

$$y(0) = 0. \tag{4.2}$$

Construction the following functional:

$$y_{n+1}(x) = y_n(x) + \frac{1}{\Gamma(\alpha+1)} \int_0^x \lambda(\tau) \left\{ \frac{d^\alpha y_n}{d\tau^\alpha} + y_n^2(\tau) - 1 \right\} (d\tau)^\alpha, \tag{4.3}$$

we have

$$\begin{aligned}
\delta y_{n+1}(x) &= \delta y_n(x) + \frac{1}{\Gamma(\alpha+1)} \delta \int_0^x \lambda(\tau) \left\{ \frac{d^\alpha y_n}{d\tau^\alpha} + y_n^2(\tau) - 1 \right\} (d\tau)^\alpha \\
&= \delta y_n + \lambda \delta y_n \big|_{\tau=x} - \frac{1}{\Gamma(\alpha+1)} \int_0^x \frac{d^\alpha \lambda(\tau)}{d\tau^\alpha} \delta y_n(\tau) (d\tau)^\alpha.
\end{aligned} \tag{4.4}$$

Similarly, we can get the coefficients of δy_n to zero:

$$1 + \lambda(\tau)\big|_{\tau=x} = 0, \qquad \frac{d^\alpha \lambda(\tau)}{d\tau^\alpha} = 0. \tag{4.5}$$

The generalized Lagrange multiplier can be identified by the above equations:

$$\lambda(x) = -1. \tag{4.6}$$

substituting (4.6) into (4.3) produces the iteration formulation as follows:

$$y_{n+1}(x) = y_n(x) - \frac{1}{\Gamma(\alpha+1)} \int_0^x \left\{ \frac{d^\alpha y}{d\tau^\alpha} + y^2(\tau) - 1 \right\} (d\tau)^\alpha. \tag{4.7}$$

Taking the initial value $y_0(x) = 0$, we can derive

$$\begin{aligned}
y_1(x) &= y_0(x) - \frac{1}{\Gamma(\alpha+1)} \int_0^x \left\{ \frac{d^\alpha y_0}{d\tau^\alpha} + y_0^2(\tau) - 1 \right\} (d\tau)^\alpha \\
&= \frac{x^\alpha}{\Gamma(\alpha+1)}, \\
y_2(x) &= y_1(x) - \frac{1}{\Gamma(\alpha+1)} \int_0^x \left\{ \frac{d^\alpha y_1}{d\tau^\alpha} + y_1^2(\tau) - 1 \right\} (d\tau)^\alpha \\
&= \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+1)} \int_0^x \left\{ 1 + \frac{x^{2\alpha}}{\Gamma^2(\alpha+1)} - 1 \right\} (d\tau)^\alpha \\
&= \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}, \\
y_3(x) &= \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \\
&\quad - \frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)x^{7\alpha}}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)}, \\
y_4(x) &= \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \\
&\quad - \frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)x^{7\alpha}}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)} - \frac{4\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)x^{7\alpha}}{\Gamma^4(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} \\
&\quad + \frac{2\Gamma(2\alpha+1)\Gamma(6\alpha+1)\Gamma(8\alpha+1)x^{9\alpha}}{\Gamma^5(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)\Gamma(9\alpha+1)} + \frac{4\Gamma^2(2\alpha+1)\Gamma(4\alpha+1)\Gamma(8\alpha+1)x^{9\alpha}}{\Gamma^5(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(5\alpha+1)\Gamma(9\alpha+1)} \\
&\quad - \frac{2\Gamma^3(2\alpha+1)\Gamma(6\alpha+1)\Gamma(10\alpha+1)x^{11\alpha}}{\Gamma^6(\alpha+1)\Gamma^3(3\alpha+1)\Gamma(7\alpha+1)\Gamma(11\alpha+1)} - \frac{4\Gamma^2(2\alpha+1)\Gamma^2(4\alpha+1)\Gamma(10\alpha+1)x^{11\alpha}}{\Gamma^6(\alpha+1)\Gamma^2(3\alpha+1)\Gamma^2(5\alpha+1)\Gamma(11\alpha+1)} \\
&\quad + \frac{4\Gamma^3(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)\Gamma(12\alpha+1)x^{13\alpha}}{\Gamma^7(\alpha+1)\Gamma^3(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)\Gamma(13\alpha+1)} \\
&\quad - \frac{\Gamma^4(2\alpha+1)\Gamma^2(6\alpha+1)\Gamma(14\alpha+1)x^{15\alpha}}{\Gamma^8(\alpha+1)\Gamma^4(3\alpha+1)\Gamma^2(7\alpha+1)\Gamma(15\alpha+1)} \\
&\vdots
\end{aligned} \tag{4.8}$$

