Hindawi Publishing Corporation
The Scientific World Journal
Volume 2015, Article ID 424306, 7 pages
http://dx.doi.org/10.1155/2015/424306

Research Article

An Existence Theorem for Fractional q-Difference Inclusions with Nonlocal Substrip Type Boundary Conditions

Ahmed Alsaedi,[1] Sotiris K. Ntouyas,[2] and Bashir Ahmad[1]

[1]*Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2]*Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*

Correspondence should be addressed to Ahmed Alsaedi; aalsaedi@hotmail.com

Received 2 August 2014; Revised 22 September 2014; Accepted 22 September 2014

Academic Editor: Erdal Karapinar

By employing a nonlinear alternative for contractive maps, we investigate the existence of solutions for a boundary value problem of fractional q-difference inclusions with nonlocal substrip type boundary conditions. The main result is illustrated with the aid of an example.

1. Introduction

In this paper, we consider the following boundary value problem of fractional q-difference inclusions with nonlocal and substrip type boundary conditions:

$$
\begin{aligned}
&{}^{c}D_{q}^{\upsilon}x\left(t\right) \in F\left(t, x\left(t\right)\right), \quad t \in \left[0,1\right], \ 1 < \upsilon \leq 2, \ 0 < q < 1,\\
&x\left(0\right) = g\left(x\right), \qquad x\left(\omega\right) = b\int_{\delta}^{1} x\left(s\right) d_{q}s, \quad 0 < \omega < \delta < 1,
\end{aligned} \tag{1}
$$

where ${}^{c}D_{q}^{\upsilon}$ denotes the Caputo fractional q-derivative of order υ, $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $g : C([0,1],\mathbb{R}) \to \mathbb{R}$ is a given continuous function, and b is a real constant. Here, we emphasize that the nonlocal conditions are regarded as more plausible than the standard initial conditions for the description of some physical phenomena. In (1), $g(x)$ may be understood as $g(x) = \sum_{j=1}^{p} \alpha_j x(t_j)$, where α_j, $j = 1, \ldots, p$, are given constants and $0 < t_1 < \cdots < t_p \leq 1$. For more details, we refer to the work by Byszewski and Lakshmikantham [1, 2].

Boundary value problems with integral boundary conditions constitute an important class of problems and arise in the mathematical modeling of various phenomena such as heat conduction, wave propagation, gravitation, chemical engineering, underground water flow, thermoelasticity, and plasma physics. They include two-point, three-point, multipoint, and nonlocal boundary value problems.

The topic of fractional differential equations has been of great interest for many researchers in view of its theoretical development and widespread applications in various fields of science and engineering such as control, porous media, electromagnetic, and other fields [3, 4]. For some recent results and applications, we refer the reader to a series of papers ([5–13]) and the references cited therein.

Fractional q-difference (q-fractional) equations are regarded as fractional analogue of q-difference equations. Motivated by recent interest in the study of fractional-order differential equations, the topic of q-fractional equations has attracted the attention of many researchers. The details of some recent work on the topic can be found in ([14–20]). For notions and basic concepts of q-fractional calculus, we refer to a recent text [21].

The present work is motivated by a recent paper of the authors [22], where the problem (1) was considered for a single valued case. To the best of our knowledge, this is the first paper dealing with fractional q-difference inclusions in the given framework. Moreover, the main result of our paper can be regarded as an improvement and extension of some known results; see, for instance, [18, 19].

The paper is organized as follows. Section 2 contains some fundamental concepts of fractional q-calculus. In Section 3, we show the existence of solutions for the problem (1) by

means of the nonlinear alternative for contractive mappings. Finally, an example illustrating the applicability of our result is presented.

2. Preliminaries

First of all, we recall the notations and terminology for q-fractional calculus [21, 23, 24].

For a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, a q-real number denoted by $[a]_q$ is defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}. \tag{2}$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{i=0}^{k-1} \left(1 - aq^i\right), \quad k \in \mathbb{N} \cup \{\infty\}. \tag{3}$$

The q-analogue of the exponent $(x-y)^k$ is

$$(x-y)^{(0)} = 1, \quad (x-y)^{(k)} = \prod_{j=0}^{k-1} \left(x - yq^j\right), \quad k \in \mathbb{N}, \quad x, y \in \mathbb{R}. \tag{4}$$

The q-gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y) = \frac{(1-q)^{(y-1)}}{(1-q)^{y-1}}, \tag{5}$$

where $y \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. Observe that $\Gamma_q(y+1) = [y]_q \Gamma_q(y)$.

