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Research Article

On the Deformation Theory of Structure Constants for Associative Algebras

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An algebraic scheme for constructing deformations of structure constants for associative algebras generated by deformation driving algebras (DDAs) is discussed. An ideal of left divisors of zero plays a central role in this construction. Deformations of associative three-dimensional algebras with the DDA being a three-dimensional Lie algebra and their connection with integrable systems are studied.

1. Introduction

An idea to study deformations of structure constants for associative algebras goes back to the classical works of Gerstenhaber [1, 2]. As one of the approaches to deformation theory he suggested "to take the point of view that the objects being deformed are not merely algebras, but essentially algebras with a fixed basis" and to treat "the algebraic set of all structure constants as parameter space for deformation theory" [2].

Thus, following this approach, one chooses the basis $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_N$ for a given algebra A, takes the structure constants C^n_{jk} defined by the multiplication table

$$\mathbf{P}_j \mathbf{P}_k = \sum_{n=0}^{N} C^n_{jk} \mathbf{P}_n, \quad j, k = 0, 1, \ldots, N, \tag{1.1}$$

and looks for their deformations $C^n_{jk}(x)$, where $(x) = (x^1, \ldots, x^M)$ is the set of deformation parameters, such that the associativity condition

$$\sum_{m=0}^{N} C^m_{jk}(x) C^n_{ml}(x) = \sum_{m=0}^{N} C^m_{kl}(x) C^n_{jm}(x) \tag{1.2}$$

or similar equation is satisfied.

A remarkable example of deformations of this type with $M = N + 1$ has been discovered by Witten [3] and Dijkgraaf et al. [4]. They demonstrated that the function F which defines the correlation functions $\langle \Phi_j \Phi_k \Phi_l \rangle = \partial^3 F / \partial x^j \partial x^k \partial x^l$ and so forth in the deformed two-dimensional topological field theory obeys the associativity equation (1.2) with the structure constants given by

$$C^l_{jk} = \sum_{m=0}^{N} \eta^{lm} \frac{\partial^3 F}{\partial x^j \partial x^k \partial x^m}, \tag{1.3}$$

where the constants are $\eta^{lm} = (g^{-1})^{lm}$ and $g_{lm} = \partial^3 F / \partial x^0 \partial x^l \partial x^m$ where the variable x^0 is associated with the units element. Each solution of the WDVV equations (1.2) and (1.3) describes a deformation of the structure constants of the $N + 1$- dimensional associative algebra of primary fields Φ_j.

The interpretation and formalization of the WDVV equation in terms of Frobenius manifolds proposed by Dubrovin [5, 6] provides us with a method to describe class of deformations of the so-called Frobenius algebras. An extension of this approach to general algebras and corresponding F-manifolds has been given by Hertling and Manin [7]. The beautiful and rich theory of Frobenius and F-manifolds has various applications from the singularity theory to quantum cohomology (see, e.g., [6, 8, 9]).

An alternative approach to the deformation theory of the structure constants for commutative associative algebras has been proposed recently in [10–14]. Within this method the deformations of the structure constants are governed by the so-called central system (CS). Its concrete form depends on the class of deformations under consideration and CS contains, as particular reductions, many integrable systems like WDVV equation, oriented associativity equation, and integrable dispersionless, dispersive, and discrete equations (Kadomtsev-Petviashvili equation, etc.). The common feature of the coisotropic, quantum, discrete deformations considered in [10–14] is that for all of them elements p_j of the basis and deformation parameters x_j form a certain algebra (Poisson, Heisenberg, etc.). A general class of deformations considered in [13] is characterized by the condition that the ideal $J = \langle f_{jk} \rangle$ generated by the elements $f_{jk} = -p_j p_k + \sum_{l=0}^{N} C^l_{jk}(x) p_l$ representing the multiplication table (1) is the Poisson ideal. It was shown that this class contains a subclass of so-called integrable deformations for which the CS has a simple and nice geometrical meaning.

In the present paper we will discuss a purely algebraic formulation of such integrable deformations. We will consider the case when the algebra generating deformations of the structure constants, that is, the algebra formed by the elements p_j of the basis and deformation parameters x_k(deformation driving algebra (DDA)), is a Lie algebra. The basic idea is to require that all elements $f_{jk} = -p_j p_k + \sum_{l=0}^{N} C^l_{jk}(x) p_l$ are left divisors of zero and that they generate the ideal J of left divisors of zero. This requirement gives rise to the central system which governs deformations generated by DDA. This central system of equations for structure constants differs, in general, from the associativity condition. So, deformed algebras form families of commutative but not necessarily associative algebras.

Here we will study the deformations of the structure constants for the three-dimensional algebra in the case when the DDA is given by one of the three-dimensional Lie algebras. Such deformations are parametrized by a single deformation variable x. Depending on the choice of DDA and identification of p_1, p_2, and x with the elements of DDA, the corresponding CS takes the form of the system of ordinary differential equations or the system of discrete equations (multidimensional mappings). In the first case the CS contains the third-order ODEs from the Chazy-Bureau list as the particular examples. This approach provides us also with the Lax form of the above equations and their first integrals.

The paper is organized as follows. General formulation of the deformation theory for the structure constants is presented in Section 2. Quantum, discrete, and coisotropic deformations are discussed in Section 3. Three-dimensional Lie algebras as DDAs are analyzed in Section 4. Deformations generated by general DDAs are studied in Section 5. Deformations driven by the nilpotent and solvable DDAs are considered in Sections 6 and 7, respectively.

2. Deformations of the Structure Constants Generated by DDA

So, we consider a finite-dimensional commutative algebra A with (or without) unit element $\mathbf{P}_0$ in the fixed basis composed by the elements $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_N$. The multiplication table (1) defines the structure constants C^l_{jk}. The commutativity of the basis implies that $C^l_{jk} = C^l_{kj}$. In the presence of the unit element one has $C^l_{j0} = \delta^l_j$ where δ^l_j is the Kronecker symbol.

Following Gerstenhaber's suggestion [1, 2] we will treat the structure constants C^l_{jk} as the objects to deform and will denote the deformation parameters by $x^1, x^2, \dots, x^M$. For the undeformed structure constants the associativity conditions (1.2) are nothing else than the compatibility conditions for the table of multiplication (1.1). In the construction of deformations we should first specify a "deformed" version of the multiplication table and then require that this realization is self-consistent and meaningful.

Thus, to define deformations one has the following.

(1) We associate a set of elements $p_0, p_1, \dots, p_N, x^1, x^2, \dots, x^M$ with the elements of the basis $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_N$ and deformation parameters $x^1, x^2, \dots, x^M$.