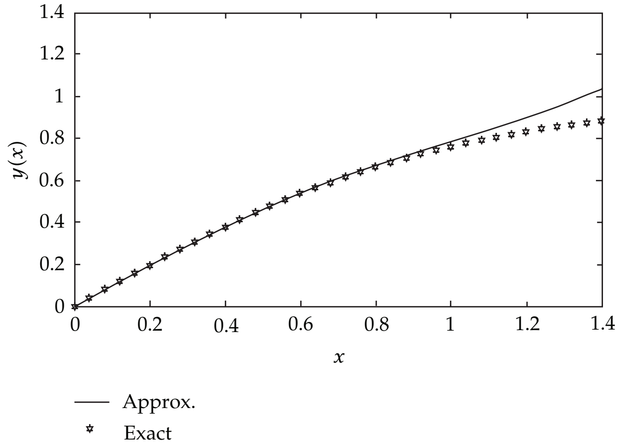

Figure 1: The graph indicates the solution $y(x)$ for (4.1), when $\alpha = 1$.

Then, the approximate solutions in a series form are

$$\begin{aligned} y(x) &= \lim_{n\to\infty} y_n(x) \\ &= \frac{x^{\alpha}}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \\ &\quad - \frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)x^{7\alpha}}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)} - \frac{4\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)x^{7\alpha}}{\Gamma^4(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} + \cdots. \end{aligned} \tag{4.9}$$

As $\alpha = 1$ is

$$y(x) = \lim_{n\to\infty} y_n(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{63} + \cdots. \tag{4.10}$$

The exact solution of (4.1) is $y(x) = (e^{2x} - 1)/(e^{2x} + 1)$, when $\alpha = 1$.

Figure 1 indicates the solution obtained using FVIM versus the exact solution when $\alpha = 1$. Figure 2 is plotted for approximate solution of time-fractional Riccati differential equation for $\alpha = 0.7$, 0.8, 0.9, and 1. Equation (4.1) is solved by using the homotopy perturbation method (HPM) [24]. FVIM solutions indicate that the present algorithm performs extreme efficiency, simplicity, and reliability. The results obtained from FVIM are fully compatible with those of the HPM.

Table 1 shows the approximate solutions for (4.1) obtained for different values of α using the variational iteration method and HPM [24]. From the numerical results in Table 1, it is clear that the approximate solutions are in high agreement with the exact solutions, when $\alpha = 1$, and the solution continuously depends on the time-fractional derivative. Example 4.1 has been solved using HAM [21], ADM [27], VIM [26], and HPM [23–25].

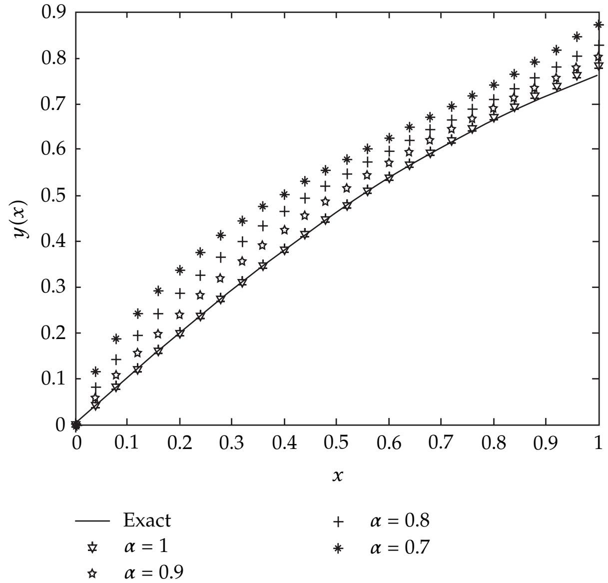

Figure 2: Plots of approx. solution $y_3(x)$ for different values of α.