Definition 1 (see [23]). Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\beta \geq 0$ is $(I_q^0 f)(t) = f(t)$ and

$$I_q^\beta f(t) := \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s)\, d_q s = t^\beta (1-q)^\beta \sum_{k=0}^{\infty} q^k \frac{\left(q^\beta;q\right)_k}{(q;q)_k} f\left(tq^k\right), \quad \beta > 0, \quad t \in [0,1]. \tag{6}$$

Observe that $\beta = 1$ in Definition 1 yields q-integral:

$$I_q f(t) := \int_0^t f(s)\, d_q s = t(1-q) \sum_{k=0}^{\infty} q^k f\left(tq^k\right). \tag{7}$$

For more details on q-integrals and fractional q-integrals, see Sections 1.3 and 4.2, respectively, in [21].

Remark 2. The q-fractional integration possesses the semigroup property ([21, Proposition 4.3]):

$$I_q^\gamma I_q^\beta f(t) = I_q^{\beta+\gamma} f(t); \quad \gamma, \beta \in \mathbb{R}^+. \tag{8}$$

Further, it has been shown in Lemma 6 of [24] that

$$I_q^\beta \left((x)^{(\sigma)}\right) = \frac{\Gamma_q(\sigma+1)}{\Gamma_q(\beta+\sigma+1)} (x)^{(\beta+\sigma)}, \quad 0 < x < a, \quad \beta \in \mathbb{R}^+, \quad \sigma \in (-1, \infty). \tag{9}$$

Before giving the definition of fractional q-derivative, we recall the concept of q-derivative.

We know that the q-derivative of a function $f(t)$ is defined as

$$\left(D_q f\right)(t) = \frac{f(t) - f(qt)}{t - qt}, \quad t \neq 0, \qquad \left(D_q f\right)(0) = \lim_{t \to 0} \left(D_q f\right)(t). \tag{10}$$

Furthermore,

$$D_q^0 f = f, \quad D_q^n f = D_q\left(D_q^{n-1} f\right), \quad n = 1, 2, 3, \ldots. \tag{11}$$

Definition 3 (see [21]). The Caputo fractional q-derivative of order $\beta > 0$ is defined by

$${}^c D_q^\beta f(t) = I_q^{\lceil\beta\rceil - \beta} D_q^{\lceil\beta\rceil} f(t), \tag{12}$$

where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

Next, we recall some properties involving Riemann-Liouville q-fractional integral and Caputo fractional q-derivative ([21, Theorem 5.2]):

$$I_q^\beta\, {}^c D_q^\beta f(t) = f(t) - \sum_{k=0}^{\lceil\beta\rceil - 1} \frac{t^k}{\Gamma_q(k+1)} \left(D_q^k f\right)(0^+), \quad \forall t \in (0, a], \quad \beta > 0; \qquad {}^c D_q^\beta\, I_q^\beta f(t) = f(t), \quad \forall t \in (0, a], \ \beta > 0. \tag{13}$$

Lemma 4 (see [22]). *Let $y \in C([0,1], \mathbb{R})$. Then, the following problem*

$${}^c D_q^\upsilon x(t) = y(t), \quad 1 < \upsilon \leq 2, \qquad x(0) = y_0, \quad x(\omega) = b \int_\delta^1 x(s)\, d_q s, \quad y_0 \in \mathbb{R}, \ t \in [0,1], \tag{14}$$

is equivalent to an integral equation:

$$x(t) = \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} y(s)\, d_q s + \frac{t}{\vartheta} \left\{ b \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} y(u)\, d_q u \right) d_q s - \int_0^\omega \frac{(\omega - qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} y(s)\, d_q s \right\} + y_0 \left[1 + \frac{t}{\vartheta} (b(1-\delta) - 1) \right], \tag{15}$$

where

$$\vartheta = \omega - \frac{b\left(1-\delta^2\right)}{1+q} \neq 0. \tag{16}$$

3. Existence Results

First of all, we outline some basic definitions and results for multivalued maps [25, 26].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y$ is closed$\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y$ is bounded$\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y$ is compact$\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y$ is compact and convex$\}$. A multivalued map $G : X \to \mathcal{P}(X)$

(i) is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$;

(ii) is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);

(iii) is called upper semicontinuous (u.s.c.) on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;

(iv) is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$;

(v) is said to be measurable if, for every $y \in \mathbb{R}$, the function

$$t \longmapsto d\left(y, G(t)\right) = \inf\left\{\left|y - z\right| : z \in G(t)\right\} \tag{17}$$

is measurable;

(vi) has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by Fix G.