(2) We consider the Lie algebra B of the dimension $N+M$ with the basis elements $e_1, \dots, e_{N+M}$ obeying the commutation relations:

$$[e_\alpha, e_\beta] = \sum_{\gamma=1}^{N+M} C_{\alpha\beta\gamma} e_\gamma, \quad \alpha, \beta = 1, 2, \dots, N+M. \tag{2.1}$$

(3) We identify the elements $p_1, \dots, p_N, x^1, x^2, \dots, x^M$ with the elements $e_1, \dots, e_{N+M}$, thus defining the deformation driving algebra (DDA). Different identifications define different DDAs. We assume that the element p_0 commutes with all elements of DDA and we put $p_0 = 1$. The commutativity of the basis in the algebra A implies the commutativity between p_j, and in this paper we assume the same property for all x^k. So, we will consider the DDAs defined by the commutation relations of the type

$$[p_j, p_k] = 0, \quad \left[x^j, x^k\right] = 0, \quad \left[p_j, x^k\right] = \sum_l \alpha^k_{jl} x^l + \sum_l \beta^{kl}_j p_l, \tag{2.2}$$

where α^k_{jl} and β^{kl}_j are some constants.

(4) We consider the elements

$$f_{jk} = -p_j p_k + \sum_{l=0}^{N} C^l_{jk}(x) p_l, \quad j, k = 1, \dots, N \tag{2.3}$$

of the universal enveloping algebra $U(B)$ of the algebra DDA(B). These f_{jk} "represent" the table (1) in $U(B)$.

(5) We require that all f_{jk} are left zero divisors and have a common right zero divisor.

In this case f_{jk} generate the left ideal J of left zero divisors. We remind that non-zero elements a and b are called left and right divisors of zero if $ab = 0$ (see e.g., [15]).

Definition 2.1. The structure constants $C^l_{jk}(x)$ are said to define deformations of the algebra A generated by given DDA if all f_{jk} are left zero divisors with common right zero divisor.

To justify this definition we first observe that the simplest possible realization of the multiplication table (1) in $U(B)$ given by the equations $f_{jk} = 0, j, k = 1, \dots, N$ is too restrictive in general. Indeed, for instance, for the Heisenberg algebra B [12] such equations imply that $[p_l, C^m_{jk}(x)] = \partial C^m_{jk}/\partial x^l = 0$ and, hence, all C^m_{jk} are constants. So, one should look for a weaker realization of the multiplication table. A condition that all f_{jk} are left zero divisors is a natural candidate. The condition of compatibility of the corresponding equations $f_{jk} \cdot \Psi_{jk} = 0,\ j, k = 1, \dots, N$ where Ψ_{jk} are right zero divisors requires that the l.h.s. of these equations and, hence, Ψ_{jk} should have a common divisor (see, e.g., [15]). We restrict ourselves to the case when $\Psi_{jk} = \Psi \cdot \Phi_{jk},\ j, k = 1, \dots, N$ where Φ_{jk} are invertible elements of $U(B)$. In this case one has the set of equations

$$f_{jk} \cdot \Psi = 0, \quad j, k = 0, 1, \dots, N; \tag{2.4}$$

that is, all left zero divisors f_{jk} have common right zero divisor Ψ.

These conditions impose constraints on $C^m_{jk}(x)$. To clarify these constraints we will use the associativity of $U(B)$. First we observe that due to the relations (2.2) one has the identity $(p_0 = 1)$

$$\left[p_l, C^m_{jk}(x)\right] = \sum_{t=0}^{N} \Delta^{mt}_{jk,l}(x) p_t, \tag{2.5}$$

where $\Delta^{mt}_{jk,l}(x)$ are certain functions of $x^1, \dots, x^M$ only. Then, taking into account (2.2) and associativity of $U(B)$, one obtains

$$(p_j p_k) p_l - p_j (p_k p_l) = \sum_{s,t=0}^{N} K^{st}_{klj} \cdot f_{st} + \sum_{t=0}^{N} \Omega^t_{klj}(x) \cdot p_t, \quad j, k, l = 0, 1, \dots, N, \tag{2.6}$$

where

$$\begin{aligned} K^{st}_{klj} &= \frac{1}{2}\left(\delta^s_k \delta^t_l + \delta^t_k \delta^s_l\right) p_j - \frac{1}{2}\left(\delta^s_k \delta^t_j + \delta^t_k \delta^s_j\right) p_l + \frac{1}{2}\left(\delta^s_j C^t_{kl} + \delta^t_j C^s_{kl}\right) \\ &\quad - \frac{1}{2}\left(\delta^s_l C^t_{kj} + \delta^t_l C^s_{kj}\right) + \Delta^{st}_{kl,j} - \Delta^{st}_{kj,l}, \\ \Omega^t_{klj}(x) &= \sum_s C^s_{jk} C^t_{ls} - \sum_s C^s_{lk} C^t_{js} + \sum_{s,n}\left(\Delta^{sn}_{kj,l} - \Delta^{sn}_{kl,j}\right) C^t_{sn}. \end{aligned} \tag{2.7}$$

Thus, the identity (2.6) gives

$$\sum_{s,t=0}^{N} K_{klj}^{st} \cdot f_{st} + \sum_{t=0}^{N} \Omega_{klj}^{t}(x) \cdot p_t = 0, \quad j,k,l = 0,1,\ldots,N. \tag{2.8}$$

Due to the relations (2.4), (2.8) implies that

$$\left(\sum_{t=0}^{N} \Omega_{klj}^{t}(x) \cdot p_t\right)\Psi = 0. \tag{2.9}$$

These equations are satisfied if

$$\Omega_{klj}^{t}(x) = \sum_{s} C_{jk}^{s} C_{ls}^{t} - \sum_{s} C_{lk}^{s} C_{js}^{t} + \sum_{s,n}\left(\Delta_{kj,l}^{sn} - \Delta_{kl,j}^{sn}\right) C_{sn}^{t} = 0, \quad j,k,l,t = 0,1,\ldots,N. \tag{2.10}$$

This system of equations plays a central role in our approach. If Ψ has no left zero divisors linear in p_j, the relation (2.10) is the necessary condition for existence of a common right zero divisor for f_{jk} since $U(B)$ has no zero elements linear in p_j (see e.g., [16]).

