

Research Article

Interval Oscillation Criteria for Second-Order Forced Functional Dynamic Equations on Time Scales

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This paper is concerned with oscillation of second-order forced functional dynamic equations of the form $(r(t)(x^\Delta(t))^\gamma)^\Delta + \sum_{i=0}^n q_i(t)|x(\delta_i(t))|^{\alpha_i} \operatorname{sgn} x(\delta_i(t)) = e(t)$ on time scales. By using a generalized Riccati technique and integral averaging techniques, we establish new oscillation criteria which handle some cases not covered by known criteria.

1. Introduction

The theory of time scales was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify the continuous and discrete analysis. Not only can this theory of the so-called “dynamic equations” unify theories of differential equations and difference equations but also it can extend these classical cases to cases “in between,” for example, to the so-called q -difference equations. A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} with the topology and ordering inherited from \mathbb{R} , and the cases when this time scale is equal to \mathbb{R} or to the integers \mathbb{Z} represent the classical theories of differential and difference equations. Of course many other interesting time scales exist, and they give rise to plenty of applications. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various dynamic equations on time scales, and we refer the readers to [1–20].

In 2006, Saker [1] provided sufficient conditions for the boundedness of the solutions of forced dynamic equations of the form:

$$x^{\Delta\Delta}(t) + q^\sigma(t) f(x(t)) = e(t). \quad (1)$$

In 2007, Sun and Wong [2] considered interval oscillation of second-order forced ordinary differential equations with mixed nonlinearities:

$$\begin{aligned} & (p(t)x'(t))' + q(t)x(t) \\ & + \sum_{i=1}^n q_i(t)|x(t)|^{\alpha_i} \operatorname{sgn} x(t) = e(t), \end{aligned} \quad (2)$$

where $\alpha_1 > \dots > \alpha_m > 1 > \alpha_{m+1} > \dots > \alpha_n$.

In 2008, Erbe et al. [3] established oscillation criteria for the forced second-order nonlinear dynamic equation:

$$(p(t)x^\Delta(t))^\Delta + q(t)|x^\sigma(t)|^\gamma \operatorname{sgn} x^\sigma(t) = e(t), \quad \gamma \geq 1. \quad (3)$$

In 2009, Li and Chen [4] considered oscillation of second-order functional differential equations with mixed nonlinearities:

$$\begin{aligned} & (p(t)x'(t))' + q(t)x(t-\tau) \\ & + \sum_{i=1}^n q_i(t)|x(t-\tau)|^{\alpha_i} \operatorname{sgn} x(t-\tau) = e(t), \quad \tau \geq 0. \end{aligned} \quad (4)$$

In 2010, Lin et al. [5] considered forced oscillation of second-order half-linear dynamic equations on time scales:

$$(r(t) [x^\Delta(t)]^\alpha)^\Delta + p(t) x^\alpha(\sigma(t)) = f(t), \quad (5)$$

where α is a quotient of odd positive integers. Also, Erbe et al. [6] obtained some interval oscillation criteria for forced second order nonlinear delay dynamic equations with oscillatory potential of the form:

$$(r(t) x^\Delta(t))^\Delta + q(t) f(x(\tau(t))) = e(t). \quad (6)$$

In 2011, Hassan et al. [7] discussed oscillation of the following forced second-order differential equations with mixed nonlinearities:

$$\begin{aligned} & (a(t) (x'(t))^\gamma)^\gamma + p_0(t) x^\gamma(t) \\ & + \sum_{i=1}^n p_i(t) |x(t)|^{\alpha_i} \operatorname{sgn} x(t) = e(t), \end{aligned} \quad (7)$$

$$\begin{aligned} & (a(t) (x'(t))^\gamma)^\gamma + p_0(t) x^\gamma(g_0(t)) \\ & + \sum_{i=1}^n p_i(t) |x(g_i(t))|^{\alpha_i} \operatorname{sgn} x(g_i(t)) = e(t), \end{aligned}$$

where γ is a quotient of odd positive integers, $\alpha_i > 0$, $i = 1, 2, \dots, n$ and $\alpha_1 > \dots > \alpha_m > \gamma > \alpha_{m+1} > \dots > \alpha_n$ with $a(t), e(t), p_i(t) \in C([t_0, \infty), \mathbb{R})$, $a(t) > 0$, $g_i(t) : \mathbb{R} \rightarrow \mathbb{R}$ is positive and nondecreasing continuous and $\lim_{t \rightarrow \infty} g_i(t) = \infty$ for $i = 1, 2, \dots, n$. The authors established some sufficient conditions for the oscillation of (7), which did not assume that $e(t)$ and $p_i(t)$ ($i = 1, 2, \dots, n$) are of definite sign.

