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Research Article

H_∞ Control for Nonlinear Systems with Time-Varying Delay Using Matrix-Based Quadratic Convex Approach

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H_∞ control problem for nonlinear system with time-varying delay is considered by using a set of improved Lyapunov-Krasovskii functionals including some integral terms, and a matrix-based on quadratic convex, combined with Wirtinger's inequalities and some useful integral inequality. H_∞ controller is designed via memoryless state feedback control and new sufficient conditions for the existence of the H_∞ state feedback for the system are given in terms of linear matrix inequalities (LMIs). Numerical examples are given to illustrate the effectiveness of the obtained result.

1. Introduction

The phenomena of time delays are often encountered in many practical systems such as process control systems, manufacturing systems, networked control systems, and economic systems. The existence of these delays may be the source of instability and serious deterioration in the performance of the closed-loop systems. In real world systems especially, the delay should be assumed to be time-varying satisfying $0 \leq \tau_1 \leq \tau(t) \leq \tau_2$ and τ_1 is not necessarily restricted to be 0, namely, interval time-varying delay. Stability analysis of time-delay system has been investigated extensively in the past decades [1–25].

As of time delays, it is well known that the nonlinear perturbations can also cause instability and poor performance of practical systems. Therefore, the stability problem of time-delay systems with nonlinear perturbations has received increasing attention; see [9, 13, 16] and the references cited therein.

H_∞ control problem has been widely used to minimize the effects of the external disturbances. The purpose of the problem is to design an H_∞ controller to robustly stabilize the systems while guaranteeing a prescribed level of disturbance attenuation γ in the H_∞ sense for the systems with external disturbances. A delay-dependent H_∞ controller ensures asymptotic stability and a prescribed H_∞ performance level of the closed-loop systems. The H_∞ performance indexes and the upper bound of the delay are usually two performance indexes to be used to evaluate the conservatism of the derived condition. The conservatism of the delay-dependent H_∞ control is measured by the allowable delay size or H_∞ performance level bound obtained.

Recently, an improved robust stability and H_∞ performance analysis criterion has been reported [3–5, 7, 10–12, 15, 17, 18]. In [17], H_∞ control problem for uncertain linear system with state delay and parameter uncertainties has been studied, but the time-varying delay is only bounded above by a constant. H_∞ control problem for system with interval time-varying delay has been considered in [7], by employing free weighting matrices approach. However, some useful terms in estimating the derivative of Lyapunov-Krasovskii functional are ignored which might lead to some conservatism. In [15], H_∞ control problem for nonlinear systems with interval time-varying delay has been studied by using Jensen's inequality to estimate some integral terms of Lyapunov-Krasovskii functional and deriving delay-dependent sufficient condition for the existence of H_∞ control by using reciprocally convex combination technique.

In the study of time-delay system, several approaches have been proposed in order to reduce conservatism.

For example, it is well known that choosing appropriate Lyapunov-Krasovskii functional and using improved bounding techniques to estimate time-derivative of Lyapunov-Krasovskii functional lead to improvement of stability region. Furthermore, free weighting matrices approach; delay decomposition approach; and convex optimization and reciprocally convex optimization techniques have been widely used to reduce conservatism of stability criterion; see [3, 7, 8, 10, 11, 13, 14, 16]. Recently, the so-called matrix-based quadratic convex approach has been introduced to derive stability criterion for time-delay system which was shown to reduce conservatism; for example, it gives better maximum allowable upper bound for time-varying delay than some other existing approaches; see [20–22].

Motivated by the above discussions, in this paper, matrix-based quadratic convex approach will be used to study H_∞ control problem for system with interval time-varying delay and nonlinear perturbations. To the best of our knowledge, this is one of the first reports of such investigation. By introducing new augmented Lyapunov-Krasovskii functional which has not been considered yet in stability analysis of H_∞ control problem, a delay-dependent stability criterion and H_∞ performance analysis are derived in terms of linear matrix inequalities (LMIs). This new Lyapunov-Krasovskii functional consists of integral terms of the form $\int_{t-h}^{t}(h-t-s)^j\dot{x}^T(s)R_j\dot{x}(s)\,ds$ ($j = 1,2$) which allows us to use the matrix-based quadratic convex approach introduced in [20–22]. With the use of this new Lyapunov-Krasovskii functional, matrix-based quadratic convex approach combined with some improved bounding techniques for integral terms such as Wirtinger-based integral inequality [14, 20], some new cross terms will be introduced which enhance the feasible stability criterion. Through two numerical examples, it is shown that the obtained stability criterion may give a larger maximum allowable upper bound of time-varying delay than some existing results.

2. Mathematical Model and Preliminaries

The following notations will be used in this paper: $\mathscr{R}^+$ denotes the set of all nonnegative real numbers; $\mathscr{R}^n$ denotes the n-dimensional space with the Euclidean norm $\|\cdot\|$; $\mathscr{M}^{n\times r}$ denotes the space of all matrices of $(n\times r)$-dimensions.

A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\operatorname{Re}\lambda;\ \lambda \in \lambda(A)\}$.

$x_t := \{x(t+s) : s \in [-h, t_0]\}$, $\|x_t\| = \sup_{s\in[-h,t_0]}\|x(t+s)\|$; $\mathscr{C}([t_0,t],\mathscr{R}^n)$ denotes the set of all $\mathscr{R}^n$-valued continuous functions on $[t_0,t]$; $\mathscr{L}_2([t_0,t],\mathscr{R}^m)$ denotes the set of all the $\mathscr{R}^m$-valued square integrable functions on $[t_0,t]$.

Matrix A is called semipositive definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in \mathscr{R}^n$; A is positive definite ($A > 0$) if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$. The symmetric term in a matrix is denoted by $*$.

Consider the following system with time-varying delays and control input:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Dx(t-\tau(t)) + Bu(t)\\ &\quad + Cw(t) + f(t, x(t), x(t-\tau(t)), u(t), w(t)),\\ z(t) &= Ex(t) + Gx(t-\tau(t))\\ &\quad + Fu(t) + g(t, x(t), x(t-\tau(t)), u(t)),\end{aligned} \tag{1}$$

$$\begin{gathered} x(t_0+\theta) = \phi(\theta),\\ \theta \in [-\tau_2, 0], \quad (t_0,\phi) \in \mathscr{R}^+ \times \mathscr{C}([-\tau_2, t_0], \mathscr{R}^n),\end{gathered} \tag{2}$$

where $x(t) \in \mathscr{R}^n$ is the state; $u(t) \in \mathscr{R}^m$ is the control input; $w(t) \in \mathscr{L}_2([0,\infty],\mathscr{R}^r)$ is a disturbance input; and $z(t) \in \mathscr{R}^s$ is the observation output. The delay $\tau(t)$ is time-varying continuous function which satisfies

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \qquad \mu_1 \leq \dot{\tau}(t) \leq \mu_2. \tag{3}$$

Let $x^\tau = x(t-\tau(t))$. The nonlinear functions $f(t,x,x^\tau,u,w) : \mathscr{R}^+\times\mathscr{R}^n\times\mathscr{R}^n\times\mathscr{R}^m\times\mathscr{R}^r \to \mathscr{R}^n$, $g(t,x,x^\tau,u) : \mathscr{R}^+\times\mathscr{R}^n\times\mathscr{R}^n\times\mathscr{R}^m \to \mathscr{R}^s$ satisfy the following growth condition:

$$\begin{aligned} &\exists a,b,c,d > 0 : \left\|f(t,x,x^\tau,u,w)\right\|\\ &\qquad \leq a\|x\| + b\left\|x^\tau\right\| + c\|u\| + d\|w\|,\\ &\qquad\qquad \forall (x, x^\tau, u, w),\\ &\exists a_1, b_1, c_1 > 0 : \left\|g(t,x,x^\tau,u)\right\|^2\\ &\qquad \leq a_1\|x\|^2 + b_1\left\|x^\tau\right\|^2 + c_1\|u\|^2,\\ &\qquad\qquad \forall (x, x^\tau, u).\end{aligned} \tag{4}$$

Definition 1. Given $\gamma > 0$, the H_∞ control problem for system (1) is to seek if there exists a memoryless state feedback controller $u(t) = Kx(t)$ such that we have the following.

(i) The zero solution of the closed-loop system, where $w(t) = 0$,

$$\dot{x} = (A+BK)x(t) + Dx(t-\tau(t)) + f(t,x,x^\tau,u,0), \tag{5}$$

is asymptotically stable.