At $N \geq 3$ it is also a sufficient condition. Indeed, if $C_{jk}^{m}(x)$ are such that (2.10) is satisfied, then

$$\sum_{s,t=0}^{N} K_{klj}^{st} \cdot f_{st} = 0, \quad j,k,l = 0,1,\ldots,N. \tag{2.11}$$

Generically, it is the system of $(1/2)N^2(N-1)$ linear equations for $N(N+1)/2$ unknowns f_{st} with noncommuting coefficients K_{klj}^{st}. At $N \geq 3$ for generic (nonzeros, nonzero divisors) $K_{klj}^{st}(x,p)$ the system (2.11) implies that

$$\alpha_{jk} f_{jk} = \beta_{lm} f_{lm}, \quad j,k,l,m = 1,\ldots,N, \tag{2.12}$$

$$\gamma_{jk} f_{jk} = 0, \quad j,k = 1,\ldots,N, \tag{2.13}$$

where α_{jk}, β_{lm}, and γ_{jk} are certain elements of $U(B)$ (see e.g., [17, 18]). Thus, all f_{jk} are right zero divisors. They are also left zero divisors. Indeed, due to Ado's theorem (see e.g., [16]) finite-dimensional Lie algebra B and, hence, $U(B)$ are isomorphic to matrix algebras. For the matrix algebras zero divisors (matrices with vanishing determinants) are both right and left zero divisors [15]. Then, under the assumption that all α_{jk} and β_{lm} are not zero divisors, the relations (2.12) imply that the right divisor of one of f_{jk} is also the right zero divisor for the others.

At $N = 2$ one has only two relations of the type (2.12) and a right zero divisor of one of f_{11}, f_{12}, f_{22} is the right zero divisor of the others. We note that it is not easy to control assumptions mentioned above. Nevertheless, (2.4) and (2.10) certainly are fundamental one for the whole approach.

We will refer to the system (2.10) as the Central System (CS) governing deformations of the structure constants of the algebra A generated by a given DDA. Its concrete form depends strongly on the form of the brackets $[p_t, C^l_{jk}(x)]$ which are defined by the relations (2.2) for the elements of the basis of DDA. For stationary solutions ($\Delta^{mt}_{jk,l} = 0$) the CS (2.10) is reduced to the associativity conditions (1.2).

3. Quantum, Discrete, and Coisotropic Deformations

Coisotropic, quantum, and discrete deformations of associative algebras considered in [10–14] represent particular realizations of the above general scheme associated with different DDAs.

For the *quantum deformations* one has $M = N$ and the deformation driving algebra is given by the Heisenberg algebra [12]. The elements of the basis of the algebra A and deformation parameters are identified with the elements of the Heisenberg algebra in such a way that

$$[p_j, p_k] = 0, \quad \left[x^j, x^k\right] = 0, \quad \left[p_j, x^k\right] = \hbar\delta^k_j, \quad j,k = 1,\ldots,N, \tag{3.1}$$

where $\hbar$ is the real constant (Planck's constant in physics). For the Heisenberg DDA

$$\Delta^{mt}_{jk,l} = \hbar\delta^t_0 \frac{\partial C^m_{jk}(x)}{\partial x^l}, \tag{3.2}$$

and consequently

$$\Omega^n_{klj}(x) = \hbar\frac{\partial C^n_{jk}}{\partial x^l} - \hbar\frac{\partial C^n_{kl}}{\partial x^j} + \sum_{m=0}^{N}\left(C^m_{jk}C^n_{ml} - C^m_{kl}C^n_{jm}\right) = 0, \quad j,k,l,n = 0,1,\ldots,N. \tag{3.3}$$

Quantum CS (3.3) governs deformations of structure constants for associative algebra driven by the Heisenberg DDA. It has a simple geometrical meaning of vanishing Riemann curvature tensor for torsionless Christoffel symbols Γ^l_{jk} identified with the structure constants ($C^l_{jk} = \hbar\Gamma^l_{jk}$) [12].

In the representation of the Heisenberg algebra (3.1) by operators acting in a linear space H left divisors of zero are realized by operators with nonempty kernel. The ideal J is the left ideal generated by operators f_{jk} which have nontrivial common kernel or, equivalently, for which equations

$$f_{jk}|\Psi\rangle = 0, \quad j,k = 1,2,\ldots,N \tag{3.4}$$

have nontrivial common solutions $|\Psi\rangle \subset H$. The compatibility condition for (3.4) is given by the CS (3.3). The common kernel of the operators f_{jk} forms a subspace H_Γ in the linear space H. So, in the approach under consideration the multiplication table (1) is realized only on H_Γ, but not on the whole H. Such type of realization of the constraints is well known in quantum theory as Dirac's recipe for quantization of the first-class constraints [19]. In quantum theory context equation (3.4) serves to select the physical subspace in the whole Hilbert space. Within the deformation theory one may refer to the subspace H_Γ as the "structure constants" subspace. In [12] the recipe (3.4) was the starting point for construction of the quantum deformations.

Quantum CS (3.3) contains various classes of solutions which describe different classes of deformations. An important subclass is given by isoassociative deformations, that is, by deformations for which the associativity condition (1.2) is valid for all values of deformation parameters. For such quantum deformations the structure constants should obey the following equations:

$$\frac{\partial C^n_{jk}}{\partial x^l} - \frac{\partial C^n_{kl}}{\partial x^j} = 0, \quad j,k,l,n = 1,\ldots,N. \tag{3.5}$$

These equations imply that $C^n_{jk} = \partial^2\Phi^n/\partial x^j \partial x^k$ where Φ^n are some functions while the associativity condition (1.2) takes the following form:

$$\sum_{m=0}^{N} \frac{\partial^2\Phi^m}{\partial x^j \partial x^k} \frac{\partial^2\Phi^n}{\partial x^m \partial x^l} = \sum_{m=0}^{N} \frac{\partial^2\Phi^m}{\partial x^l \partial x^k} \frac{\partial^2\Phi^n}{\partial x^m \partial x^j}. \tag{3.6}$$

It is the oriented associativity equation introduced in [5, 20]. Under the gradient reduction $\Phi^n = \sum_{l=0}^{N} \eta^{nl}(\partial F/\partial x^l)$ equation (3.7) becomes the WDVV equations (1.2) and (1.3).

Non-isoassociative deformations for which the condition (3.5) is not valid are of interest too. They are described by some well-known integrable soliton equations [12]. In particular, there are Boussinesq equation among them for $N = 2$ and Kadomtsev-Petviashvili (KP) hierarchy for the infinite-dimensional algebra of polynomials in the Faa' de Bruno basis [12]. In the latter case the deformed structure constants are given by

$$C^l_{jk} = \delta^l_{j+k} + H^k_{j-l} + H^j_{k-l}, \quad j,k,l = 0,1,2,\ldots \tag{3.7}$$

with

$$H^j_k = \frac{1}{\hbar} P_k\left(-\hbar\widetilde{\partial}\right) \frac{\partial \log\tau}{\partial x^j}, \quad j,k = 1,2,3,\ldots, \tag{3.8}$$

where τ is the famous tau-function for the KP hierarchy and $P_k(-\hbar\widetilde{\partial}) \doteq P_k(-\hbar(\partial/\partial x^1), (-1/2)\hbar(\partial/\partial x^2), -(1/3)\hbar(\partial/\partial x^3), \ldots)$ where $P_k(t_1, t_2, t_3, \ldots)$ are Schur polynomials defined by the generating formula $\exp(\sum_{k=1}^{\infty} \lambda^k t_k) = \sum_{k=0}^{\infty} \lambda^k P_k(t)$.