Table 1: Approximate solutions for (4.1).

t	Ref. $[24]_{\alpha=0.5}$	$y_3(x)_{\alpha=0.5}$	$[24]_{\alpha=0.75}$	$y_3(x)_{\alpha=0.75}$	$[24]_{\alpha=1}$	$y_3(x)_{\alpha=1}$	$y(x)_{\text{exact}}$
0	0	0	0	0	0	0	0
0.1	0.273875	0.086513	0.184795	0.190102	0.099668	0.099667	0.099667
0.2	0.454125	0.161584	0.313795	0.310033	0.197375	0.197375	0.197375
0.3	0.573932	0.238256	0.414562	0.405062	0.291312	0.291320	0.291312
0.4	0.644422	0.321523	0.492889	0.483479	0.379944	0.380005	0.379948
0.5	0.674137	0.413682	0.462117	0.550470	0.462078	0.462375	0.462117
0.6	0.671987	0.515445	0.597393	0.610344	0.536857	0.537923	0.537049
0.7	0.648003	0.626403	0.631772	0.666961	0.603631	0.606768	0.604367
0.8	0.613306	0.745278	0.660412	0.723760	0.661706	0.669695	0.664036
0.9	0.579641	0.870074	0.687960	0.783638	0.709919	0.728139	0.716297
1.0	0.558557	0.998176	0.718260	0.848783	0.746032	0.784126	0.761594

Example 4.2. Let usconsider the fractional Riccati differential equation, we get

$$\frac{d^\alpha y}{dx^\alpha} = 2y(x) - y^2(x) + 1, \qquad 0 < \alpha \leq 1, \tag{4.11}$$

with initial conditions

$$y(0) = 0. \tag{4.12}$$

Construction the following functional:

$$y_{n+1}(x) = y_n(x) + \frac{1}{\Gamma(\alpha+1)}\int_0^x \lambda(\tau)\left\{\frac{d^\alpha y_n}{d\tau^\alpha} - 2y_n(\tau) + y_n^2(\tau) - 1\right\}(d\tau)^\alpha. \tag{4.13}$$

we have

$$\begin{aligned}\delta y_{n+1}(x) &= \delta y_n(x) + \frac{1}{\Gamma(\alpha+1)}\delta\int_0^x \lambda(\tau)\left\{\frac{d^\alpha y_n}{d\tau^\alpha} - 2y_n(\tau) + y_n^2(\tau) - 1\right\}(d\tau)^\alpha \\ &= \delta y_n + \lambda\delta y_n\big|_{\tau=x} - \frac{1}{\Gamma(\alpha+1)}\int_0^x \frac{d^\alpha\lambda(\tau)}{d\tau^\alpha}\delta y_n(\tau)(d\tau)^\alpha.\end{aligned} \tag{4.14}$$

Similarly, we can get the coefficients of δy_n to zero:

$$1 + \lambda(\tau)|_{\tau=x} = 0, \qquad \frac{d^\alpha\lambda(\tau)}{d\tau^\alpha} = 0. \tag{4.15}$$

The generalized Lagrange multiplier can be identified by the above equations:

$$\lambda(x) = -1, \tag{4.16}$$

substituting (4.16) into (4.13) produces the iteration formulation as follows:

$$y_{n+1}(x) = y_n(x) - \frac{1}{\Gamma(\alpha+1)}\int_0^x \left\{\frac{d^\alpha y_n}{d\tau^\alpha} - 2y_n(\tau) + y_n^2(\tau) - 1\right\}(d\tau)^\alpha. \tag{4.17}$$

Taking the initial value $y_0(x) = 0$, we can derive

$$\begin{aligned}y_1(x) &= y_0(x) - \frac{1}{\Gamma(\alpha+1)}\int_0^x \left\{\frac{d^\alpha y_0}{d\tau^\alpha} - 2y_0(\tau) + y_0^2(\tau) - 1\right\}(d\tau)^\alpha \\ &= \frac{x^\alpha}{\Gamma(\alpha+1)}, \\ y_2(x) &= y_1(x) - \frac{1}{\Gamma(\alpha+1)}\int_0^x \left\{\frac{d^\alpha y_1}{d\tau^\alpha} + y_1^2(\tau) - 1\right\}(d\tau)^\alpha \\ &= \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+1)}\int_0^x \left\{1 - \frac{2\tau^\alpha}{\Gamma(\alpha+1)} + \frac{\tau^{2\alpha}}{\Gamma^2(\alpha+1)} - 1\right\}(d\tau)^\alpha \\ &= \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}, \\ y_3(x) &= \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{4x^{3\alpha}}{\Gamma(3\alpha+1)} \\ &\quad - \frac{4\Gamma(3\alpha+1)x^{4\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)} - \frac{2\Gamma(2\alpha+1)x^{4\alpha}}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}\end{aligned}$$