(ii) The H_∞ performance

$$\|z(t)\|_2 < \gamma\|w(t)\|_2 \tag{6}$$

of the closed-loop system (5) is guaranteed for all nonzero $w(t) \in \mathscr{L}_2([0,\infty],\mathscr{R}^r)$ and a prescribed $\gamma > 0$ under the condition $x(t) = 0, \forall t \in [-\tau_2, t_0]$. In this case, we say that the feedback control $u(t) = Kx(t)$ asymptotically stabilizes the system.

We introduce the following technical lemmas, which will be used in the proof of our results.

Lemma 2 (see [14]). *For a given matrix $R > 0$, the following inequality holds for any continuously differentiable function $w : [a,b] \to \mathscr{R}^n$:*

$$\int_a^b \dot{\omega}^T(u) R\dot{\omega}(u)\, du \geq \frac{1}{b-a}\left(\Gamma_1^T R\Gamma_1 + 3\Gamma_2^T R\Gamma_2\right), \quad (7)$$

where

$$\Gamma_1 := \omega(b) - \omega(a)$$
$$\Gamma_2 := \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(u)\, du. \quad (8)$$

Remark 3. It is clear to see that the inequality in Lemma 2 provides a tighter lower bound for $\int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du$ than Jensen's inequality since $3\Gamma_2^T R\Gamma_2 > 0$ for $\Gamma_2 \neq 0$. Thus, inequality (7) is an improvement over Jensen's inequality.

Before we introduce some useful integral inequalities, we denote

$$\nu_1(t) := \frac{1}{\tau_2 - \tau(t)}\int_{t-\tau_2}^{t-\tau(t)} y(s)\, ds,$$
$$\nu_2(t) := \frac{1}{\tau(t) - \tau_1}\int_{t-\tau(t)}^{t-\tau_1} y(s)\, ds, \quad (9)$$
$$\nu_3(t) := \frac{1}{\tau_1}\int_{t-\tau_1}^{t} y(s)\, ds.$$

Lemma 4 (see [20]). *For a given scalar $\tau_1 \geq 0$ and any $n\times n$ real matrices $Y_1 > 0$ and $Y_2 > 0$ and a vector $\dot{y} : [-\tau_1, 0] \to \mathscr{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any vector-valued function $\pi_1(t) : [0,\infty) \to \mathscr{R}^k$ and matrices $M_1 \in \mathscr{R}^{k\times k}$ and $N_1 \in \mathscr{R}^{k\times n}$ satisfying $\begin{bmatrix} M_1 & N_1 \\ N_1^T & Y_1 \end{bmatrix} \geq 0$,*

$$\wp_1 := \int_{t-\tau_1}^{t} (\tau_1 - t + s)\, \dot{y}^T(s) Y_1 \dot{y}(s)\, ds$$
$$\geq -\frac{\tau_1^2}{2}\pi_1^T(t) M_1 \pi_1(t) - 2\tau_1 \pi_1^T N_1\left[y(t) - \nu_3(t)\right], \quad (10)$$
$$\wp_2 := \int_{t-\tau_1}^{t} (\tau_1 - t + s)^2\, \dot{y}^T(s) Y_2 \dot{y}(s)\, ds$$
$$\geq \tau_1\left[y(t) - \nu_3(t)\right]^T Y_2\left[y(t) - \nu_3(t)\right],$$

where $\nu_3(t)$ is defined in (9).

Lemma 5 (see [21]). *Let $\tau(t)$ be a continuous function satisfying $0 \leq \tau_1 \leq \tau(t) \leq \tau_2$. For any $n\times n$ real matrix $R_2 > 0$ and a vector $\dot{y} : [-\tau_2, 0] \to \mathscr{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $\phi_{i1} \in \mathscr{R}^q$ and real matrices $Z_i \in \mathscr{R}^{q\times q}$, $F_i \in \mathscr{R}^{q\times n}$ satisfying $\begin{bmatrix} Z_i & F_i \\ F_i^T & R_2 \end{bmatrix} \geq 0$, $(i = 1, 2)$,*

$$-\int_{t-\tau_2}^{t-\tau_1} (\tau_2 - t + s)\, \dot{y}^T(s) R_2 \dot{y}(s)\, ds$$
$$\leq \frac{1}{2}(\tau_2 - \tau(t))^2 \phi_{11}^T Z_1 \phi_{11} + 2(\tau_2 - \tau(t))\phi_{11}^T F_1 \phi_{12}$$
$$+ \frac{1}{2}\left[(\tau_2 - \tau_1)^2 - (\tau_2 - \tau(t))^2\right]\phi_{21}^T Z_2 \phi_{21} \quad (11)$$
$$+ 2\phi_{21}^T F_2\left[(\tau_2 - \tau(t))\phi_{22} + (\tau(t) - \tau_1)\phi_{23}\right],$$

where

$$\phi_{12} := y(t - \tau(t)) - \nu_1(t),$$
$$\phi_{22} := y(t - \tau_1) - x(t - \tau(t)), \quad (12)$$
$$\phi_{23} := y(t - \tau_1) - \nu_2(t),$$

with $\nu_i(t)$ $(i = 1, 2)$ being defined in (9).

Lemma 6 (see [20]). *Let $\tau(t)$ be a continuous function satisfying $0 \leq \tau_1 \leq \tau(t) \leq \tau_2$. For any $n\times n$ real matrix $R_1 > 0$ and a vector $\dot{y} : [-\tau_2, 0] \to \mathscr{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $2n\times 2n$ real matrix S_1 satisfying $\begin{bmatrix} \widetilde{R}_1 & S_1 \\ S_1^T & \widetilde{R}_1 \end{bmatrix} \geq 0$,*

$$-(\tau_2 - \tau_1)\int_{t-\tau_2}^{t-\tau_1} \dot{y}^T(s) R_1 \dot{y}(s)\, ds$$
$$\leq 2\psi_{11}^T S_1 \psi_{21} - \psi_{11}^T \widetilde{R}_1 \psi_{11} - \psi_{21}^T \widetilde{R}_1 \psi_{21}, \quad (13)$$

where $\widetilde{R}_1 := \mathrm{diag}\{R_1, 3R_1\}$ and

$$\psi_{11} := \begin{bmatrix} y(t - \tau(t)) - y(t - \tau_2) \\ y(t - \tau(t)) + y(t - \tau_2) - 2\nu_1(t) \end{bmatrix},$$
$$\psi_{21} := \begin{bmatrix} y(t - \tau_1) - y(t - \tau(t)) \\ y(t - \tau_1) + y(t - \tau(t)) - 2\nu_2(t) \end{bmatrix}, \quad (14)$$

with $\nu_i(t)$ $(i = 1, 2)$ being defined in (9).

Remark 7. In Lemma 6, when $\tau_1 = 0$, the inequality reduces to similar one in [14]. It contains slack matrix variable S_1 with dimension $2n \times 2n$ comparing to slack matric variable with dimension $2n \times 5n$ introduced in [8].

Lemma 8 (see [20]). *Let χ_0, χ_1, and χ_2 be $m \times m$ real symmetric matrices and a continuous function τ satisfy $\tau_1 \leq \tau \leq \tau_2$, where τ_1 and τ_2 are constants satisfying $0 \leq \tau_1 \leq \tau_2$. If $\chi_0 \geq 0$, then*

$$\tau^2\chi_0 + \tau\chi_1 + \chi_2 < 0\,(\leq 0), \quad \forall\tau \in [\tau_1, \tau_2],$$
$$\Longleftrightarrow \tau_i^2\chi_0 + \tau_i\chi_1 + \chi_2 < 0\,(\leq 0), \quad (i = 1, 2), \quad (15)$$

or

$$\tau^2\chi_0 + \tau\chi_1 + \chi_2 > 0\,(\geq 0), \quad \forall\tau \in [\tau_1, \tau_2]$$
$$\Longleftrightarrow \tau_i^2\chi_0 + \tau_i\chi_1 + \chi_2 > 0\,(\geq 0), \quad (i = 1, 2). \quad (16)$$