Discrete deformations of noncommutative associative algebras are generated by the DDA with $M = N$ and commutation relations

$$[p_j, p_k] = 0, \quad \left[x^j, x^k\right] = 0, \quad \left[p_j, x^k\right] = \delta^k_j p_j, \quad j,k = 1,\ldots,N. \tag{3.9}$$

In this case

$$\Delta^{mt}_{jk,l} = \delta^t_l (T_l - 1) C^m_{jk}(x), \quad j,k,l,m,t = 0,1,2,\ldots,N, \tag{3.10}$$

where for an arbitrary function $\varphi(x)$ the action of T_j is defined by $T_j\varphi(x^0,\ldots,x^j,\ldots,x^N) = \varphi(x^0,\ldots,x^j+1,\ldots,x^N)$. The corresponding CS is of the form

$$C_l T_l C_j - C_j T_j C_l = 0, \quad j,l = 0,1,\ldots,N, \tag{3.11}$$

where the matrices C_j are defined as $(C_j)^l_k = C^l_{jk}$, $j,k,l = 0,1,\ldots,N$. The discrete CS (3.11) governs discrete deformations of associative algebras. The CS (3.11) contains, as particular cases, the discrete versions of the oriented associativity equation, WDVV equation, Boussinesq equation, and discrete KP hierarchy and Hirota-Miwa bilinear equations for KP τ-function [13].

For *coisotropic deformations* of commutative algebras [10, 11] again $M = N$, but the DDA is the Poisson algebra with p_j and x^k identified with the Darboux coordinates, that is,

$$\{p_j, p_k\} = 0, \quad \left\{x^j, x^k\right\} = 0, \quad \left\{p_j, x^k\right\} = -\delta^k_j, \quad j,k = 0,1,\ldots,N, \tag{3.12}$$

where $\{,\}$ is the standard Poisson bracket. The algebra $U(B)$ is the commutative ring of functions and divisors of zero are realized by functions with zeros. So, the functions f_{jk} should be functions with common set Γ of zeros. Thus, in the coisotropic case the multiplication table (1) is realized by the following set of equations [10]:

$$f_{jk} = 0, \quad j,k = 0,1,2,\ldots,N. \tag{3.13}$$

The compatibility condition for these equations is (see e.g., [10])

$$\{f_{jk}, f_{nl}\}|_{\Gamma} = 0, \quad j,k,l,n = 1,2,\ldots,N. \tag{3.14}$$

The set Γ is the coisotropic submanifold in $R^{2(N+1)}$. The condition (3.14) gives rise to the following system of equations for the structure constants:

$$[C,C]^m_{jklr} \doteqdot \sum_{s=0}^{N}\left(C^m_{sj}\frac{\partial C^s_{lr}}{\partial x^k} + C^m_{sk}\frac{\partial C^s_{lr}}{\partial x^j} - C^m_{sr}\frac{\partial C^s_{jk}}{\partial x^l} - C^m_{sl}\frac{\partial C^s_{jk}}{\partial x^r} + C^s_{lr}\frac{\partial C^m_{jk}}{\partial x^s} - C^s_{jk}\frac{\partial C^m_{lr}}{\partial x^s}\right) = 0 \tag{3.15}$$

while the equations $\Omega^n_{klj}(x) = 0$ have the form of associativity conditions (1.2):

$$\Omega^n_{klj}(x) = \sum_{m=0}^{N}\left(C^m_{jk}(x)C^n_{ml}(x) - C^m_{kl}(x)C^n_{jm}(x)\right) = 0. \tag{3.16}$$

Equations (3.15) and (3.16) form the CS for coisotropic deformations [10]. In this case C^l_{jk} is transformed as the tensor of the type $(1,2)$ under the general transformations of coordinates x^j, and the whole CS of (3.15) and (3.16) is invariant under these transformations [14]. The bracket $[C,C]^m_{jklr}$ has appeared for the first time in [21] where the so-called differential concomitants were studied. It was shown in [16] that this bracket is a tensor only if the tensor C^l_{jk} obeys the algebraic constraint (3.16). In [7] the CS of (3.15) and (3.16) has appeared implicitly as the system of equations which characterizes the structure constants for F-manifolds. In [10] it has been derived as the CS governing the coisotropic deformations of associative algebras.

The CS of (3.15) and (3.16) contains the oriented associativity equation, the WDVV equation, dispersionless KP hierarchy, and equations from the genus zero universal Whitham hierarchy as the particular cases [10, 11]. Yano manifolds and Yano algebroids associated with the CS of (3.15) and (3.16) are studied in [14].

We would like to emphasize that for all deformations considered above the stationary solutions of the CSs obey the global associativity condition (1.2).

4. Three-Dimensional Lie Algebras as DDA

In the rest of the paper we will study deformations of associative algebras generated by three-dimensional real Lie algebra L. The complete list of such algebras contains 9 algebras (see e.g. [16]). Denoting the basis elements by e_1, e_2, e_3, one has the following nonequivalent cases:

(1) abelian algebra L_1,

(2) general algebra L_2: $[e_1, e_2] = e_1,\ [e_2, e_3] = 0,\ [e_3, e_1] = 0$,

(3) nilpotent algebra L_3 : $[e_1, e_2] = 0,\ [e_2, e_3] = e_1,\ [e_3, e_1] = 0$,

(4)–(7) four nonequivalent solvable algebras: $[e_1, e_2] = 0,\ [e_2, e_3] = \alpha e_1 + \beta e_2,\ [e_3, e_1] = \gamma e_1 + \delta e_2$ with $\alpha\delta - \beta\gamma \neq 0$,

(8)-(9) simple algebras $L_8 = \mathrm{so}\,(3)$ and $L_9 = \mathrm{so}\,(2, 1)$.

In virtue of the one-to-one correspondence between the elements of the basis in DDA and the elements p_j, x^k an algebra L should have an abelian subalgebra and only one of its elements may play the role of the deformation parameter x. For the original algebra A and the algebra B one has two options.

(1) A is a two-dimensional algebra without unit element and $B = L$.

(2) A is a three-dimensional algebra with the unit element and $B = L_0 \oplus L$ where L_0 is the algebra generated by the unity element p_0.