In 2013, Anderson and Saker [8] establish some oscillation criteria for the second-order nonlinear Emden-Fowler functional dynamic equation with oscillatory potential and forcing term on time scales of the form:

$$(rx^\Delta)^\Delta(t) + q(t) |x(\tau(t))|^\gamma x(\tau(t)) = e(t), \quad t \in \mathbb{T}, \quad (8)$$

where \mathbb{T} is a time scale unbounded above, $\gamma > 1$, the potentials r and q and the forcing function e are right dense continuous with $r > 0$, and $\tau : \mathbb{T} \rightarrow \mathbb{T}$ satisfies $\lim_{t \rightarrow \infty} \tau(t) = \infty$. They also did not assume that e and q are of definite sign.

In this paper, motivated by [1–8] and others, we study the second-order nonlinear dynamic equation:

$$(r(t) (x^\Delta(t))^\gamma)^\Delta + \sum_{i=0}^n q_i(t) |x(\delta_i(t))|^{\alpha_i} \operatorname{sgn} x(\delta_i(t)) = e(t) \quad (9)$$

on a time scale \mathbb{T} , where $e(t), q_i(t) \in C_{\text{rd}}(\mathbb{T}, \mathbb{R})$, $\gamma > 0$ is a quotient of odd positive integers, $\alpha_0 = \gamma$, and $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a real n -tuple satisfying $\alpha_1 > \alpha_2 > \dots > \alpha_m > \gamma > \alpha_{m+1} > \dots > \alpha_n > 0$ ($n > m \geq 1$), $i = 1, 2, \dots, n$.

This paper is organized as follows. In the next section, we give some preliminaries and lemmas. In Sections 3 and 4, we will use the Riccati transformation technique to prove our main results. In Section 5, we present two examples to illustrate our results.

2. Preliminaries and Lemmas

For convenience, we recall some concepts related to time scales. More details can be found in [9].

Definition 1. Let \mathbb{T} be a time scale; for $t \in \mathbb{T}$ the forward jump operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, the backward jump operator by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, and the graininess function by $\mu(t) := \sigma(t) - t$, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. If $\sigma(t) > t$, t is said to be right scattered; otherwise, it is right dense. If $\rho(t) < t$, t is said to be left scattered; otherwise, it is left dense. The set \mathbb{T}^κ is defined as follows. If \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2. For a function $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the delta-derivative $f^\Delta(t)$ of $f(t)$ to be the number (provided that it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some δ) such that

$$\begin{aligned} & |[f(\sigma(t)) - f(s)] - f^\Delta(t) [\sigma(t) - s]| \\ & \leq \varepsilon |\sigma(t) - s| \quad \forall s \in U. \end{aligned} \quad (10)$$

We say that f is delta differentiable (or in short: differentiable) on \mathbb{T}^κ provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

It is easily seen that if f is continuous at $t \in \mathbb{T}$ and t is right scattered, and then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \quad (11)$$

Moreover, if t is right dense, then f is differential at t iff the limit

$$\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} \quad (12)$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}. \quad (13)$$

In addition, if $f^\Delta \geq 0$, then f is nondecreasing. A useful formula is as follows:

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \quad \text{where } f^\sigma(t) := f(\sigma(t)). \quad (14)$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable functions f and g :

$$\begin{aligned} & (fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \\ & \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \end{aligned} \quad (15)$$

Definition 3. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function, f is called right dense continuous (rd-continuous) if it is continuous at

right-dense points in \mathbb{T} , and its left-sided limits exist (finite) at left-dense points in \mathbb{T} . A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an antiderivative of f provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. By the antiderivative, the Cauchy integral of f is defined as $\int_a^b f(s)\Delta s = F(b) - F(a)$, and $\int_a^\infty f(s)\Delta s = \lim_{t \rightarrow \infty} \int_a^t f(s)\Delta s$.

In (9), we assume that \mathbb{T} is a time scale satisfying $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$, and

$$(h_1) \delta_i(t) \in C_{rd}(\mathbb{T}, \mathbb{T}), \delta_i^\Delta(t) > 0 \text{ and } \lim_{t \rightarrow \infty} \delta_i(t) = \infty, \\ i = 0, 1, \dots, n;$$

$$(h_2) r(t) \in C_{rd}(\mathbb{T}, (0, \infty)), \text{ and } q_i(t), e(t) \in C_{rd}(\mathbb{T}, \mathbb{R}), i = 0, 1, \dots, n.$$

By a solution of (9), we mean a nontrivial real-valued function $x(t) \in C_{rd}^1[t_0, \infty)$ satisfying $r(t)(x^\Delta(t))^\gamma \in C_{rd}^1[t_0, \infty)$ and (9). Our attention is restricted to those solutions of (9) that exist on some half-line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t \geq t_0\} > 0$ for any $t_0 \geq t_x$. A solution $x(t)$ of (9) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