3. Main Results

In this section, we give a design of memoryless H_∞ feedback control for system (1). First, we present delay-dependent asymptotical stabilizability analysis conditions for the nonlinear system with time-varying delay (1). Now, we operate the matrix-based quadratic convex approach with the integral inequalities in [20] to formulate a new stability criterion for system (1). For our goal, we choose the following Lyapunov-Krasovskii functional:

$$V(t, x_t, \dot{x}_t) = V_1(t) + V_2(t) + V_3(t), \tag{17}$$

where x_t denotes the function $x(t)$ defined on the interval $[t-\tau_2, t]$. Set $P_1 = P^{-1}$, $y(t) = P_1 x(t)$, $\tau_{21} := \tau_2 - \tau_1$, and

$$\begin{aligned}
V_1(t) &:= y^T(t) P y(t) + \int_{t-\tau_1}^{t} \dot{y}^T(s) Q_0 \dot{y}(s)\, ds, \\
V_2(t) &:= \int_{t-\tau_1}^{t} \left[y^T(t) \;\; y^T(s)\right] Q_1 \left[y^T(t) \;\; y^T(s)\right]^T ds \\
&\quad + \int_{t-\tau(t)}^{t-\tau_1} \left[y^T(t) \;\; y^T(s)\right] Q_2 \left[y^T(t) \;\; y^T(s)\right]^T ds \\
&\quad + \int_{t-\tau_2}^{t-\tau(t)} \left[y^T(t) \;\; y^T(s)\right] Q_3 \left[y^T(t) \;\; y^T(s)\right]^T ds, \\
V_3(t) &:= \int_{t-\tau_1}^{t} \left[\tau_1 (\tau_1 - t + s) \dot{y}^T(s) W_1 \dot{y}(s) \right. \\
&\quad \left. + (\tau_1 - t + s)^2 \dot{y}^T(s) W_2 \dot{y}(s)\right] ds \\
&\quad + \int_{t-\tau_2}^{t-\tau_1} \left[\tau_{21} (\tau_2 - t + s) \dot{y}^T(s) R_1 \dot{y}(s) \right. \\
&\quad \left. + (\tau_2 - t + s)^2 \dot{y}^T(s) R_2 \dot{y}(s)\right] ds,
\end{aligned} \tag{18}$$

where $Q_j > 0$, $(j = 0, 1, 2, 3)$, $W_1 > 0$, $W_2 > 0$, $R_1 > 0$, $R_2 > 0$, and P are real matrices to be determined. Before introducing the main theorem, for simplicity, we set

$$\varepsilon = a + b + c + \frac{4d^2}{\gamma}. \tag{19}$$

Theorem 9. *Given $\gamma > 0$, then system (1) is asymptotically stabilizable and satisfies $\|z(t)\|_2 < \gamma \|w(t)\|_2$ for all nonzero $w \in \mathscr{L}_2[0, \infty)$ if there exist positive definite matrices P, $Q_j > 0$, $(j = 0, 1, 2, 3)$, W_1, W_2, R_1, R_2, S_1, Z_1, Z_2, Z_3, N_1, N_2, N_3, and Y such that the following LMIs hold:*

$$\begin{gathered}
\begin{bmatrix} \widetilde{R}_1 & S_1 \\ S_1^T & \widetilde{R}_1 \end{bmatrix} \geq 0, \qquad \begin{bmatrix} Z_i & N_i \\ N_i^T & R_2 \end{bmatrix} \geq 0, \quad (i = 1, 2), \\
\begin{bmatrix} Z_3 & N_3 \\ N_3^T & W_2 \end{bmatrix} \geq 0, \quad Z_1 \geq Z_2,
\end{gathered} \tag{20}$$

$$\begin{gathered}
\Xi_2(\tau_1, \mu_1) + \Xi_3(\tau_1) + \widehat{\Xi}_4 < 0, \\
\Xi_2(\tau_1, \mu_2) + \Xi_3(\tau_1) + \widehat{\Xi}_4 < 0, \\
\Xi_2(\tau_2, \mu_1) + \Xi_3(\tau_2) + \widehat{\Xi}_4 < 0, \\
\Xi_2(\tau_2, \mu_2) + \Xi_3(\tau_2) + \widehat{\Xi}_4 < 0,
\end{gathered} \tag{21}$$

where $\widetilde{R}_1 = \operatorname{diag}\{R_1, 3R_1\}$ and

$$\Xi_2(\tau(t), \dot{\tau}(t)) := \Xi_{20} + [\tau(t) - \tau_1]\, \Xi_{21} + [\tau_2 - \tau(t)]\, \Xi_{22}, \tag{22}$$

$$\begin{aligned}
\Xi_3(\tau(t)) := {} & \widetilde{\varphi}_1^T S_1 \widetilde{\varphi}_2 + \widetilde{\varphi}_2^T S_1^T \widetilde{\varphi}_1 - \widetilde{\varphi}_1^T \widetilde{R}_1^T \widetilde{\varphi}_1 \\
& + (\tau_2 - \tau(t))^2 (Z_1 - Z_2) + (\tau_2 - \tau(t))\, \Xi_{31} \\
& + (\tau(t) - \tau_1)\, \Xi_{32} + \tau_{21}^2 Z_2 - \widetilde{\varphi}_2^T \widetilde{R}_1 \widetilde{\varphi}_2,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\widehat{\Xi}_4 := {} & \tau_1^2 Z_3 - \widetilde{\varphi}_3^T \widetilde{W}_1 \widetilde{\varphi}_3 \\
& + e_9^T (\tau_1^2 W_1 + \tau_1^2 W_2) e_9 + 2\tau_1 N_3 (e_1 - e_7) \\
& + e_8^T (\tau_{21}^T R_1 + \tau_{21}^T R_2) e_8 + 2\tau_1 (e_1 - e_7)^T N_3^T \\
& + e_1^T \left(AP + PA^T + (BY + Y^T B^T) + \frac{4}{\gamma} CC^T + \varepsilon I \right) e_1 \\
& + e_1^T (PD) e_2 + e_2^T (D^T P) e_1 + e_1^T (AP + YB^T) e_9 \\
& + e_9^T (PA^T + Y^T B) e_1 + e_2^T (DP) e_9 + e_9^T (PD^T) e_2 \\
& + e_8^T (Q_0) e_8 + e_9^T \left(-2P + \frac{4}{\gamma} CC^T + \varepsilon I + Q_0 \right) e_9 \\
& + e_1^T (P) e_{10} + e_{10}^T (P) e_1 + e_1^T (Y^T) e_{11} + e_{11}^T (Y) e_1 \\
& + e_1^T (PE^T) e_{12} + e_{12}^T (EP) e_1 \\
& + e_2^T (PG^T) e_{13} + e_{13}^T (GP) e_2 \\
& + e_2^T (P) e_{14} + e_{14}^T (P) e_2 + e_{10}^T \left(-\frac{1}{2a + 4a_1} I \right) e_{10} \\
& + e_{11}^T \left(-\frac{1}{2 + 2c + 4c_1} I \right) e_{11} + e_{12}^T \left(-\frac{I}{3} \right) e_{12} \\
& + e_{13}^T \left(-\frac{I}{3} \right) e_{13} + e_{14}^T \left(-\frac{1}{2b + 4b_1} I \right) e_{14},
\end{aligned} \tag{24}$$

with $e_i \in \mathscr{R}^{n\times 14n}$ $(i = 1, 2, \ldots, 14)$ denoting the ith row-block vector of the $14n \times 14n$ identity matrix $\widetilde{W}_1 = \mathrm{diag}\{W_1, 3W_1\}$ and

$$\begin{aligned}
\Xi_{20} :=\ & \left[e_1^T \ \ e_3^T\right](Q_2 - Q_1)\left[e_1^T \ \ e_3^T\right]^T \\
& + \tau_1\left[e_9^T \ \ 0\right]Q_1\left[e_1^T \ \ e_7^T\right]^T + \tau_1\left[e_1^T \ \ e_7^T\right]Q_1\left[e_9^T \ \ 0\right]^T \\
& - (1 - \dot{\tau}(t))\left[e_1^T \ \ e_2^T\right](Q_2 - Q_3)\left[e_1^T \ \ e_2^T\right]^T \\
& - \left[e_1^T \ \ e_4^T\right]Q_3\left[e_1^T \ \ e_4^T\right]^T + \left[e_1^T \ \ e_1^T\right]Q_1\left[e_1^T \ \ e_1^T\right]^T, \\
\Xi_{21} :=\ & \left[e_1^T \ \ e_6^T\right]Q_2\left[e_9^T \ \ 0\right]^T + \left[e_9^T \ \ 0\right]Q_2\left[e_1^T \ \ e_6^T\right]^T, \\
\Xi_{22} :=\ & \left[e_1^T \ \ e_5^T\right]Q_3\left[e_9^T \ \ 0\right]^T + \left[e_9^T \ \ 0\right]Q_3\left[e_1^T \ \ e_5^T\right]^T, \\
\Xi_{31} :=\ & 2N_1(e_2 - e_5) + 2N_2(e_3 - e_2) \\
& + 2(e_3 - e_2)^T N_2^T + 2(e_2 - e_5)^T N_1^T, \\
\Xi_{32} :=\ & 2N_1(e_3 - e_6) + 2(e_3 - e_6)^T N_1^T, \\
\widetilde{\varphi}_1 :=\ & col\{e_2 - e_4, e_2 + e_4 - 2e_5\}, \\
\widetilde{\varphi}_2 :=\ & col\{e_3 - e_2, e_3 + e_2 - 2e_6\}, \\
\widetilde{\varphi}_3 :=\ & col\{e_1 - e_3, e_1 + e_3 - 2e_7\}.
\end{aligned} \tag{25}$$