For each $x \in C([0,1], \mathbb{R})$, define the set of selections of F by

$$S_{F,x} := \left\{v \in L^1([0,1], \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1]\right\}. \tag{18}$$

Definition 5. A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be a Carathéodory function if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$.

Further, a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,1], \mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t) \tag{19}$$

for all $\|x\| \leq \alpha$ and for a.e. $t \in [0,1]$.

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y,\ y \in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

Lemma 6 (see [25, Proposition 1.2]). *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$, that is, for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$, and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semicontinuous.*

Lemma 7 (see [27]). *Let X be a separable Banach space. Let $F : [0,1] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory function. Then, the operator*

$$\Theta \circ S_F : C([0,1], X) \longrightarrow \mathcal{P}_{cp,c}(C([0,1], X)),$$
$$x \longmapsto (\Theta \circ S_F)(x) = \Theta\left(S_{F,x,y}\right) \tag{20}$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

Definition 8. A function $x \in AC^1([0,1], \mathbb{R})$ is called a solution of problem (1) if there exists a function $f \in L^1([0,1], \mathbb{R})$ with $f(t) \in F(t, x(t))$, a.e. on $[0,1]$ such that $D_q^v x(t) = f(t)$, a.e. on $[0,1]$ and $x(0) = g(x)$ and $x(\omega) = b\int_\delta^1 x(s) d_q s$.

To prove our main result in this section we will use the following form of the nonlinear alternative for contractive maps [28, Corollary 3.8].

Theorem 9. *Let X be a Banach space and let D be a bounded neighborhood of $0 \in X$. Let $\chi_1 : X \to \mathcal{P}_{cp,c}(X)$ and $\chi_2 : \overline{D} \to \mathcal{P}_{cp,c}(X)$ be two multivalued operators such that*

(a) *χ_1 is contraction,*

(b) *χ_2 is u.s.c and compact.*

Then, if $\chi = \chi_1 + \chi_2$, either

(i) *χ has a fixed point in $\overline{D}$ or*

(ii) *there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda\chi(u)$.*

Theorem 10. *Assume that*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,1], \mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\left\{\left|y\right| : y \in F(t,x)\right\} \leq p(t)\psi(\|x\|) \quad \text{for each } (t,x) \in [0,1] \times \mathbb{R}; \tag{21}$$

(H_3) *$g : C([0,1], \mathbb{R}) \to \mathbb{R}$ is a continuous function satisfying the condition*

$$\left|g(u) - g(v)\right| \leq \ell\|u - v\| \quad \forall u, v \in C([0,1], \mathbb{R}),\ \ell > 0; \tag{22}$$

(H_4) *there exists a number* $M > 0$ *such that*

$$\frac{(1-\ell k_0)M}{\|p\|\psi(M)\mu_0} > 1, \tag{23}$$

with $\ell k_0 < 1$, *where*

$$\mu_0 := \frac{1}{\Gamma_q(\upsilon+1)} + \frac{1}{|\vartheta|}\left\{\frac{|b|\left(1-\delta^{\upsilon+1}\right)}{\Gamma_q(\upsilon+2)} + \frac{\omega^{\upsilon}}{\Gamma_q(\upsilon+1)}\right\},$$
$$k_0 := 1 + \frac{1}{|\vartheta|}|b(1-\delta)-1|. \tag{24}$$

Then, the problem (1) has at least one solution on $[0,1]$.

Proof. To transform the problem (1) into a fixed point problem, we define an operator $\mathcal{F} : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ as

$$\begin{aligned}\mathcal{F}(x) = \Bigg\{ & h \in C([0,1],\mathbb{R}) : h(t) \\ & = \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(s)\, d_q s \\ & \quad + \frac{t}{\vartheta}\left\{ b\int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(u)\, d_q u \right) d_q s \right. \\ & \qquad \left. - \int_0^\omega \frac{(\omega-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(s)\, d_q s \right\} \\ & \quad + \left[1 + \frac{t}{\vartheta}(b(1-\delta)-1)\right] g(x), \\ & t \in [0,1],\ f \in S_{F,x} \Bigg\}. \end{aligned} \tag{25}$$