After the choice of B one should establish a correspondence between p_1, p_2, x and e_1, e_2, e_3 defining DDA. For each algebra L_k there are obviously, in general, six possible identifications if one avoids linear superpositions. Some of them are equivalent. The incomplete list of nonequivalent identifications is as follows

(1) algebra $L_1 : p_1 = e_1,\ p_2 = e_2,\ x = e_3$; DDA is the commutative algebra with

$$[p_1, p_2] = 0, \quad [p_1, x] = 0, \quad [p_2, x] = 0, \tag{4.1}$$

(2) algebra L_2:

case (a) $p_1 = -e_2,\ p_2 = e_3,\ x = e_1$; the corresponding DDA is the algebra L_{2a} with the commutation relations:

$$[p_1, p_2] = 0, \quad [p_1, x] = x, \quad [p_2, x] = 0, \tag{4.2}$$

case (b) $p_1 = e_1,\ p_2 = e_3,\ x = e_2$; the corresponding DDA L_{2b} is defined by

$$[p_1, p_2] = 0, \quad [p_1, x] = p_1, \quad [p_2, x] = 0, \tag{4.3}$$

(3) algebra L_3: $p_1 = e_1,\ p_2 = e_2,\ x = e_3$; DDA L_3 is

$$[p_1, p_2] = 0, \quad [p_1, x] = 0, \quad [p_2, x] = p_1, \tag{4.4}$$

(4) solvable algebra L_4 with $\alpha = 0,\ \beta = 1,\ \gamma = -1,\ \delta = 0 : p_1 = e_1,\ p_2 = e_2,\ x = e_3$; DDA L_4 is

$$[p_1, p_2] = 0, \quad [p_1, x] = p_1, \quad [p_2, x] = p_2, \tag{4.5}$$

(5) solvable algebra L_5 at $\alpha = 1,\ \beta = 0,\ \gamma = 0,\ \delta = 1 : p_1 = e_1,\ p_2 = e_2,\ x = e_3$; DDA L_5 is

$$[p_1, p_2] = 0, \quad [p_1, x] = p_1, \quad [p_2, x] = -p_2. \tag{4.6}$$

For the second choice of the algebra $B = L_0 \oplus L$ mentioned above the table of multiplication (1.1) consists of the trivial part $\mathbf{P}_0\mathbf{P}_j = \mathbf{P}_j\mathbf{P}_0 = \mathbf{P}_j$, $j = 0, 1, 2$ and the nontrivial part:

$$\begin{aligned} \mathbf{P}_1^2 &= A\mathbf{P}_0 + B\mathbf{P}_1 + C\mathbf{P}_2, \\ \mathbf{P}_1\mathbf{P}_2 &= D\mathbf{P}_0 + E\mathbf{P}_1 + G\mathbf{P}_2, \\ \mathbf{P}_2^2 &= K\mathbf{P}_0 + M\mathbf{P}_1 + N\mathbf{P}_2. \end{aligned} \tag{4.7}$$

For the first choice $B = K$ the multiplication table is given by (4.7) with $A = D = K = 0$.

It is convenient also to arrange the structure constants $A, B, \ldots, N$ into the matrices C_1, C_2 defined by $(C_j)_k^l = C_{jk}^l$. One has

$$C_1 = \begin{pmatrix} 0 & A & D \\ 1 & B & E \\ 0 & C & G \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & D & K \\ 0 & E & M \\ 1 & G & N \end{pmatrix}. \tag{4.8}$$

In terms of these matrices the associativity conditions (1.2) are written as

$$C_1 C_2 = C_2 C_1. \tag{4.9}$$

Simple algebras L_8 and L_9 do not contain two commuting elements to be identified with p_1 and p_2, and, hence, they cannot be DDA. Deformations generated by algebras L_6 and L_7 will be considered elsewhere.

5. Deformations Generated by General DDAs

(1) Commutative DDA (4.1) does not force any deformation of structure constants. So, we begin with the three-dimensional commutative algebra A and DDA L_{2a} defined by the commutation relations (4.2). These relations imply that for an arbitrary function $\varphi(x)$

$$[p_j, \varphi(x)] = \Delta_j \varphi(x), \quad j = 1, 2, \tag{5.1}$$

where $\Delta_1 = (x\partial/\partial x), \Delta_2 = 0$. Consequently, one has the following CS:

$$\Omega^n_{klj}(x) = \Delta_l C^n_{jk} - \Delta_j C^n_{kl} + \sum_{m=0}^{2} \left(C^m_{jk} C^n_{lm} - C^m_{kl} C^n_{jm} \right) = 0, \quad j, k, l, n = 0, 1, 2. \tag{5.2}$$

In terms of the matrices C_1 and C_2 defined above this CS has a form of the Lax equation:

$$x \frac{\partial C_2}{\partial x} = [C_2, C_1]. \tag{5.3}$$

The CS (5.3) has all remarkable standard properties of the Lax equations (see e.g. [20, 21]): it has three independent first integrals:

$$I_1 = \operatorname{tr} C_2, \qquad I_2 = \frac{1}{2} \operatorname{tr} (C_2)^2, \qquad I_3 = \frac{1}{3} \operatorname{tr} (C_2)^3, \tag{5.4}$$

and it is equivalent to the compatibility condition of the linear problems:

$$\begin{aligned} C_2 \Phi &= \lambda \Phi, \\ x \frac{\partial \Phi}{\partial x} &= -C_1 \Phi, \end{aligned} \tag{5.5}$$

where Φ is the column with three components and λ is a spectral parameter. Though the evolution in x described by the second linear problem (5.5) is too simple, nevertheless the CS (5.2) or (5.3) has the meaning of the isospectral deformations of the matrix C_2 that is typical to the class of integrable systems (see e.g. [22, 23]).

CS (5.3) is the system of six equations for the structure constants D, E, G, L, M, N with free A, B, C:

$$\begin{aligned} D' &= DB + KC - AE - DG, \\ K' &= DE + KG - AM - DN, \\ E' &= MC - EG - D, \\ M' &= E^2 + MG - BM - EN - K, \\ G' &= GB + NC - CE - G^2 + A, \\ N' &= GE - CM + D, \end{aligned} \tag{5.6}$$

where $D' = x\partial D/\partial x$ and so forth. Here we will consider only simple particular cases of the CS (5.6). First it corresponds to the constraint $A = 0$, $B = 0$, $C = 0$, that is, to the nilpotent $\mathbf{P}_1$. The corresponding solution is

$$D = \frac{\beta}{\ln x}, \quad E = -\beta + \frac{\gamma}{\ln x}, \quad G = \frac{1}{\ln x}, \quad K = \alpha\beta + 2\beta^2 + \delta \ln x - \frac{\beta\gamma}{\ln x},$$
$$M = \alpha\gamma + 3\beta\gamma + \mu \ln x - \delta(\ln x)^2 - \frac{\gamma^2}{\ln x}, \quad N = \alpha + \beta - \frac{\gamma}{\ln x}, \tag{5.7}$$

where $\alpha, \beta, \gamma, \delta, \mu$ are arbitrary constants. The three integrals for this solution are

$$I_1 = \alpha, \quad I_2 = \frac{1}{2}\alpha^2 + 3\beta^2 + 2\alpha\beta + \mu,$$
$$I_3 = \frac{1}{3}\left((\alpha+\beta)^3 - \beta^3\right) + (\alpha+\beta)(\mu + \beta(\alpha + 2\beta)) - \gamma\delta. \tag{5.8}$$