Moreover, the feedback control is given by

$$u(t) = YP^{-1}x(t), \quad t \geq 0. \tag{26}$$

Proof. Taking the derivative of V along the solution of system (1), we obtain

$$\begin{aligned}
\dot{V}_1 =\ & 2y^T(t)P\dot{y}(t) + \dot{y}^T(t)Q_0\dot{y}(t) \\
& - \dot{y}^T(t-\tau_1)Q_0\dot{y}(t-\tau_1),
\end{aligned} \tag{27}$$

$$\begin{aligned}
\dot{V}_2 =\ & \left[y^T(t) \ \ y^T(t)\right]Q_1\left[y^T(t) \ \ y^T(t)\right]^T \\
& - \left[y^T(t) \ \ y^T(t-\tau_1)\right]Q_1\left[y^T(t) \ \ y^T(t-\tau_1)\right]^T \\
& + \int_{t-\tau_1}^{t} 2\left[y^T(t) \ \ y^T(s)\right]Q_1\left[\dot{y}(t)^T \ \ 0\right]^T ds \\
& + \left[y^T(t) \ \ y^T(t-\tau_1)\right]Q_2 \\
& \cdot \left[y^T(t) \ \ y^T(t-\tau_1)\right]^T \\
& - (1 - \dot{\tau}(t))\left[y^T(t) \ \ y^T(t-\tau(t))\right]Q_2 \\
& \cdot \left[y^T(t) \ \ y^T(t-\tau(t))\right]^T \\
& + 2\int_{t-\tau(t)}^{t-\tau_1}\left[y^T(t) \ \ y^T(s)\right]Q_2\left[\dot{y}^T(t) \ \ 0\right]^T ds \\
& - \left[y^T(t) \ \ y^T(t-\tau_2)\right]Q_3\left[y^T(t) \ \ y^T(t-\tau_2)\right]^T \\
& + (1 - \dot{\tau}(t))\left[y^T(t) \ \ y^T(t-\tau(t))\right]Q_3 \\
& \cdot \left[y^T(t) \ \ y^T(t-\tau(t))\right]^T \\
& + \int_{t-\tau_2}^{t-\tau(t)} 2\left[y^T(t) \ \ y^T(s)\right]Q_2\left[\dot{y}^T(t) \ \ 0\right]^T ds, \\
\dot{V}_3 =\ & \tau_1\dot{y}^T(t)\tau_1W_1\dot{y}(t) + \tau_1^2\dot{y}^T(t)W_2\dot{y}(t) \\
& - \int_{t-\tau_1}^{t}\dot{y}^T(s)\tau_1W_1\dot{y}(s)\,ds \\
& - 2\int_{t-\tau_1}^{t}(\tau_1 - t + s)\dot{y}^T(s)W_2\dot{y}(s)\,ds \\
& + \tau_{21}\dot{y}^T(t-\tau_1)\tau_{21}R_1\dot{y}(t-\tau_1) \\
& + \tau_{21}^2\dot{y}^T(t-\tau_1)R_2\dot{y}(t-\tau_1) \\
& - \tau_{21}\int_{t-\tau_2}^{t-\tau_1}\dot{y}^T(s)R_1\dot{y}(s)\,ds \\
& - 2\int_{t-\tau_2}^{t-\tau_1}(\tau_2 - t + s)\dot{y}^T(s)R_2\dot{y}(s)\,ds.
\end{aligned} \tag{28}$$

From (4) and Cauchy inequality, we get the following inequalities:

$$\begin{aligned}
& 2x^TP_1f(t, x, x_h, u, w) \\
& \quad \leq 2\|P_1x\|\,\|f(t, x, x_h, u, w)\| \\
& \quad \leq 2\|P_1x\|\left(a\|x\| + b\|x_h\| + c\|u\| + d\|w\|\right) \\
& \quad \leq a\|P_1x\|^2 + a\|x\|^2 + b\|P_1x\|^2 + b\|P_1x_h\|^2 + c\|P_1x\|^2 \\
& \qquad + c\|u\|^2 + \frac{4d^2}{\gamma}\|P_1x\|^2 + \frac{\gamma}{4}\|w\|^2 \\
& \quad = a\|x\|^2 + b\|x_h\|^2 + c\|u\|^2 + \frac{\gamma}{4}\|w\|^2 + \varepsilon\|P_1x\|^2.
\end{aligned} \tag{29}$$

Similarly,

$$\begin{aligned}
& 2\dot{x}^TP_1f(t, x, x_\tau, u, w) \\
& \quad \leq a\|x\|^2 + b\|x_\tau\|^2 + c\|u\|^2 + \frac{\gamma}{4}\|w\|^2 + \varepsilon\|P_1\dot{x}\|^2, \\
& 2x(t)^TP_1Cw \leq \frac{\gamma}{4}\|w\|^2 + \frac{4}{\gamma}x(t)^TP_1CC^TP_1x(t), \\
& 2\dot{x}(t)^TP_1Cw \leq \frac{\gamma}{4}\|w\|^2 + \frac{4}{\gamma}\dot{x}(t)^TP_1CC^TP_1\dot{x}(t).
\end{aligned} \tag{30}$$

By using the following identity relation:

$$-2\dot{x}^T(t)P_1[\dot{x}(t)-Ax(t)-Dx(t-\tau(t)) \\ -Bu(t)-Cw(t)-f(\cdot)]=0, \tag{31}$$

we obtain the following:

$$\begin{aligned} 0 &= -2\dot{x}^T(t)P_1[\dot{x}(t)-Ax(t)-Dx(t-\tau(t)) \\ &\qquad -Bu(t)-Cw(t)-f(\cdot)] \\ &\le -2\dot{x}^T(t)P_1\left[\dot{x}(t)-Ax(t)-Dx(t-\tau(t)) \right. \\ &\qquad \left. -BYP_1x(t)\right] \\ &\quad + 2\dot{x}^T(t)P_1Cw(t)+2\dot{x}^T(t)P_1f(\cdot) \\ &\le -2\dot{y}^T(t)P\dot{y}(t)+2\dot{y}^T(t)APy(t) \\ &\quad + 2\dot{y}^T(t)DPy(t-\tau(t))+2\dot{y}^T(t)BYy(t) \\ &\quad + \frac{4}{\gamma}\dot{y}(t)^TCC^T\dot{y}(t)+ay(t)^TP^2y(t) \\ &\quad + by^T(t-\tau(t))P^2y(t-\tau(t))+cy^T(t)Y^TYy(t) \\ &\quad + \frac{\gamma}{2}w(t)^Tw(t)+\varepsilon\dot{y}^T(t)\dot{y}(t). \end{aligned} \tag{32}$$