$$
\begin{aligned}
&- \frac{4\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^2(2\alpha+1)\Gamma(5\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \\
&- \frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)x^{7\alpha}}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)} \\
&\vdots
\end{aligned}
\tag{4.18}
$$

Then, the approximate solutions in a series form are

$$
\begin{aligned}
y(x) = \operatorname*{Lim}_{n\to\infty} y_n(x) = {} & \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{4x^{3\alpha}}{\Gamma(3\alpha+1)} \\
& - \frac{4\Gamma(3\alpha+1)x^{4\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)} - \frac{2\Gamma(2\alpha+1)x^{4\alpha}}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)} \\
& - \frac{4\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^2(2\alpha+1)\Gamma(5\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \\
& - \frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)x^{7\alpha}}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)} + \cdots .
\end{aligned}
\tag{4.19}
$$

As $\alpha = 1$ is

$$
y(x) = \operatorname*{Lim}_{n\to\infty} y_n(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{x^7}{63} + \cdots . \tag{4.20}
$$

The exact solution of (4.11) is $y(x) = 1 + \sqrt{2}\tanh(\sqrt{2}t + (1/2)\log((\sqrt{2}-1)/(\sqrt{2}+1)))$, when $\alpha = 1$.

Figure 3 is plotted for approximate solution of time-fractional Riccati differential equation found in Example 4.2. In Figure 4, we have shown the graphic of approximate solution of (4.11) for $\alpha = 0.7$, 0.8, 0.9, and 1. Figures 2 and 4 show that a decrease in the fractional order α causes an increase in the function.

Table 2 indicates the approximate solutions for (4.11) obtained for different values of α using the variational iteration method and HPM [24]. From the numerical results in Table 2, it is clear that the approximate solutions are in high agreement with the exact solutions, when $\alpha = 1$, and the solution continuously depends on the time-fractional derivative.

Example 4.3. Let usconsider the fractional Riccati differential equation, we get

$$
\frac{d^{\alpha}y}{dx^{\alpha}} = x^2 + y^2(x), \qquad 0 < \alpha \le 1, \tag{4.21}
$$

with initial conditions:

$$
y(0) = 1. \tag{4.22}
$$

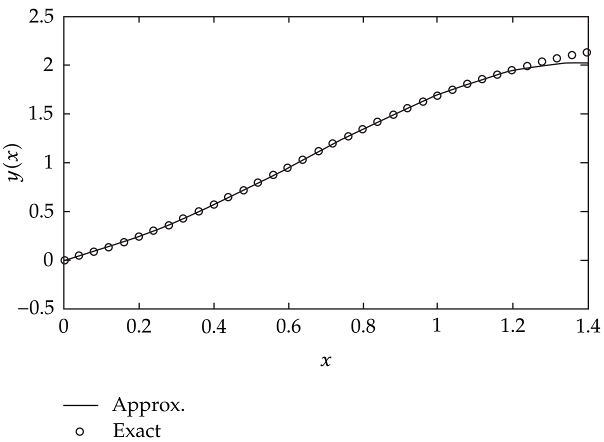

Figure 3: The graph indicates the solution $y(x)$ for (4.11), when $\alpha = 1$.