Next, we introduce two operators $\mathcal{F}_1 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ and $\mathcal{F}_2 : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ as follows:

$$\mathcal{F}_1 x(t) = \left[1 + \frac{t}{\vartheta}(b(1-\delta)-1)\right] g(x), \quad t \in [0,1],$$

$$\begin{aligned}\mathcal{F}_2(x) = \Bigg\{ & h \in C([0,1],\mathbb{R}) : h(t) \\ & = \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(s)\, d_q s \\ & \quad + \frac{t}{\vartheta}\left\{ b\int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(u)\, d_q u \right) d_q s \right. \\ & \qquad \left. - \int_0^\omega \frac{(\omega-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(s)\, d_q s \right\}, \\ & t \in [0,1],\ f \in S_{F,x} \Bigg\}. \end{aligned} \tag{26}$$

Observe that $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$. We will show that the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy all the conditions of Theorem 9 on $[0,1]$. For the sake of clarity, we split the proof into a number of steps and claims.

Step 1. $\mathcal{F}_1$ is a contraction on $C([0,1],\mathbb{R})$. This is a consequence of (H_3). Indeed, we have

$$\begin{aligned}|\mathcal{F}_1 x(t) - \mathcal{F}_1 y(t)| &= \left[1 + \frac{t}{\vartheta}(b(1-\delta)-1)\right]|g(x)-g(y)| \\ &\le \left[1 + \frac{1}{|\vartheta|}|b(1-\delta)-1|\right]\ell\|x-y\| \\ &\le k_0\ell\|x-y\|. \end{aligned} \tag{27}$$

Taking supremum over $t \in [0,1]$, we have

$$\|\mathcal{F}_1 x - \mathcal{F}_1 y\| \le k_0\ell\|x-y\|, \quad \ell k_0 < 1. \tag{28}$$

Step 2. $\mathcal{F}_2$ is compact, convex valued, and completely continuous. This will be established in several claims.

Claim 1. $\mathcal{F}_2$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For that, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \mathcal{F}_2(x)$, $x \in B_\rho$, we have

$$\begin{aligned}|h(t)| &\le \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)}|f(s)|\, d_q s \\ &\quad + \frac{1}{|\vartheta|}\left\{|b|\int_\delta^1\left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)}|f(u)|\, d_q u\right) d_q s \right. \\ &\qquad \left. + \int_0^\omega \frac{(\omega-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)}|f(s)|\, d_q s\right\} \\ &\le \|p\|\psi(\rho)\left[\frac{1}{\Gamma_q(\upsilon+1)} + \frac{1}{|\vartheta|}\left\{\frac{|b|\left(1-\delta^{\upsilon+1}\right)}{\Gamma_q(\upsilon+2)} \right.\right. \\ &\qquad \left.\left. + \frac{\omega^{\upsilon}}{\Gamma_q(\upsilon+1)}\right\}\right] \\ &\le \|p\|\psi(\rho)\mu_0, \end{aligned} \tag{29}$$

and consequently, for each $h \in \mathcal{F}_2(B_\rho)$, we have

$$\|h\| \le \|p\|\psi(\rho)\mu_0. \tag{30}$$

Claim 2. $\mathcal{F}_2$ maps bounded sets into equicontinuous sets. As before, let B_ρ be a bounded set and let $h \in \mathcal{F}_2(x)$ for $x \in B_\rho$.

Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\rho$. Then, for each $h \in \mathscr{F}_2(x)$, we obtain