From (27) and (29), (32), $\dot{V}_1$ is estimated as

$$\begin{aligned} \dot{V}_1 &\le y^T(t)\left[AP+PA^T+\left(BY+Y^TB^T\right)\right. \\ &\qquad \left. +\frac{4}{\gamma}CC^T+aP^2+cY^TY+\varepsilon\right]y(t) \\ &\quad + \frac{\gamma}{2}\left\|w(t)^2\right\|+2y^T(t)[DP]y^T(t-\tau(t)) \\ &\quad + by^T(t-\tau(t))P^2y \\ &\quad \cdot\left(t-\tau(t)-2\dot{y}^T(t)P\dot{y}(t)\right. \\ &\qquad + 2\dot{y}^T(t)APy(t)+2\dot{y}^T(t)DPy(t-\tau(t)) \\ &\qquad \left. +2\dot{y}^T(t)BYy(t)\right) \\ &\quad + \frac{4}{\gamma}\dot{y}(t)^TCC^T\dot{y}(t)+ay(t)^TP^2y(t) \\ &\quad + by^T(t-\tau(t))P^2y(t-\tau(t))+cy^T(t)Y^TYy(t) \\ &\quad + \frac{\gamma}{2}w(t)^Tw(t)+\varepsilon\dot{y}^T(t)\dot{y}(t) \\ &\quad + \dot{y}^T(t)Q_0\dot{y}(t)-\dot{y}^T(t-\tau_1)Q_0\dot{y}(t-\tau_1) \\ &= \xi^T(t)\Xi_1\xi(t), \end{aligned} \tag{33}$$

where

$$\begin{aligned} \xi(t) &:= \operatorname{col}\{y(t),y(t-\tau(t)),y(t-\tau_1),y(t-\tau_2), \\ &\qquad \nu_1(t),\nu_2(t),\nu_3(t),\dot{y}(t-\tau_1),\dot{y}(t)\}, \\ \Xi_1 &:= e_1^T\left(AP+PA^T+\left(BY+Y^TB^T\right)\right. \\ &\qquad \left. +\frac{4}{\gamma}CC^T+2aP^2+2cY^TY+\varepsilon I\right)e_1 \\ &\quad + e_1^T(PD)e_2+e_2^T\left(D^TP\right)e_1+e_1^T\left(AP+YB^T\right)e_9 \\ &\quad + e_9^T\left(PA^T+Y^TB\right)e_1+e_2^T(DP)e_9+e_9^T\left(PD^T\right)e_2 \\ &\quad + e_9^T\left(-2P+\frac{4}{\gamma}CC^T+\varepsilon I+Q_0\right)e_9 \\ &\quad + e_2^T\left(2bP^2\right)e_2+e_8^T(-Q_0)e_8. \end{aligned} \tag{34}$$

With the consideration of the three terms of $\dot{V}_2(t)$, we obtained the following inequality:

$$\begin{aligned} &\int_{t-\tau_1}^{t} 2\left[y^T(t)\;\; y^T(s)\right]Q_1\left[\dot{y}(t)^T\;\; 0\right]^T ds \\ &\quad \le 2\left[\int_{t-\tau_1}^{t} y^T(t)\,ds\;\; \int_{t-\tau_1}^{t} y^T(s)\,ds\right]Q_1\left[\dot{y}^T(t)\;\; 0\right]^T \\ &\quad = 2\tau_1\left[y^T(t)\;\; \nu_3^T\right]Q_1\left[\dot{y}^T(t)\;\; 0\right]^T, \\ &\int_{t-\tau(t)}^{t-\tau_1} 2\left[y^T(t)\;\; y^T(s)\right]Q_2\left[\dot{y}(t)^T\;\; 0\right]^T ds \\ &\quad \le 2\left[\int_{t-\tau(t)}^{t-\tau_1} y^T(t)\,ds\;\; \int_{t-\tau(t)}^{t-\tau_1} y^T(s)\,ds\right]Q_2\left[\dot{y}^T(t)\;\; 0\right]^T \\ &\quad = 2\left(\tau(t)-\tau_1\right)\left[y^T(t)\;\; \nu_2^T\right]Q_2\left[\dot{y}^T(t)\;\; 0\right]^T, \\ &\int_{t-\tau_2}^{t-\tau(t)} 2\left[y^T(t)\;\; y^T(s)\right]Q_3\left[\dot{y}(t)^T\;\; 0\right]^T ds \\ &\quad \le 2\left[\int_{t-\tau_2}^{t-\tau(t)} y^T(t)\,ds\;\; \int_{t-\tau_2}^{t-d(t)} y^T(s)\,ds\right]Q_3\left[\dot{y}^T(t)\;\; 0\right]^T \\ &\quad = 2\left(\tau_2-\tau(t)\right)\left[y^T(t)\;\; \nu_1^T\right]Q_3\left[\dot{y}^T(t)\;\; 0\right]. \end{aligned} \tag{35}$$

Therefore, the estimation of $\dot{V}_2(t)$ is as follows:

$$\begin{aligned} \dot{V}_2(t) &\le \Xi_{20}+\left(\tau(t)-\tau_1\right)\Xi_{21}+\left(\tau_2-\tau(t)\right)\Xi_{22} \\ &= \xi^T(t)\,\Xi_2\left(\tau(t),\dot{\tau}(t)\right)\xi(t), \end{aligned} \tag{36}$$

where Ξ_2 is defined in (22). Similarly, $\dot{V}_3(t)$ is estimated as

$$\dot{V}_3(t)=\xi^T(t)\,\Xi_{30}\xi(t)+\delta_1(t)+\delta_2(t), \tag{37}$$

where

$$\Xi_{30} := e_9^T\left(\tau_1^2W_1 + \tau_1^2W_2\right)e_9 + e_8^T\left(\tau_{21}^2R_1 + \tau_{21}^2R_2\right)e_8,$$
$$\delta_1(t) = -\tau_{21}\int_{t-\tau_2}^{t-\tau_1}\dot{y}^T(s)R_1\dot{y}(s)\,ds - 2\int_{t-\tau_2}^{t-\tau_1}(\tau_2 - t + s)\dot{y}^T(s)R_2\dot{y}(s)\,ds, \quad (38)$$
$$\delta_2(t) = -\tau_1\int_{t-\tau_1}^{t}\dot{y}^T(s)W_1\dot{y}(s)\,ds - 2\int_{t-\tau_1}^{t}(\tau_1 - t + s)\dot{y}^T(s)W_2\dot{y}(s)\,ds.$$

By Lemmas 5 and 6, we obtain the following:

$$-(\tau_2 - \tau_1)\int_{t-\tau_2}^{t-\tau_1}\dot{y}^T(s)R_1\dot{y}(s)\,ds \leq 2\psi_{11}^TS_1\psi_{21} - \psi_{11}^T\widetilde{R}_1\psi_{11} - \psi_{21}^T\widetilde{R}_1\psi_{21}, \quad (39)$$

where $\widetilde{R}_1 := \operatorname{diag}\{R_1, 3R_1\}$ and

$$\psi_{11} := \begin{bmatrix} y(t-\tau(t)) - y(t-\tau_2) \\ y(t-\tau(t)) + y(t-\tau_2) - 2\nu_1(t)\end{bmatrix},$$
$$\psi_{21} := \begin{bmatrix} y(t-\tau_1) - y(t-\tau(t)) \\ y(t-\tau_1) + y(t-\tau(t)) - 2\nu_2(t)\end{bmatrix},$$
$$\begin{aligned}-2\int_{t-\tau_2}^{t-\tau_1}&(\tau_2 - t + s)\dot{y}^T(s)R_2\dot{y}(s)\,ds \\ &\leq -2\left\{\frac{1}{2}(\tau_2 - \tau(t))^2\xi^T(t)Z_1\xi(t)\right. \\ &\quad + 2(\tau_2 - \tau(t))\xi^T(t)N_1\left[y(t-\tau(t)) - \nu_1\right] \\ &\quad + \frac{1}{2}\left[\tau_{21}^2(\tau_2 - \tau(t))^2\right]\xi^T(t)Z_2\xi(t) + 2\xi^T(t)N_2 \\ &\quad \cdot\left[(\tau_2 - \tau(t))\left[y(t-\tau_1) - y(t-\tau(t))\right]\right. \\ &\quad \left.\left. + (\tau(t) - \tau_1)\left[y(t-\tau_1) - \nu_2\right]\right]\right\}.\end{aligned} \quad (40)$$