The second example is given by the constraint $B = 0$, $C = 1$, $G = 0$ for which the quantum CS (3.3) is equivalent to the Boussinesq equation [12]. Under this constraint the CS (5.6) is reduced to the single equation:

$$E'' - 6E^2 + 4\alpha E + \beta = 0, \tag{5.9}$$

and the other structure constants are given by

$$A = 2E - \alpha, \quad B = 0, \quad C = 1, \quad D = \gamma - \frac{1}{2}E', \quad G = 0,$$
$$K = -E^2 + \alpha E + \frac{1}{2}\beta, \quad M = \gamma + \frac{1}{2}E', \quad N = \alpha - N, \tag{5.10}$$

where α, β, γ are arbitrary constants. The corresponding first integrals are

$$I_1 = \alpha, \quad I_2 = \frac{1}{2}\left(\beta + \alpha^2\right), \quad I_3 = \frac{1}{3}\alpha^3 + \gamma^2 + \frac{1}{2}\alpha\beta - \frac{1}{4}(E')^2 + E^3 - \alpha E^2 - \frac{1}{2}\beta E. \tag{5.11}$$

Integral I_3 reproduces the well-known first integral of (5.9). Solutions of (5.9) are given by elliptic integrals (see e.g., [24]). Any such solution together with the formulae (5.10) describes deformation of the three-dimensional algebra A driven by DDA L_{2a}.

Now we will consider deformations of the two-dimensional algebra A without unit element according to the first option mentioned in the previous section. In this case the CS has the form (5.3) with the 2×2 matrices

$$C_1 = \begin{pmatrix} B & E \\ C & G \end{pmatrix}, \qquad C_2 = \begin{pmatrix} E & M \\ G & N \end{pmatrix} \tag{5.12}$$

or in components

$$\begin{aligned} E' &= MC - EG, \\ M' &= E^2 + MG - BM - EN, \\ G' &= GB + NC - CE - G^2, \\ N' &= GE - CM. \end{aligned} \tag{5.13}$$

In this case there are two independent integrals of motion:

$$I_1 = E + N, \qquad I_2 = \frac{1}{2}\left(E^2 + N^2 + 2MG\right). \tag{5.14}$$

The corresponding spectral problem is given by (5.5). Eigenvalues of the matrix C_2, that is, $\lambda_{1,2} = (1/2)(E+N\pm\sqrt{(E-N)^2+4GM})$ are invariant under deformations and $\det C_2 = (1/2)I_1^2 - I_2$. We note also an obvious invariance of (5.6) and (5.13) under the rescaling of x.

The system of (5.13) contains two arbitrary functions B and C. In virtue of the possible rescaling $\mathbf{P}_1 \to \mu_1\mathbf{P}_1, \mathbf{P}_2 \to \mu_2\mathbf{P}_2$ of the basis for the algebra A with two arbitrary functions μ_1, μ_2, one has four nonequivalent choices (1) $B = 0, C = 0$, (2) $B = 1, C = 0$, (3) $B = 0, C = 1$, and (4) $B = 1, C = 1$.

In the case $B = 0, C = 0$ (nilpotent $\mathbf{P}_1$) the solution of the system (5.13) is

$$B = 0, \quad C = 0, \quad E = \frac{\beta}{\ln x}, \quad G = \frac{1}{\ln x}, \quad M = \gamma \ln x - \frac{\beta^2}{\ln x} + \alpha\beta, \quad N = -\frac{\beta}{\ln x} + \alpha, \tag{5.15}$$

where α, β, γ are arbitrary constants. For this solution the integrals are equal to $I_1 = \alpha, I_2 = \gamma + (1/2)\alpha^2$, and $\lambda_{1,2} = (1/2)(\alpha + \sqrt{\alpha^2 + 4\gamma})$.

At $B = 1, C = 0$ the system (5.13) has the following solution:

$$\begin{gathered} B = 1, \quad C = 0, \quad E = \frac{\gamma}{x+\beta}, \quad G = \frac{x}{x+\beta}, \\ M = \delta + \left(\alpha\gamma + \beta\delta - \frac{\gamma^2}{\beta}\right)\frac{1}{x} + \frac{\gamma^2}{\beta(x+\beta)}, \quad N = -\frac{\gamma}{x+\beta} + \alpha, \end{gathered} \tag{5.16}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants. The integrals are $I_1 = \alpha, I_2 = \delta + (1/2)\alpha^2$. The formulae (5.15) and (5.16) provide us with explicit deformations of the structure constants.

In the last two cases the CS (5.13) is equivalent to the simple third-order ordinary differential equations. At $B = 0, C = 1$ with additional constraint $I_1 = 0$ one gets

$$G''' + 2G^2G' + 4(G')^2 + 2GG'' = 0 \tag{5.17}$$

while at $B = 1, C = 1$, and $I_1 = 0$ the system (5.13) becomes

$$G''' + 2G^2G' + 4(G')^2 + 2GG'' - G' = 0. \tag{5.18}$$

The second integral for these ODEs is

$$I_2 = -\frac{1}{2}G^4 + \frac{1}{2}(G')^2 - 2G^2G' - GG'' + \frac{1}{2}BG^2. \tag{5.19}$$

Equation (5.17) with $G' = \partial G/\partial y$ is the Chazy V equation from the well-known Chazy-Bureau list of the third-order ODEs having Painlevé property [25, 26]. The integral (5.19) is known too (see e.g. [27]).

The appearance of the Chazy V equation among the particular cases of the system (5.13) indicates that for other choices of B and C the CS (5.13) may be equivalent to the other notable third-order ODEs. It is really the case. Here we will consider only the reduction $C = 1$ with $I_1 = N + E = 0$. In this case the system (5.13) is reduced to the following equation:

$$G''' + 2G^2G' + 4(G')^2 + 2GG'' - 2G'\Phi - G\Phi' = 0, \tag{5.20}$$

where $\Phi = B' + (1/2)B^2$. The second integral is

$$I_2 = -\frac{1}{2}G^4 + \frac{1}{2}(G')^2 - 2G^2G' - GG'' + \Phi G^2, \tag{5.21}$$

and $\lambda_{1,2} = \pm\sqrt{I_2/2}$.