$$\begin{aligned}
&\left|(\mathscr{F}_1 x)(t_2) - (\mathscr{F}_1 x)(t_1)\right| \\
&\leq \frac{1}{\Gamma_q(\upsilon)} \int_0^{t_1} \left[(t_2 - qs)^{(\upsilon-1)} - (t_1 - qs)^{(\upsilon-1)}\right] \left|f(s)\right| d_q s \\
&\quad + \frac{1}{\Gamma_q(\upsilon)} \int_{t_1}^{t_2} (t_2 - qs)^{(\upsilon-1)} \left|f(s)\right| d_q s \\
&\quad + \frac{|t_2 - t_1|}{|\vartheta|} \left\{ |b| \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma(\upsilon)} \left|f(u)\right| d_q u \right) d_q s \right. \\
&\qquad \left. + \int_0^\omega \frac{(\omega - qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} \left|f(s)\right| d_q s \right\} \\
&\leq \frac{\|p\| \psi(\rho)}{\Gamma_q(\upsilon)} \int_0^{t_1} \left[(t_2 - qs)^{(\upsilon-1)} - (t_1 - qs)^{(\upsilon-1)}\right] d_q s \\
&\quad + \frac{\|p\| \psi(\rho)}{\Gamma_q(\upsilon)} \int_{t_1}^{t_2} (t_2 - qs)^{(\upsilon-1)} d_q s \\
&\quad + \frac{\|p\| \psi(\rho) |t_2 - t_1|}{|\vartheta|} \left\{ |b| \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} d_q u \right) d_q s \right. \\
&\qquad \left. + \int_0^\omega \frac{(\omega - qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} d_q s \right\},
\end{aligned} \tag{31}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Therefore, it follows by the Arzelá-Ascoli theorem that $\mathscr{F}_2 : C([0,1], \mathbb{R}) \to \mathscr{P}(C[0,1], \mathbb{R})$ is completely continuous.

Claim 3. $\mathscr{F}_2$ has a closed graph. Let $x_n \to x_*$, $h_n \in \mathscr{F}_2(x_n)$, and $h_n \to h_*$. Then, we need to show that $h_* \in \mathscr{F}_2(x_*)$. Associated with $h_n \in \mathscr{F}_2(x_n)$, there exists $f_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned}
h_n(t) &= \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f_n(s)\, d_q s \\
&\quad + \frac{t}{\vartheta} \left\{ b \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f_n(u)\, d_q u \right) d_q s \right. \\
&\quad \left. - \int_0^\omega \frac{(\omega - qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f_n(s)\, d_q s \right\}.
\end{aligned} \tag{32}$$

Then, we have to show that there exists $f_* \in S_{F,x_*}$ such that, for each $t \in [0,1]$,

$$\begin{aligned}
h_*(t) &= \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f_*(s)\, d_q s \\
&\quad + \frac{t}{\vartheta} \left\{ b \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f_*(u)\, d_q u \right) d_q s \right. \\
&\quad \left. - \int_0^\omega \frac{(\omega - qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f_*(s)\, d_q s \right\}.
\end{aligned} \tag{33}$$

Let us consider the continuous linear operator $\Theta : L^1([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ defined by

$$\begin{aligned}
f \longmapsto \Theta(f)(t) &= \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(s)\, d_q s \\
&\quad + \frac{t}{\vartheta} \left\{ b \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(u)\, d_q u \right) d_q s \right. \\
&\quad \left. - \int_0^\omega \frac{(\omega - qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(s)\, d_q s \right\}.
\end{aligned} \tag{34}$$

Observe that

$$\begin{aligned}
&\left|h_n(t) - h_*(t)\right| \\
&= \left| \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} \left(f_n(s) - f_*(s)\right) d_q s \right. \\
&\quad + \frac{t}{\vartheta} \left\{ b \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} \left(f_n(u) - f_*(u)\right) d_q u \right) d_q s \right. \\
&\quad \left. \left. - \int_0^\omega \frac{(\omega - qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} \left(f_n(s) - f_*(s)\right) d_q s \right\} \right| \longrightarrow 0,
\end{aligned} \tag{35}$$

as $n \to \infty$. Thus, it follows by Lemma 7 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned}
h_*(t) &= \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f_*(s)\, d_q s \\
&\quad + \frac{t}{\vartheta} \left\{ b \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f_*(u)\, d_q u \right) d_q s \right. \\
&\quad \left. - \int_0^\omega \frac{(\omega - qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f_*(s)\, d_q s \right\},
\end{aligned} \tag{36}$$

for some $f_* \in S_{F,x_*}$. Hence, $\mathscr{F}_2$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathscr{F}_2$ is compact valued.