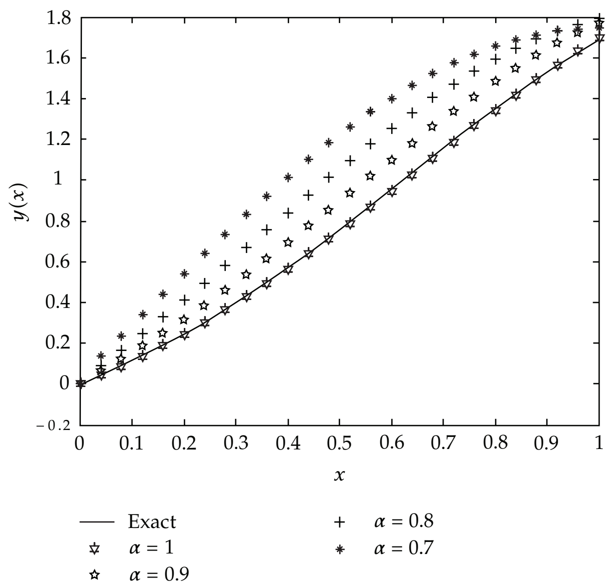

Figure 4: Plots of approx. solution $y_3(x)$ for different values of α.

Construction the following functional:

$$y_{n+1}(x) = y_n(x) + \frac{1}{\Gamma(\alpha+1)} \int_0^x \lambda(\tau) \left\{ \frac{d^\alpha y_n}{d\tau^\alpha} - \tau^2 - y_n^2(\tau) \right\} (d\tau)^\alpha. \tag{4.23}$$

we have

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) + \frac{1}{\Gamma(\alpha+1)} \delta \int_0^x \lambda(\tau) \left\{ \frac{d^\alpha y_n}{d\tau^\alpha} - \tau^2 - y_n^2(\tau) \right\} (d\tau)^\alpha \\ &= \delta y_n + \lambda \delta y_n \big|_{\tau=x} - \frac{1}{\Gamma(\alpha+1)} \int_0^x \frac{d^\alpha \lambda(\tau)}{d\tau^\alpha} \delta y_n(\tau) (d\tau)^\alpha. \end{aligned} \tag{4.24}$$

Table 2: Approximate solutions for (4.11).

t	Ref. [24]$_{\alpha=0.5}$	$y_3(x)_{\alpha=0.5}$	[24]$_{\alpha=0.75}$	$y_3(x)_{\alpha=0.75}$	[24]$_{\alpha=1}$	$y_3(x)_{\alpha=1}$	$y(x)_{\text{exact}}$
0	0	0	0	0	0	0	0
0.1	0.321730	0.577431	0.216866	0.244460	0.110294	0.110266	0.110295
0.2	0.629666	0.912654	0.428892	0.469709	0.241965	0.241585	0.241976
0.3	0.940941	1.166253	0.654614	0.698718	0.395106	0.393515	0.395104
0.4	1.250737	1.353549	0.891404	0.924319	0.568115	0.564013	0.567812
0.5	1.549439	1.482633	1.132763	1.137952	0.757564	0.749528	0.756014
0.6	1.825456	1.559656	1.370240	1.331462	0.958259	0.945155	0.953566
0.7	2.066523	1.589984	1.594278	1.497600	1.163459	1.144826	1.152946
0.8	2.260633	1.578559	1.794879	1.630234	1.365240	1.341552	1.346363
0.9	2.396839	1.530028	1.962239	1.724439	1.554960	1.527690	1.526911
1.0	2.466004	1.448805	2.087384	1.776542	1.723810	1.695238	1.689498

Similarly, we can get the coefficients of δy_n to zero:

$$1 + \lambda(\tau)|_{\tau=x} = 0, \qquad \frac{d^{\alpha}\lambda(\tau)}{d\tau^{\alpha}} = 0. \tag{4.25}$$

The generalized Lagrange multiplier can be identified by the above equations:

$$\lambda(x) = -1. \tag{4.26}$$

substituting (4.26) into (4.23) produces the iteration formulation as follows:

$$y_{n+1}(x) = y_n(x) - \frac{1}{\Gamma(\alpha+1)} \int_0^x \left\{ \frac{d^{\alpha} y_n}{d\tau^{\alpha}} - \tau^2 - y_n^2(\tau) \right\} (d\tau)^{\alpha}. \tag{4.27}$$