Choosing particular B or Φ, one gets equations from the Chazy-Bureau list. Indeed, at $\Phi = 0$ one has the Chazy V equation (5.17). Choosing $\Phi = G'$, one gets the Chazy VII equation:

$$G''' + 2G^2G' + 2(G')^2 + GG'' = 0. \tag{5.22}$$

At $B = 2G$ (5.20) becomes the Chazy VIII equation:

$$G''' - 6G^2G' = 0. \tag{5.23}$$

Choosing the function Φ such that

$$\left(6\Phi e^{(1/3)G}\right)' = 2G^2G' + (G')^2 + 4GG'', \tag{5.24}$$

one gets the Chazy III equation:

$$G''' - 2GG'' + 3(G')^2 = 0. \tag{5.25}$$

In the above particular cases the integral I_2 (5.21) is reduced to those given in [27].

All Chazy equations presented above have the Lax representation (5.3) with $E = -N = -(1/2)(G' + G^2 + GB)$, $M = -(1/2)(G'' + 3GG' + G^3 + G^2B + (GB)')$, $C = 1$, and the proper choice of B.

Solutions of all these Chazy equations provide us with the deformations of the structure constants (5.12) for the two-dimensional algebra A generated by the DDA L_{2a}.

(2) Now we pass to the DDA L_{2b}. The commutation relations (4.3) imply that

$$[p_1, \varphi(x)] = (T-1)\varphi(x) \cdot p_1, \quad [p_2, \varphi(x)] = 0, \tag{5.26}$$

where $\varphi(x)$ is an arbitrary function and $T\varphi(x) = \varphi(x+1)$. Using (5.26), one finds the corresponding CS:

$$\sum_{m=0}^{2} \left((\Delta_l + 1) C^m_{jk}(x) \cdot C^n_{lm}(x) = (\Delta_j + 1) C^m_{kl}(x) \cdot C^n_{jm}(x) \right), \quad j,k,l,n = 0,1,2, \tag{5.27}$$

where $\Delta_1 = T - 1$, $\Delta_2 = 0$. In terms of the matrices C_1 and C_2, this CS is

$$C_1 T C_2 = C_2 C_1. \tag{5.28}$$

For nondegenerated matrix C_1 one has

$$T C_2 = C_1^{-1} C_2 C_1. \tag{5.29}$$

The CS (5.29) is the discrete version of the Lax equation (5.3) and has similar properties. It has three independent first integrals:

$$I_1 = \operatorname{tr} C_2, \qquad I_2 = \frac{1}{2} \operatorname{tr} (C_2)^2, \qquad I_3 = \frac{1}{3} \operatorname{tr} (C_2)^3, \tag{5.30}$$

and it represents itself the compatibility condition for the linear problems:

$$\begin{aligned} \Phi C_2 &= \lambda \Phi, \\ T\Phi &= \Phi C_1. \end{aligned} \tag{5.31}$$

Note that $\det C_2$ is the first integral too.

The CS (5.28) is the discrete dynamical system in the space of the structure constants. For the two-dimensional algebra A with matrices (5.12) it is

$$\begin{aligned} BTE + ETG &= EB + MC, \\ BTM + ETN &= E^2 + MG, \\ CTE + GTG &= BG + CN, \\ CTM + GTN &= EG + NG, \end{aligned} \tag{5.32}$$

where B and C are arbitrary functions. For nondegenerated matrix C_1, that is, at $BG - CE \neq 0$, one has the resolved form (5.29), that is,

$$TE = \frac{GM - EN}{BG - CE} C, \qquad TG = B + \frac{BN - CM}{BG - CE} C,$$
$$TM = \frac{GM - EN}{BG - CE} G, \qquad TN = E + \frac{BN - CM}{BG - CE} G. \tag{5.33}$$

This system defines discrete deformations of the structure constants.

6. Nilpotent DDA

For the nilpotent DDA L_3, in virtue of the defining relations (4.5), one has

$$[p_1, \varphi(x)] = 0, \qquad [p_2, \varphi(x)] = \frac{\partial \varphi}{\partial x} \cdot p_1 \tag{6.1}$$

or

$$[p_j, \varphi(x)] = \frac{\partial \varphi}{\partial x} \cdot \sum_{k=1}^{2} a_{jk} p_k, \tag{6.2}$$

where $a_{21} = 1, a_{11} = a_{12} = a_{22} = 0$. Using (6.2), one gets the following CS:

$$\sum_{q=1}^{2} a_{lq} \sum_{m=0}^{2} C_{qm}^{n} \frac{\partial C_{jk}^{m}}{\partial x} - \sum_{q=1}^{2} a_{jq} \sum_{m=0}^{2} C_{qm}^{n} \frac{\partial C_{kl}^{m}}{\partial x} + \sum_{m=0}^{2} \left(C_{jk}^{m} C_{lm}^{n} - C_{kl}^{m} C_{jm}^{n} \right) = 0, \quad j,k,l,n = 0,1,2. \tag{6.3}$$

In the matrix form it is

$$C_1 \frac{\partial C_1}{\partial x} = [C_1, C_2]. \tag{6.4}$$

For invertible matrix C_1

$$\frac{\partial C_1}{\partial x} = C_1^{-1} [C_1, C_2]. \tag{6.5}$$

This system of ODEs has three independent first integrals:

$$I_1 = \operatorname{tr} C_1, \qquad I_2 = \frac{1}{2} \operatorname{tr} (C_1)^2, \qquad I_3 = \frac{1}{3} \operatorname{tr} (C_1)^3, \tag{6.6}$$

and it is equivalent to the compatibility condition for the linear system:

$$C_1\Phi = \lambda\Phi,$$
$$C_1\frac{\partial\Phi}{\partial x} + C_2\Phi = 0. \tag{6.7}$$

So, as in the previous section the CS (6.4) describes isospectral deformations of the matrix C_1. This CS governs deformations generated by L_3.