Thus,

$$\begin{aligned}\delta_1(t) &\leq 2\psi_{11}^TS_1\psi_{21} - \psi_{11}^T\widetilde{R}_1\psi_{11} - \psi_{21}^T\widetilde{R}_1\psi_{21} \\ &\quad - 2\left\{\frac{1}{2}(\tau_2 - \tau(t))^2\xi^T(t)Z_1\xi(t)\right. \\ &\quad + 2(\tau_2 - \tau(t))\xi^T(t)N_1\left[y(t-\tau(t)) - \nu_1\right] \\ &\quad + \frac{1}{2}\left[\tau_{21}^2(\tau_2 - \tau(t))^2\right]\xi^T(t)Z_2\xi(t) + 2\xi^T(t)N_2 \\ &\quad \cdot\left[(\tau_2 - \tau(t))\left[y(t-\tau_1) - y(t-\tau(t))\right]\right. \\ &\quad \left.\left. + (\tau(t) - \tau_1)\left[y(t-\tau_1) - \nu_2\right]\right]\right\}. \\ &= \xi^T(t)\Xi_3(\tau(t))\xi(t),\end{aligned} \quad (41)$$

where $\Xi_3(\tau(t))$ is given in (23). From Lemmas 2 and 4, we obtain

$$\begin{aligned}-\tau_1\int_{t-\tau_1}^{t}&\dot{y}^T(s)W_1\dot{y}(s)\,ds \\ &\leq \left[y(t) - y(t-\tau_1)\right]^TW_1\left[y(t) - y(t-\tau_1)\right] \\ &\quad + 3\widetilde{\Omega}_1^TW_1\widetilde{\Omega}_1, \\ -2\int_{t-\tau_1}^{t}&(\tau_1 - t + s)\dot{y}^T(s)W_2\dot{y}(s)\,ds \\ &\leq -\tau_1^2\xi^T(t)Z_3\xi(t) - 2\tau_1\xi^T(t)N_3\left[y(t) - \nu_3\right],\end{aligned} \quad (42)$$

from which it follows that

$$\begin{aligned}\delta_2(t) &\leq \left[y(t) - y(t-\tau_1)\right]^TW_1\left[y(t) - y(t-\tau_1)\right] \\ &\quad + 3\widetilde{\Omega}_1^TW_1\widetilde{\Omega}_1 - \tau_1^2\xi^T(t)Z_3\xi(t) \\ &\quad - 2\tau_1\xi^T(t)N_3\left[y(t) - \nu_3\right], \\ &= \xi^T(t)\Xi_{33}\xi(t),\end{aligned} \quad (43)$$

where

$$\begin{aligned}\widetilde{\Omega}_1 &= y(t) + y(t-\tau_1) - 2\nu_3, \\ \widetilde{\Omega}_2 &= y(t-\tau(t)) + y(t+\tau_2) - \nu_1, \\ \widetilde{\Omega}_3 &= y(t-\tau_1) + y(t-\tau(t)) - \nu_2, \\ \Xi_{33} &:= -\widetilde{\varphi}_3^T\operatorname{diag}\{W_1, 3W_1\}\widetilde{\varphi}_3 + \tau_1^2Z_3 \\ &\quad + 2\tau_1N_3(e_1 - e_7) + 2\tau_1(e_1 - e_7)^TN_3^T.\end{aligned} \quad (44)$$

Hence, from (41) and (43), we obtain

$$\dot{V}_3 \leq \xi^T(t)\left[\Xi_3(\tau(t)) + \Xi_4\right]\xi(t), \quad (45)$$

where $\Xi_4 := \Xi_{30} + \Xi_{33}$. From (33), (36), and (45), we obtain $\dot{V}(t, y_t, \dot{y}_t)$ along the solution of system (1) as

$$\dot{V}(t, x_t) \leq \xi^T\Delta(\tau(t), \dot{\tau}(t))\xi(t), \quad (46)$$

where

$$\Delta(\tau(t), \dot{\tau}(t)) = \Xi_2(\tau(t), \dot{\tau}(t)) + \Xi_3(\tau(t)) + (\Xi_1 + \Xi_4). \quad (47)$$

Therefore, we have

$$\begin{aligned}
\dot{V}(t,x_t) \le{}& \xi^T(t)\,\Delta(\tau(t),\dot\tau(t))\,\xi(t) + \gamma\|w(t)\|^2 \\
&+ y^T(t)\left[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY\right]y(t) \\
&- y^T(t)\left[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY\right]y(t) \\
&+ y^T(t-\tau(t))\left[3PG^TGP + 4b_1P^2\right]y(t-\tau(t)) \\
&- y^T(t-\tau(t))\left[3PG^TGP + 4b_1P^2\right]y(t-\tau(t)) \\
={}& \xi^T(t)\,\widehat\Delta(\tau(t),\dot\tau(t))\,\xi(t) + \gamma\|w(t)\|^2 \\
&- y^T(t)\left[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY\right]y(t) \\
&- y^T(t-\tau(t))\left[3PG^TGP + 4b_1P^2\right]y(t-\tau(t)),
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
\widehat\Delta(\tau(t),\dot\tau(t)) &= \Xi_2(\tau(t),\dot\tau(t)) + \Xi_3(\tau(t)) + \left(\widehat\Xi_1 + \Xi_4\right) \\
&= \Xi_2(\tau(t),\dot\tau(t)) + \Xi_3(\tau(t)) + \widehat\Xi_4, \\
\widehat\Xi_1 &= \Xi_1 + e_1^T\left((2+4c_1)Y^TY + 3PE^TEP + 4a_1\right)e_1 \\
&\quad + e_2^T\left(3PE^TEP + 4b_1P^2\right)e_2,
\end{aligned} \tag{49}$$

and $\widehat\Xi_4 = \widehat\Xi_1 + \Xi_4$ is defined in (24). Observe that $\widehat\Delta(\tau(t),\dot\tau(t))$ may be rewritten as

$$\widehat\Delta(\tau(t),\dot\tau(t)) = \tau^2(t)\,\Delta_0 + \tau(t)\,\Delta_1 + \Delta_2, \tag{50}$$

where $\Delta_0 = Z_1 - Z_2$ and Δ_1, Δ_2 are $\tau(t)$-independent real matrices. By Lemma 8, if $Z_1 - Z_2 \ge 0$ and the inequalities in (21) hold, then $\widehat\Delta(\tau(t),\dot\tau(t)) < 0$, $\forall\tau(t) \in [\tau_1,\tau_2]$, $\forall\dot\tau(t) \in [\mu_1,\mu_2]$. Moreover, $\widehat\Delta(\tau(t),\dot\tau(t))$ may be rewritten as a convex combination of $\dot\tau(t)$ as follows:

$$\widehat\Delta(\tau(t),\dot\tau(t)) = (1-\dot\tau(t))\,\Box_0 + \dot\tau(t)\,\Box_1 + \Box_2, \tag{51}$$

where $\Box_0 = Q_2 - Q_3$ and $\Box_1$, $\Box_2$ are $\dot\tau(t)$-independent real matrices. By utilizing the Schur complement lemma, it follows from (21), (50), and (51) that $\widehat\Delta(\tau(t),\dot\tau(t)) < 0$ holds, from which it follows from inequality (48) that

$$\begin{aligned}
\dot{V}(t,x_t) \le{}& \gamma w(t)^T w(t) \\
&- y^T(t)\left[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY\right]y(t) \\
&- y^T(t-\tau(t))\left[3PG^TGP + 4b_1P^2\right]y(t-\tau(t)).
\end{aligned} \tag{52}$$

Letting $w(t) = 0$ and from

$$\begin{aligned}
&- y^T(t)\left[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY\right]y(t) \le 0, \\
&- y^T(t-\tau(t))\left[3PG^TGP + 4b_1P^2\right]y(t-\tau(t)) \le 0,
\end{aligned} \tag{53}$$

there exists a scalar $\varepsilon_3 > 0$ such that

$$\dot{V}(t,x_t) \le -\varepsilon_3\|x(t)\|^2 < 0, \quad \forall t \ge 0. \tag{54}$$

Therefore, system (1) with $w(t) \equiv 0$ is asymptotically stable. To complete the proof of theorem, next we consider the H_∞ performance $\|z\|_2 < \gamma\|w\|_2$. By assuming that $x(t) = 0$, $t \in [-\tau_2, t_0]$, it follows from definition of $z(t)$ that

$$\begin{aligned}
\|z\|^2 \le{}& \|E(x)\|^2 + \|Gx(t-\tau(t))\|^2 + \|u(t)\|^T \\
&+ 2x^T(t)E^TGx(t-\tau(t)) + 2x^T(t)E^Tg(\cdot) \\
&+ 2x^T(t-\tau(t))E^Tg(\cdot) + 2u^T(t)F^Tg(\cdot) + \|g(\cdot)\|^2 \\
\le{}& 3\|Ex(t)\|^2 + 3\|Gx(t-\tau(t))\|^2 \\
&+ 2\|u(t)\|^2 + 4\|g(\cdot)\|^2 \\
\le{}& x^T(t)\left[3E^TE + 4a_1\right]x(t) + x^T(t-\tau(t)) \\
&\cdot\left[3G^TG + 4b_1\right]x(t-\tau(t)) + \left[2+4c_1\right]\|u(t)\|^2 \\
={}& y^T(t)\left[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY\right]y(t) \\
&+ y^T(t-\tau(t))\left[3PG^TGP + 4b_1P^2\right]y(t-\tau(t)).
\end{aligned} \tag{55}$$