Thus, the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy hypotheses of Theorem 9 and hence, by its application, it follows that either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \lambda\mathcal{F}_1(x) + \lambda\mathcal{F}_2(x)$ for $\lambda \in (0,1)$, then there exists $f \in S_{F,x}$ such that $x = \lambda\mathcal{F}(x)$, that is,

$$\begin{aligned} x(t) = \lambda \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(s)\, d_q s \\ + \lambda \frac{t}{\vartheta} \left\{ b \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(u)\, d_q u \right) d_q s \right. \\ \left. - \int_0^\omega \frac{(\omega-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} f(s)\, d_q s \right\} \\ + \lambda \left[1 + \frac{t}{\vartheta}(b(1-\delta)-1) \right] g(x); \quad t \in [0,1]. \end{aligned} \tag{37}$$

In consequence, we have

$$\begin{aligned} |x(t)| &\le \int_0^t \frac{(t-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} |f(s)|\, d_q s \\ &\quad + \frac{1}{|\vartheta|} \left\{ |b| \int_\delta^1 \left(\int_0^s \frac{(s-qu)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} |f(u)|\, d_q u \right) d_q s \right. \\ &\quad \left. + \int_0^\omega \frac{(\omega-qs)^{(\upsilon-1)}}{\Gamma_q(\upsilon)} |f(s)|\, d_q s \right\} \\ &\quad + \left[1 + \frac{1}{|\vartheta|} |b(1-\delta)-1| \right] \ell \|x\| \\ &\le \|p\| \psi(\|x\|) \left[\frac{1}{\Gamma_q(\upsilon+1)} \right. \\ &\quad \left. + \frac{1}{|\vartheta|} \left\{ \frac{|b|\left(1-\delta^{\upsilon+1}\right)}{\Gamma_q(\upsilon+2)} + \frac{\omega^\upsilon}{\Gamma_q(\upsilon+1)} \right\} \right] \\ &\quad + \left[1 + \frac{1}{|\vartheta|} |b(1-\delta)-1| \right] \ell \|x\| \\ &\le \|p\| \psi(\|x\|) \mu_0 + k_0 \ell \|x\|. \end{aligned} \tag{38}$$

If condition (ii) of Theorem 9 holds, then there exist $\lambda \in (0,1)$ and $x \in \partial B_M$ with $x = \lambda\mathcal{F}(x)$, where $B_M = \{x \in C([0,1],\mathbb{R}) : \|x\| \le M\}$. Then, x is a solution of $x = \lambda\mathcal{F}(x)$ with $\|x\| = M$. Now, by the last inequality, we get

$$\frac{(1-\ell k_0) M}{\|p\| \psi(M) \mu_0} \le 1, \tag{39}$$

which contradicts (23). Hence, $\mathcal{F}$ has a fixed point on $[0,1]$ by Theorem 9, and consequently the problem (1) has a solution. This completes the proof. □

Example 11. Consider the following q-fractional boundary value problem:

$$\begin{aligned} {}^c D_q^{3/2} x(t) \in F(t, x(t)), \quad t \in [0,1], \\ x(0) = \frac{1}{2} + \frac{1}{5}\tan^{-1}\left(x\left(\frac{1}{4}\right)\right), \qquad x\left(\frac{1}{4}\right) = \frac{1}{5}\int_{3/4}^1 x(s)\, d_q s. \end{aligned} \tag{40}$$

Here, $\upsilon = 3/2$, $q = 1/2$, $b = 1/5$, $\omega = 1/4$, $\delta = 3/4$, $\ell = 1/15$, and

$$F(t,x) = \left[\frac{3}{(3+t)^2} \frac{|x|}{1+|x|} + \frac{\cos x}{(4+t)^2} + \frac{1}{2}, -\frac{1}{10} \right]. \tag{41}$$

Obviously,

$$\begin{aligned} \sup\{|u| : u \in F(t,x)\} &\le \frac{3}{(3+t)^2} + \frac{1}{(4+t)^2} + \frac{1}{2} \\ &= p(t) \psi(\|x\|). \end{aligned} \tag{42}$$

With the given values, it is found that $\vartheta = 0.191667$, $\mu_0 \approx 1.3990$, $k_0 = 3.9564$, $\|p\| = 43/48$, $\psi(\|x\|) = 1$, and $M > 1.7026$ by (H_3). Thus, all the conditions of Theorem 10 hold. In consequence, the conclusion of Theorem 10 applies to the boundary value problem (40).

Disclosure

Sotiris K. Ntouyas is a Member of Nonlinear Analysis and Applied Mathematics (NAAM) Research Group at King Abdulaziz University, Jeddah, Saudi Arabia.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia. The authors, therefore, acknowledge technical and financial support of KAU.

References

[1] L. Byszewski and V. Lakshmikantham, "Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space," *Applicable Analysis*, vol. 40, no. 1, pp. 11–19, 1991.