For the two-dimensional algebra A without unit element the CS is given by (6.4) with the matrices (5.12). First integrals in this case are $I_1 = B + G, I_2 = (1/2)(B^2 + G^2 + 2CE)$ and $\det C_1 = (1/2)I_1^2 - I_2$. Since $\det C_1$ is a constant on the solutions of the system, then at $\det C_1 \neq 0$ one can always introduce the variable y defined by $x = y\det C_1$ such that CS (6.5) takes the form

$$\begin{aligned} B' &= EBG + ENC - GMC - CE^2, \\ E' &= GBM + GEN - ECM - MG^2, \\ C' &= BCE + BG^2 + MC^2 - CEG - BNC - GB^2, \\ G' &= CMG + CE^2 - CEN - BGE, \end{aligned} \tag{6.8}$$

where $B' = \partial B/\partial y$ and so forth and M, N are arbitrary functions. At $\det C_1 = BG - CE = 1$ this system becomes

$$\begin{aligned} B' &= E + C(EN - GM), \\ E' &= M + G(EN - GM), \\ C' &= G - B + C(MC - BN), \\ G' &= -E - C(EN - GM). \end{aligned} \tag{6.9}$$

Choosing $M = N = 0$, one gets

$$B' = E, \quad E' = 0, \quad C' = G - B, \quad G' = -E. \tag{6.10}$$

The solution of this system is

$$E = \alpha, \quad B = \alpha y + \beta, \quad G = -\alpha y + \gamma, \quad C = -y^2 + (\gamma - \beta)y + \delta, \tag{6.11}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants subject to the constraint $\beta\gamma - \alpha\delta = 1$. First integrals for this solution are $I_1 = \beta + \gamma, I_2 = (1/2)(\beta^2 + \gamma^2 + 2\alpha\delta)$.

With the choice $M = 0$, $N = 1$ and under the constraint $I_1 = B + G = 0$ the system (6.8) takes the form

$$B' = (1 + C)E, \quad E' = -BE, \quad C' = -(2 + C)B. \tag{6.12}$$

This system can be written as a single equation in the different equivalent forms. One of them is

$$(E')^2 + \alpha E^4 - 2E^3 + E^2 = 0, \tag{6.13}$$

where α is an arbitrary constant and

$$B^2 = -1 - \alpha E^2 + 2E, \quad C = \alpha E - 2, \quad G = -B. \tag{6.14}$$

The second integral is equal to -1.

Solutions of (6.13) can be expressed through the elliptic integrals. Solutions of (6.13) and the formulae (6.14) define deformations of the structure constants driven by DDA L_3.

7. Solvable DDAs

(1) For the solvable DDA L_4 the relations of (4.5) imply that

$$[p_j, \varphi(x)] = (T-1)\varphi(x)p_j, \quad j = 1,2, \tag{7.1}$$

where $\varphi(x)$ is an arbitrary function and T is the shift operator $T\varphi(x) = \varphi(x+1)$. With the use of (7.1) one arrives at the following CS:

$$C_1 T C_2 = C_2 T C_1. \tag{7.2}$$

For nondegenerated matrix C_1 (7.2) is equivalent to the equation $T(C_2C_1^{-1}) = C_1^{-1}C_2$ or

$$TU = C_1^{-1} U C_1, \tag{7.3}$$

where $U \doteqdot C_2C_1^{-1}$. Using this form of the CS, one promptly concludes that the CS (7.2) has three independent first integrals:

$$I_1 = \operatorname{tr}\left(C_2C_1^{-1}\right), \qquad I_2 = \frac{1}{2}\operatorname{tr}\left(C_2C_1^{-1}\right)^2, \qquad I_3 = \frac{1}{3}\operatorname{tr}\left(C_2C_1^{-1}\right)^3, \tag{7.4}$$

and it is representable as the commutativity condition for the linear system:

$$\begin{aligned} \Phi C_2 C_1^{-1} &= \lambda \Phi, \\ T\Phi &= \Phi C_1. \end{aligned} \tag{7.5}$$

For the two-dimensional algebra A one has the CS (7.2) with the matrices (5.12). It is the system of four equations for six functions:

$$\begin{aligned}
BTE + ETG &= ETB + MTC,\\
BTM + ETN &= ETE + MTG,\\
CTE + GTG &= GTB + NTC,\\
CTM + GTN &= GTE + NTG.
\end{aligned} \tag{7.6}$$

Choosing B and C as free functions and assuming that $BG - CE \neq 0$, one can easily resolve (7.6) with respect to TE, TG, TM, TN. For instance, with $B = C = 1$ one gets the following four-dimensional mapping:

$$\begin{gathered}
TE = M - E\frac{M-N}{E-G}, \quad TG = 1 + \frac{M-N}{E-G},\\
TM = N + (N-G)\frac{M-N}{E-G} - G\left(\frac{M-N}{E-G}\right)^2,\\
TN = M + (1-E)\frac{M-N}{E-G} + \left(\frac{M-N}{E-G}\right)^2.
\end{gathered} \tag{7.7}$$

(2) In a similar manner one finds the CS associated with the solvable DDA L_5. Since in this case

$$\left[p_1, \varphi(x)\right] = (T-1)\varphi(x)p_1, \qquad \left[p_2, \varphi(x)\right] = \left(T^{-1} - 1\right)\varphi(x)p_2, \tag{7.8}$$

the CS takes the form

$$C_1 T C_2 = C_2 T^{-1} C_1. \tag{7.9}$$

For nondegenerated C_2 it is equivalent to

$$TV = C_2 V C_2^{-1}, \tag{7.10}$$

where $V \doteqdot T^{-1}C_1 \cdot C_2$. Similar to the previous case the CS has three first integrals:

$$I_1 = \operatorname{tr}(C_1TC_2), \qquad I_2 = \frac{1}{2}\operatorname{tr}(C_1TC_2)^2, \qquad I_3 = \frac{1}{3}\operatorname{tr}(C_1TC_2)^3, \tag{7.11}$$

and it is equivalent to the compatibility condition for the linear system:

$$\begin{gathered}
\left(T^{-1}C_1\right)C_2\Phi = \lambda\Phi,\\
T\Phi = C_2\Phi.
\end{gathered} \tag{7.12}$$

Note that the CS (7.9) is of the form (3.11) with $T_1 = T$, $T_2 = T^{-1}$. Thus, the deformations generated by L_5 can be considered as the reductions of the discrete deformations (3.11) under the constraint $T_1T_2C^n_{jk} = C^n_{jk}$.

A class of solutions of the CS (7.9) is given by

$$C_j = g^{-1}T_j g, \tag{7.13}$$

where g is 3×3 matrix and $T_0 = 1, T_1 = T, T_2 = T^{-1}$. Since $C^n_{jk} = C^n_{kj}$, one has $T_j g^m_l = T_l g^m_j$ and hence $g^m_j = T_j\Phi^m$ where Φ^0, Φ^1, Φ^2 are arbitrary functions. So, this subclass of deformations are defined by three arbitrary functions.

To describe the isoassociative deformations for which $C_1(x)C_2(x) = C_2(x)C_1(x)$ for all x these functions should obey the systems of equations:

$$\sum_{l,t=0}^{2} T_jT_t\Phi^n \cdot \left(g^{-1}\right)^t_l \cdot T_kT_m\Phi^l = \sum_{l,t=0}^{2} T_kT_t\Phi^n \cdot \left(g^{-1}\right)^t_l \cdot T_jT_m\Phi^l, \quad j,k,n,m = 0,1,2. \tag{7.14}$$

It is a version of the discrete oriented associativity equation.

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