Taking the initial value $y_0(x) = 1$, we can derive

$$\begin{aligned}
y_1(x) &= y_0(x) - \frac{1}{\Gamma(\alpha+1)} \int_0^x \left\{ \frac{d^{\alpha} y_0}{d\tau^{\alpha}} - \tau^2 - y_0^2(\tau) \right\} (d\tau)^{\alpha} \\
&= 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{2x^{2+\alpha}}{\Gamma(\alpha+3)}, \\
y_2(x) &= y_1(x) - \frac{1}{\Gamma(\alpha+1)} \int_0^x \left\{ \frac{d^{\alpha} y_1}{d\tau^{\alpha}} - y_1^2(\tau) - \tau^2 \right\} (d\tau)^{\alpha} \\
&= 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{2x^{2+\alpha}}{\Gamma(\alpha+3)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{4x^{2+2\alpha}}{\Gamma(2\alpha+3)}
\end{aligned}$$

$$+\frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}+\frac{4\Gamma(2\alpha+3)x^{2+3\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)}+\frac{4\Gamma(2\alpha+5)x^{4+3\alpha}}{\Gamma^2(\alpha+3)\Gamma(3\alpha+5)},$$

$$\begin{aligned}
y_3(x) = 1 &+ \frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{2x^{2+\alpha}}{\Gamma(\alpha+3)}+\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{4x^{2+2\alpha}}{\Gamma(2\alpha+3)}+\frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\\
&+\frac{4\Gamma(2\alpha+3)x^{2+3\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)}+\frac{4\Gamma(2\alpha+5)x^{4+3\alpha}}{\Gamma^2(\alpha+3)\Gamma(3\alpha+5)}+\frac{8x^{2+3\alpha}}{\Gamma(3\alpha+3)}\\
&+\frac{16\Gamma(5\alpha+3)\Gamma(2\alpha+3)x^{2+6\alpha}}{\Gamma(2\alpha+1)\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)\Gamma(6\alpha+3)}+\frac{16\Gamma(3\alpha+5)x^{4+4\alpha}}{\Gamma(\alpha+3)\Gamma(2\alpha+3)\Gamma(4\alpha+5)}\\
&+\frac{4\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^2(2\alpha+1)\Gamma(5\alpha+1)}+\frac{2\Gamma(2\alpha+1)x^{4\alpha}}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}+\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}\\
&+\frac{8\Gamma(3\alpha+3)x^{2+4\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+4)}+\frac{8\Gamma(3\alpha+3)x^{2+4\alpha}}{\Gamma(\alpha+3)\Gamma(2\alpha+1)\Gamma(4\alpha+3)}+\frac{4x^{3\alpha}}{\Gamma(3\alpha+1)}\\
&+\frac{\Gamma^2(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)}+\frac{4\Gamma(5\alpha+1)x^{6\alpha}}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)\Gamma(6\alpha+1)}+\frac{8\Gamma(2\alpha+5)x^{4\alpha+4}}{\Gamma^2(\alpha+3)\Gamma(4\alpha+5)}\\
&+\frac{4\Gamma(3\alpha+1)x^{4\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}+\frac{16\Gamma(4\alpha+5)x^{4+5\alpha}}{\Gamma^2(2\alpha+3)\Gamma(5\alpha+5)}+\frac{8\Gamma(2\alpha+3)x^{2+4\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(4\alpha+3)}\\
&+\frac{8\Gamma(2\alpha+3)\Gamma(4\alpha+3)x^{5\alpha+2}}{\Gamma^2(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)\Gamma(5\alpha+3)}+\frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+5)x^{6\alpha+2}}{\Gamma(\alpha+1)\Gamma^2(\alpha+3)\Gamma(3\alpha+5)\Gamma(5\alpha+5)}\\
&+\frac{4\Gamma(2\alpha+1)\Gamma(4\alpha+3)x^{5\alpha+2}}{\Gamma(3\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+1)\Gamma(5\alpha+3)}+\frac{16\Gamma(2\alpha+3)\Gamma(4\alpha+5)x^{5\alpha+4}}{\Gamma(\alpha+1)\Gamma^2(\alpha+3)\Gamma(3\alpha+3)\Gamma(5\alpha+5)}\\
&+\frac{8\Gamma(2\alpha+1)\Gamma(5\alpha+3)x^{6\alpha+2}}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)\Gamma(2\alpha+3)\Gamma(6\alpha+3)}+\frac{16\Gamma(2\alpha+5)\Gamma(5\alpha+5)x^{6\alpha+4}}{\Gamma(2\alpha+1)\Gamma^2(\alpha+3)\Gamma(3\alpha+5)\Gamma(6\alpha+5)}\\
&+\frac{32\Gamma(5\alpha+5)x^{6\alpha+4}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)\Gamma(6\alpha+5)}+\frac{32\Gamma(2\alpha+5)\Gamma(5\alpha+7)x^{6\alpha+6}}{\Gamma(2\alpha+3)\Gamma^2(\alpha+3)\Gamma(3\alpha+5)\Gamma(6\alpha+7)}\\
&+\frac{16\Gamma^2(2\alpha+3)\Gamma(6\alpha+5)x^{7\alpha+4}}{\Gamma^2(\alpha+1)\Gamma^2(\alpha+3)\Gamma^2(3\alpha+3)\Gamma(7\alpha+5)}+\frac{16\Gamma(4\alpha+3)x^{2+5\alpha}}{\Gamma(2\alpha+1)\Gamma(2\alpha+3)\Gamma(5\alpha+3)}\\
&+\frac{16\Gamma(2\alpha+5)\Gamma(4\alpha+7)x^{5\alpha+6}}{\Gamma^3(\alpha+3)\Gamma(3\alpha+5)\Gamma(5\alpha+7)}+\frac{16\Gamma^2(2\alpha+5)\Gamma(6\alpha+9)x^{8+7\alpha}}{\Gamma^4(\alpha+3)\Gamma(7\alpha+9)\Gamma^2(3\alpha+5)}\\
&+\frac{8\Gamma(2\alpha+1)\Gamma(2\alpha+3)\Gamma(6\alpha+3)x^{7\alpha+2}}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(3\alpha+3)\Gamma(\alpha+3)\Gamma(7\alpha+3)}\\
&+\frac{8\Gamma(2\alpha+1)\Gamma(2\alpha+5)\Gamma(6\alpha+5)x^{7\alpha+4}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)\Gamma(3\alpha+5)\Gamma^2(\alpha+3)\Gamma(7\alpha+5)}\\
&+\frac{32\Gamma(2\alpha+3)\Gamma(2\alpha+5)\Gamma(6\alpha+7)x^{7\alpha+6}}{\Gamma(\alpha+1)\Gamma(3\alpha+3)\Gamma(3\alpha+5)\Gamma^3(\alpha+3)\Gamma(7\alpha+7)}\\
&\vdots
\end{aligned} \tag{4.28}$$