[2] L. Byszewski, "Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem," *Journal of Mathematical Analysis and Applications*, vol. 162, no. 2, pp. 494–505, 1991.

[3] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[4] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[5] N. J. Ford and M. L. Morgado, "Fractional boundary value problems: analysis and numerical methods," *Fractional Calculus and Applied Analysis*, vol. 14, no. 4, pp. 554–567, 2011.

[6] B. Ahmad and J. J. Nieto, "Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions," *Boundary Value Problems*, vol. 2011, artcile 36, 2011.

[7] B. Ahmad and S. K. Ntouyas, "Existence results for nonlocal boundary value problems of fractional differential equations and inclusions with strip conditions," *Boundary Value Problems*, vol. 2012, article 55, 2012.

[8] B. Ahmad, "On nonlocal boundary value problems for nonlinear integro-differential equations of arbitrary fractional order," *Results in Mathematics*, vol. 63, no. 1-2, pp. 183–194, 2013.

[9] D. O'Regan and S. Stanek, "Fractional boundary value problems with singularities in space variables," *Nonlinear Dynamics*, vol. 71, no. 4, pp. 641–652, 2013.

[10] B. Ahmad and J. J. Nieto, "Boundary value problems for a class of sequential integrodifferential equations of fractional order," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 149659, 8 pages, 2013.

[11] L. Zhang, B. Ahmad, G. Wang, and R. P. Agarwal, "Nonlinear fractional integro-differential equations on unbounded domains in a Banach space," *Journal of Computational and Applied Mathematics*, vol. 249, pp. 51–56, 2013.

[12] X. Liu, M. Jia, and W. Ge, "Multiple solutions of a p-Laplacian model involving a fractional derivative," *Advances in Difference Equations*, vol. 2013, article 126, 2013.

[13] S. K. Ntouyas, "Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and integral boundary conditions," *Fractional Differential Calculus*, vol. 3, pp. 1–20, 2013.

[14] R. Ferreira, "Nontrivial solutions for fractional q-difference boundary value problems," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 70, pp. 1–10, 2010.

[15] C. S. Goodrich, "Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions," *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 191–202, 2011.

[16] J. Ma and J. Yang, "Existence of solutions for multi-point boundary value problem of fractional q-difference equation," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 92, pp. 1–10, 2011.

[17] J. R. Graef and L. Kong, "Positive solutions for a class of higher order boundary value problems with fractional q-derivatives," *Applied Mathematics and Computation*, vol. 218, no. 19, pp. 9682–9689, 2012.

[18] B. Ahmad, S. K. Ntouyas, and I. K. Purnaras, "Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations," *Advances in Difference Equations*, vol. 2012, article 140, 2012.

[19] B. Ahmad and S. K. Ntouyas, "Existence of solutions for nonlinear fractional q-difference inclusions with nonlocal Robin (separated) conditions," *Mediterranean Journal of Mathematics*, vol. 10, no. 3, pp. 1333–1351, 2013.

[20] X. Li, Z. Han, and S. Sun, "Existence of positive solutions of nonlinear fractional q-difference equation with parameter," *Advances in Difference Equations*, vol. 2013, article 260, 2013.

[21] M. H. Annaby and Z. S. Mansour, q-*Fractional Calculus and Equations*, vol. 2056 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2012.

[22] B. Ahmad, S. K. Ntouyas, A. Alsaedi, and Al-Hutami, "Nonlinear q-fractional differential equations with nonlocal and substrip type boundary conditions," *Electronic Journal of Qualitative Differential Equations*, vol. 26, pp. 1–12, 2014.

[23] R. P. Agarwal, "Certain fractional q-integrals and q-derivatives," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 66, pp. 365–370, 1969.

[24] P. M. Rajkovic, S. D. Marinkovic, and M. S. Stankovic, "On q-analogues of caputo derivative and mittag-leffler function," *Fractional Calculus and Applied Analysis*, vol. 10, pp. 359–373, 2007.

[25] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin, Germany, 1992.

[26] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis, Theory I*, Kluwer Academic, Dordrecht, Netherlands, 1997.

[27] A. Lasota and Z. Opial, "An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations,," *Bulletin de l'Académie Polonaise des Sciences*, vol. 13, pp. 781–786, 1965.

[28] W. V. Petryshyn and P. M. Fitzpatrick, "A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact mappings," *Transactions of the American Mathematical Society*, vol. 194, pp. 1–25, 1974.