From (52), we obtain

$$\begin{aligned}
\dot{V}(t,x_t) \le{}& \gamma w(t)^T w(t) \\
&- y^T(t)\left[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY\right]y(t) \\
&- y^T(t-\tau(t))\left[3PG^TGP + 4b_1P^2\right]y(t-\tau(t)).
\end{aligned} \tag{56}$$

From estimations of $\dot V(t,x_t)$ and $\|z(t)\|^2$ in (48) and (55), we obtain

$$\dot{V}(t,x_t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0. \tag{57}$$

Integrating both sides of the above equation from t_0 to t, we get

$$\int_{t_0}^{t}\left[\dot{V}(t,x_t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t)\right]dt < 0. \tag{58}$$

It follows that

$$\begin{aligned}
&\int_{t_0}^{t}\left[z^T(t)z(t) - \gamma^2 w^T(t)w(t)\right]dt \\
&\quad\le V\left(t_0, x_{t_0}\right) - V(t,x_t) \\
&\quad\le 0.
\end{aligned} \tag{59}$$

Therefore, under zero initial condition $x(t) = 0$, $t \in [-\tau_2, t_0]$, by letting $t \to +\infty$ in (59), we get

$$\int_{t_0}^{\infty} z^T(t)z(t)\,dt < \gamma^2\int_{t_0}^{\infty} w^T(t)w(t)\,dt, \tag{60}$$

which gives $\|z\|_2 < \gamma\|w\|_2$. This completes the proof. □

When $\tau_1 = 0$, $B = 0$, and $C = 0$, the nonlinear function is $f(t, x, x^{\tau})$, and (1) reduces to

$$\dot{x}(t) = Ax(t) + Dx(t - \tau(t)) + f(t, x(t), x(t - \tau(t))), \tag{61}$$

where $0 \le \tau(t) \le \tau_2$, $\mu_1 \le \dot{\tau}(t) \le \mu_2$. In the following, we present a stability criterion for the case when $\tau_1 = 0$. We consider the following Lyapunov-Krasovskii functional candidate:

$$\begin{aligned}
&\widehat{V}(t, x_t, \dot{x}_t) \\
&\quad = x^T(t) Px(t) \\
&\qquad + \int_{t-\tau(t)}^{t} \left[x^T(t) \;\; x^T(s)\right] Q_2 \left[x^T(t) \;\; x^T(s)\right]^T ds \\
&\qquad + \int_{t-\tau_2}^{t-\tau(t)} \left[x^T(t) \;\; x^T(s)\right] Q_3 \left[x^T(t) \;\; x^T(s)\right]^T ds \\
&\qquad + \int_{t-\tau_2}^{t} \left\{\tau_2 (\tau_2 - t + s) \dot{x}^T(s) R_1 \dot{x}(s)\right. \\
&\qquad\qquad \left. + (\tau_2 - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s)\right\} ds.
\end{aligned} \tag{62}$$

Corollary 10. *For given scalars τ_2, μ_1, μ_2, a, and b, (61) is asymptotically stable if there exist symmetric positive definite matrices P, R_1, R_2, N_1, N_2, Z_1, Z_2, Q_2, Q_3, and S_1, such that the following LMIs hold:*

$$\begin{gathered}
\tau_2 \Upsilon_1 + (1 - \mu_1) \Upsilon_3 + \Upsilon_4 < 0, \\
\tau_2 \Upsilon_1 + (1 - \mu_2) \Upsilon_3 + \Upsilon_4 < 0, \\
\tau_2 \Upsilon_2 + (1 - \mu_1) \Upsilon_3 + \Upsilon_4 < 0, \\
\tau_2 \Upsilon_2 + (1 - \mu_2) \Upsilon_3 + \Upsilon_4 < 0. \\
\begin{bmatrix} \widetilde{R}_1 & S_1 \\ S_1^T & \widetilde{R}_1 \end{bmatrix} \ge 0, \qquad Z_1 \ge Z_2, \\
\begin{bmatrix} Z_i & N_i \\ N_i^T & R_2 \end{bmatrix} \ge 0, \quad (i = 1, 2), \\
Q_j = \begin{bmatrix} Q_{j1} & Q_{j2} \\ * & Q_{j3} \end{bmatrix} \ge 0, \quad (i = 2, 3),
\end{gathered} \tag{63}$$

where $\widetilde{R}_1 \triangleq \operatorname{diag}\{R_1, 3R_1\}$

$$\begin{aligned}
\Upsilon_1 &:= \left[\tilde{e}_1^T \;\; \tilde{e}_4^T\right] Q_3 \left[\tilde{e}_6^T \;\; 0\right]^T + \left[\tilde{e}_6^T \;\; 0\right] Q_3 \left[\tilde{e}_1^T \;\; \tilde{e}_4^T\right]^T \\
&\quad + \tau_2 (Z_1 - Z_2) + 2N_1 [\tilde{e}_2 - \tilde{e}_4] \\
&\quad + 2 [\tilde{e}_2 - \tilde{e}_4]^T N_1^T + 2N_2 [\tilde{e}_1 - \tilde{e}_2] \\
&\quad + 2 [\tilde{e}_1 - \tilde{e}_2]^T N_2^T, \\
\Upsilon_2 &:= \left[\tilde{e}_1^T \;\; \tilde{e}_5^T\right] Q_2 \left[\tilde{e}_6^T \;\; 0\right]^T + \left[\tilde{e}_6^T \;\; 0\right] Q_2 \left[\tilde{e}_1^T \;\; \tilde{e}_5^T\right]^T \\
&\quad + 2N_2 [\tilde{e}_1 - \tilde{e}_5] + 2 [\tilde{e}_1 - \tilde{e}_5]^T N_2^T, \\
\Upsilon_3 &:= [\tilde{e}_1 \;\; \tilde{e}_2] (Q_3 - Q_2) [\tilde{e}_1 \;\; \tilde{e}_2]^T, \\
\Upsilon_4 &:= \tilde{e}_1^T \left(A^T P + PA\right) \tilde{e}_1 + \tilde{e}_1^T (a + b) I \tilde{e}_1, \\
&\quad + \tilde{e}_1^T (PD) \tilde{e}_2 + \tilde{e}_2^T \left(D^T P\right) \tilde{e}_1 \\
&\quad + \tilde{e}_1^T (PA) \tilde{e}_6 + \tilde{e}_6^T \left(A^T P\right) \tilde{e}_1 \\
&\quad + \tilde{e}_1^T \left(P^T\right) \tilde{e}_7 + \tilde{e}_7^T (P) \tilde{e}_1 \\
&\quad + \tilde{e}_2^T (PD) \tilde{e}_6 + \tilde{e}_6^T \left(D^T P\right) \tilde{e}_2 \\
&\quad + \tilde{e}_2^T \left(P^T\right) \tilde{e}_8 + \tilde{e}_8^T (P) \tilde{e}_2 \\
&\quad + \tilde{e}_6^T (-2P) \tilde{e}_6 + \tilde{e}_6^T (a + b) I \tilde{e}_6 \\
&\quad + \tilde{e}_7^T \left(\frac{-1}{2a}\right) I \tilde{e}_7 + \tilde{e}_8^T \left(\frac{-1}{2b}\right) I \tilde{e}_8 \\
&\quad + \left[\tilde{e}_1^T \;\; \tilde{e}_1^T\right] Q_2 \left[\tilde{e}_1^T \;\; \tilde{e}_1^T\right]^T \\
&\quad - \left[\tilde{e}_1^T \;\; \tilde{e}_3^T\right] Q_3 \left[\tilde{e}_1^T \;\; \tilde{e}_3^T\right]^T + \tau_2^2 \tilde{e}_6^T (R_1 + R_2) \tilde{e}_6 \\
&\quad + \Theta_1^T S_1 \Theta_2 + \Theta_2^T S_1^T \Theta_1 - \Theta_1^T \widetilde{R}_1 \Theta_1 - \Theta_2^T \widetilde{R}_1 \Theta_2 \\
&\quad + \tau_2^2 Z_2, \\
&\quad \Theta_1 := col\left\{(\tilde{e}_2 - \tilde{e}_3), (\tilde{e}_2 + \tilde{e}_3 - 2\tilde{e}_4)\right\}, \\
&\quad \Theta_2 := col\left\{(\tilde{e}_1 - \tilde{e}_2), (\tilde{e}_1 + \tilde{e}_2 - 2\tilde{e}_5)\right\}
\end{aligned} \tag{64}$$

with $\tilde{e}_1 = [I \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0], \ldots, \tilde{e}_8 = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; I]$.

Proof. The proof is the same as in Theorem 9 by using Lyapunov-Krasovskii functional (62). The proof is omitted. □

4. Numerical Examples

In this section, we provide numerical examples to show the effectiveness of theoretical results.