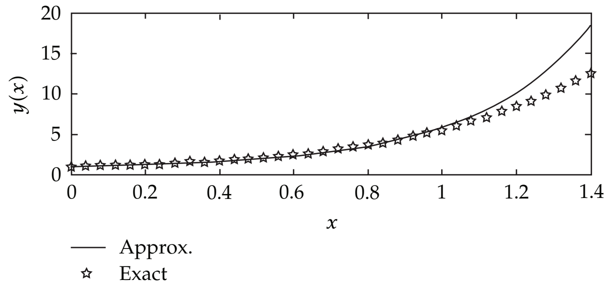

Figure 5: The graph indicates the solution $y(x)$ for (4.21), when $\alpha = 1$.

Then, the approximate solutions in a series form are

$$\begin{aligned} y(x) &= \lim_{n\to\infty} y_n(x) \\ &= 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{2x^{2+\alpha}}{\Gamma(\alpha+3)} + \frac{2x^{2\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{4x^{2+2\alpha}}{\Gamma(2\alpha+3)} \\ &\quad + \frac{\Gamma(2\alpha+1)x^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{4\Gamma(2\alpha+3)x^{2+3\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+5)x^{4+3\alpha}}{\Gamma^2(\alpha+3)\Gamma(3\alpha+5)} + \cdots . \end{aligned} \tag{4.29}$$

As $\alpha = 1$ is

$$\begin{aligned} y(x) &= \lim_{n\to\infty} y_n(x) \\ &= 1 + x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{6} + \frac{8x^5}{15} + \frac{29x^6}{90} + \frac{47x^7}{315} + \frac{41t^8}{360} + \frac{299t^9}{11340} \\ &\quad + \frac{4t^{10}}{525} + \frac{184t^{11}}{51975} + \frac{t^{12}}{2268} + \frac{4t^{13}}{12285} + \frac{t^{15}}{59535} + \cdots . \end{aligned} \tag{4.30}$$

The exact solution of (4.21) is

$$y(x) = \frac{t\left(J_{-3/4}(t^2/2)\Gamma(1/4) + 2J_{3/4}(t^2/2)\Gamma(3/4)\right)}{J_{1/4}(t^2/2)\Gamma(1/4) - 2J_{-1/4}(t^2/2)\Gamma(3/4)}, \tag{4.31}$$

where $J_v(t)$ is the Bessel function of first kind, when $\alpha = 1$.