Example 1. Consider the nonlinear system with interval time-varying delays (1) which was considered in [15], where

$$\begin{gathered}
A = \begin{bmatrix} -1.3 & 0.3 \\ 0.5 & 0.1 \end{bmatrix}, \qquad D = \begin{bmatrix} -0.01 & 0.02 \\ 0.03 & -0.04 \end{bmatrix}, \\
B = \begin{bmatrix} 0.2 & 0 \\ 0.3 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} -0.02 & 0.01 \\ 0.02 & -0.03 \end{bmatrix}, \\
E = G = \begin{bmatrix} 0.06 & -0.06 \\ -0.08 & 0.08 \end{bmatrix}, \qquad F = \begin{bmatrix} 0.8 & 0 \\ 0.6 & 0 \end{bmatrix}, \\
f(\cdot) = g(\cdot) = 0.01 \begin{bmatrix} \sqrt{x_1^2(t) + x_2^2(t - \tau(t))} \\ \sqrt{x_2^2(t) + x_1^2(t - \tau(t))} \end{bmatrix},
\end{gathered} \tag{65}$$

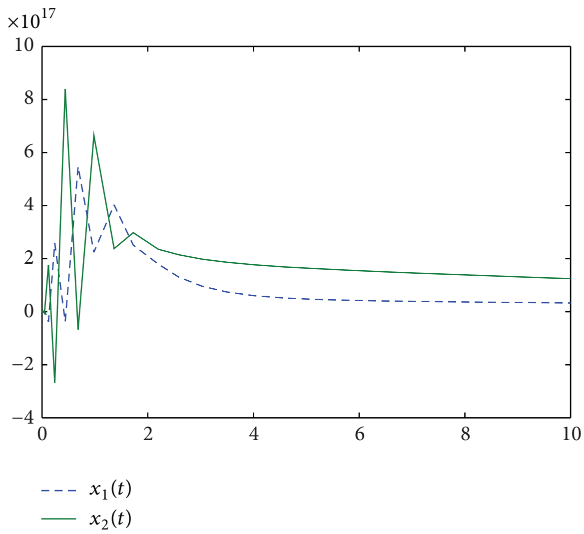

FIGURE 1: The trajectories of $x_1(t)$ and $x_2(t)$ of (1) in Example 1 without feedback control.

TABLE 1: The minimum allowable value of disturbance attenuation γ with $\mu_1 = -0.1$ and $\mu_2 = 0.1$.

$a = b = c = d = a_1 = b_1 = c_1 = 0.01$			
Method	τ_1	τ_2	$\min\gamma$
Theorem 9	0.1	0.3	0.2377
	0.1	0.5	0.2474
$a = b = c = d = a_1 = b_1 = c_1 = 0.05$			
Method	τ_1	τ_2	$\min\gamma$
Theorem 9	0.1	0.3	0.8991
	0.1	0.5	0.9643

$a = b = c = d = a_1 = b_1 = c_1 = 0.01$, $\tau_1 = 0.3$, $\tau_2 = 0.5$, $\mu_1 = -0.1$, $\mu_2 = 0.1$, and $\gamma = 4$. By using LMI Toolbox in MATLAB, the LMIs in Theorem 9 are feasible. From (26) the H_∞ controller feedback gain can be computed as

$$K = YP^{-1} = \begin{bmatrix} -3.4638 & -6.8069 \\ -4.3243 & -4.1846 \end{bmatrix}. \tag{66}$$

For simulation, we choose $\tau(t) = 0.4 + 0.1\cos(t)$, $\phi(t) = [-5\cos(t), 3\cos(t)]$, $\forall t \in [0, 10]$, and $w(t)$ is the Gaussian noise which is the set of random numbers with fluctuation range between -1 and 1. Figure 1 shows the trajectories of solutions $x_1(t)$ and $x_2(t)$ of system (1) without feedback control ($u(t) = 0$) and Figure 2 shows the trajectories of solutions $x_1(t)$ and $x_2(t)$ of the system with feedback control $u(t)$. Moreover, in Tables 1 and 2, by using Theorem 9, we give the minimum allowable value γ with $\mu_1 = -0.1$, $\mu_2 = 0.1$ and with $\mu_1 = 0.05$, $\mu_2 = 0.1$, for some given τ_1 and τ_2, respectively.

Example 2. Consider the following nonlinear system with interval time-varying delays which was considered in [18]:

$$\dot{x}(t) = Ax(t) + Dx(t - \tau(t)) + f(t, x(t), x(t - \tau(t))), \tag{67}$$

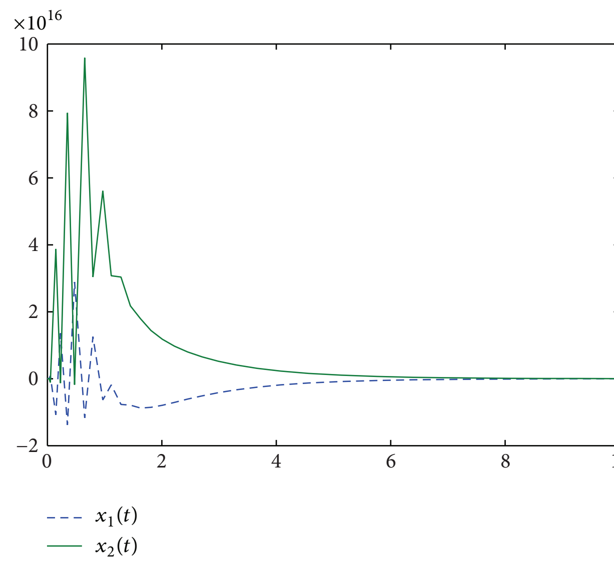

FIGURE 2: The trajectories of $x_1(t)$ and $x_2(t)$ of (1) in Example 1 with feedback control.

TABLE 2: The minimum allowable value of disturbance attenuation γ with $\mu_1 = 0.05$ and $\mu_2 = 0.1$.

$a = b = c = d = a_1 = b_1 = c_1 = 0.01$			
Method	τ_1	τ_2	$\min\gamma$
Theorem 9	0.1	0.5	0.2487

TABLE 3: The maximum allowable upper bound τ_2 in Example 2 with $\tau_1 = 0$.

	$a = 0.1$	$b = 0.1$	
Method	μ_1	μ_2	$\max\tau_2$
By Corollary 10	-0.1	0.1	2.1443
	-0.3	0.3	2.1126
	-0.5	0.5	2.0833
	-0.8	0.8	2.0473
	-1	1	2.0332

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \tag{68}$$

$a = b = 0.1$, and $0 \le \tau(t) \le \tau_2$, $\mu_1 \le \dot{\tau}(t) \le \mu_2$. By using LMI Toolbox in MATLAB, the LMIs in Corollary 10 are feasible. Table 3 shows the maximum allowable upper bound τ_2 with different values of μ_1 and μ_2.

Example 3. Consider the following nonlinear system with interval time-varying delays which was considered in [13]:

$$\dot{x}(t) = Ax(t) + Dx(t - \tau(t)) + f(t, x(t), x(t - \tau(t))), \tag{69}$$

Table 4: Comparison of maximum allowable upper bound τ_2 in Example 3 with $\tau_1 = 0$.

	$a = 0.1$	$b = 0.1$
Method	$\mu = 0.5$	$\mu = 0.9$
[25]	1.009	0.714
[23]	1.284	1.209
[13]	1.287	1.279
Corollary 10	1.7706	1.7355

where

$$A = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix}, \tag{70}$$

$a = b = 0.1$, and $0 \le \tau(t) \le \tau_2$, $\mu_1 \le \dot{\tau}(t) \le \mu_2$. By using LMI Toolbox in MATLAB, the LMIs in Corollary 10 are feasible. For comparison with other existing results, we now calculate the admissible maximum allowable upper bounds of τ_2 by setting $\mu = \mu_2 = -\mu_1$ for various values of μ. Table 4 shows that the maximum allowable upper bounds of τ_2 with given $\tau_1 = 0$ obtained from Corollary 10 are greater than the upper bounds obtained in [13, 23, 25].

5. Conclusions

In this paper, we have investigated the H_∞ control problem for a class of nonlinear systems with interval time-varying delay. A new Lyapunov-Krasovskii functional is constructed to obtain new delay-dependent sufficient condition for the H_∞ control and asymptotic stability condition in terms of LMIs. Numerical examples are given to illustrate the effectiveness of the theoretical results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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