Figure 5 is plotted for approximate solution of time-fractional Riccati differential equation found in Example 4.3. In Figure 6, we have shown the graphic of approximate solution of (4.21) for $\alpha = 0.5$, 0.65, 0.75, and 1. Figures 2, 4, and 6 show that a decrease in the fractional order α causes an increase in the function.

Table 3 indicates the approximate solutions for (4.21) obtained for different values of α using the HPM [23]. From the numerical results in Table 3, it is clear that the approximate solutions are in substantial agreement with the exact solutions, when $\alpha = 1$, and the solution continuously depends on the time-fractional derivative.

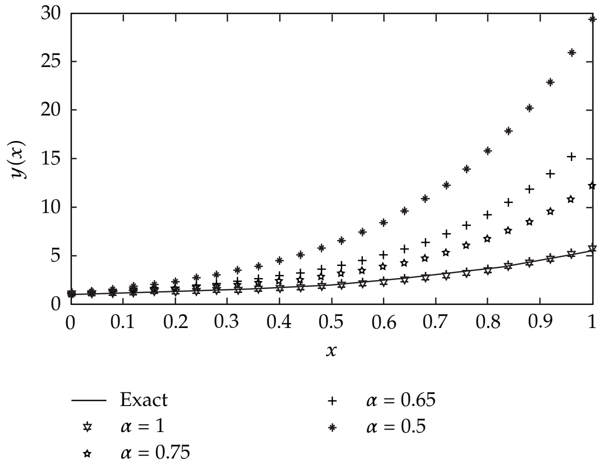

Figure 6: Plots of approx. solution $y_3(x)$ for different values of α.

Table 3: Approximate solutions for (4.21).

t	Ref. $[23]_{\alpha=0.50}$	$y_3(x)_{\alpha=0.5}$	$[23]_{\alpha=0.75}$	$y_3(x)_{\alpha=0.75}$	$[23]_{\alpha=1}$	$y_3(x)_{\alpha=1}$	$y(x)_{\text{exact}}$
0	1	1	1	1	1	1	1
0.1	1.412871	1.671055	1.219154	1.249863	1.111463	1.110922	1.111450
0.2	1.693287	2.342131	1.426528	1.513463	1.252997	1.248193	1.252533
0.3	2.035965	3.241501	1.684735	1.852218	1.439293	1.420818	1.435450
0.4	2.500705	4.468508	2.031414	2.306699	1.692749	1.641742	1.675200
0.5	3.161671	6.144389	2.517316	2.928996	2.047355	1.929432	1.989583
0.6	4.121732	8.429666	3.215023	3.789643	2.553642	2.310101	2.399200
0.7	5.523246	11.54066	4.227199	4.985354	3.284790	2.820861	2.927450
0.8	7.558999	15.77047	5.695700	6.649324	4.343940	3.514150	3.600533
0.9	10.48397	21.51700	7.811804	8.965376	5.872787	4.463992	4.447450
1.0	14.62818	29.32098	10.82770	12.18753	8.061507	5.774817	5.500000

5. Conclusions

In this paper, variational iteration method having integral w.r.t. $(d\tau)^\alpha$ has been successfully implemented to finding approximate analytical solution of fractional Riccati differential equations. Variational iteration method known as very powerful and an effective method for solving fractional Riccati differential equation. It is also a promising method to solve other nonlinear equations. In this paper, we have discussed modified variational iteration method having integral w.r.t. $(d\tau)^\alpha$ used for the first time by Jumarie. The obtained results indicate that this method is powerful and meaningful for solving the nonlinear fractional differential equations. Three examples indicate that the results of variational iteration method having integral w.r.t. $(d\tau)^\alpha$ are in excellent agreement with those obtained by HPM, ADM, and HAM, which is available in the literature.

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