For convenience, we use the notations $x(\sigma(t)) = x^\sigma(t)$, $x(\delta_i(t)) = x^{\delta_i}(t)$ ($i = 1, 2, \dots, n$), $x^\Delta(\sigma(t)) = (x^\Delta(t))^\sigma$, and, for $i = 0, 1, 2, \dots, n$ and $k = 1, 2$,

$$\xi_{i,k}(t) = \frac{\int_{\delta_i(a_k)}^{\delta_i(t)} r^{-1/\gamma}(s) \Delta s}{\int_{\delta_i(a_k)}^{\sigma(t)} r^{-1/\gamma}(s) \Delta s}, \\ \theta_{i,k}(t) = \frac{\int_{\delta_i(t)}^{\delta_i(b_k)} r^{-1/\gamma}(s) \Delta s}{\int_{\sigma(t)}^{\delta_i(b_k)} r^{-1/\gamma}(s) \Delta s}, \quad (16) \\ g_{i,k}(t) = \begin{cases} \xi_{i,k}, & \delta_i(t) < \sigma(t), \\ 1, & \delta_i(t) = \sigma(t), \\ \theta_{i,k}, & \delta_i(t) > \sigma(t). \end{cases}$$

Now, we give the first lemma.

Lemma 4. *Let conditions (h_1) and (h_2) hold. Furthermore, assume that, for any $T \geq t_0$, there exist constants $a_k, b_k \in [T, \infty)_{\mathbb{T}}$ ($a_k < b_k, k = 1, 2$) such that*

$$q_i(t) \geq 0 \text{ for } t \in [G_1(a_1), G_2(b_1)] \cup [G_1(a_2), G_2(b_2)], \\ (-1)^k e(t) \geq 0 \text{ for } t \in [G_1(a_k), G_2(b_k)], \quad (17)$$

where $G_1(t) = \min\{t, \delta_i(t)\}$, $G_2(t) = \max\{t, \delta_i(t)\}$, $i = 0, 1, \dots, n$.

If $x(t)$ is a nonoscillatory solution of (9), then, for $t \in [a_k, b_k]$, we have $x(\delta_i(t)) \geq g_{i,k}(t)x(\sigma(t))$, $i = 0, 1, 2, \dots, n$, $k = 1, 2$.

Proof. If $x(t)$ is an eventually positive solution of (9), then by (h_1) there exists a $T \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \quad x(\delta_i(t)) > 0 \text{ for } t \geq T, i = 0, 1, 2, \dots, n. \quad (18)$$

By assumption, we can choose $b_1 > a_1 > T$ such that $q_i(t) \geq 0$ and $e(t) \leq 0$ on $[G_1(a_1), G_2(b_1)]$. By (9) we have

$$(r(t)(x^\Delta(t))^\gamma)^\Delta \leq 0 \text{ for } t \in [G_1(a_1), G_2(b_1)], \quad (19)$$

which implies that $r(t)(x^\Delta(t))^\gamma$ is decreasing on $[a_1, b_1] \subset [G_1(a_1), G_2(b_1)]$.

Case 1. $t \in [a_1, b_1]$ and $\delta_i(t) < \sigma(t)$. In this case, we get

$$x^\sigma(t) - x^{\delta_i}(t) = \int_{\delta_i(t)}^{\sigma(t)} \frac{(r(s)(x^\Delta(s))^\gamma)^{1/\gamma}}{r^{1/\gamma}(s)} \Delta s \\ \leq (r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma)^{1/\gamma} \int_{\delta_i(t)}^{\sigma(t)} \frac{\Delta s}{r^{1/\gamma}(s)}. \quad (20)$$

It follows that

$$\frac{x^\sigma(t)}{x^{\delta_i}(t)} < 1 + \frac{(r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma)^{1/\gamma}}{x^{\delta_i}(t)} \int_{\delta_i(t)}^{\sigma(t)} \frac{\Delta s}{r^{1/\gamma}(s)}. \quad (21)$$

Also, since $\delta_i^\Delta(t) \geq 0$, we see $\delta_i(t) \geq \delta_i(a_1)$ for $t \in [a_1, b_1]$. So, we get

$$x^{\delta_i}(t) > x^{\delta_i}(t) - x(\delta_i(a_1)) \\ = \int_{\delta_i(a_1)}^{\delta_i(t)} \frac{(r(s)(x^\Delta(s))^\gamma)^{1/\gamma}}{r^{1/\gamma}(s)} \Delta s \\ \geq (r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma)^{1/\gamma} \int_{\delta_i(a_1)}^{\delta_i(t)} \frac{\Delta s}{r^{1/\gamma}(s)}, \quad (22)$$

which implies that

$$\frac{(r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma)^{1/\gamma}}{x^{\delta_i}(t)} < \frac{1}{\int_{\delta_i(a_1)}^{\delta_i(t)} (\Delta s/r^{1/\gamma}(s))}. \quad (23)$$

It follows that

$$\frac{x^\sigma(t)}{x^{\delta_i}(t)} < 1 + \frac{\int_{\delta_i(t)}^{\sigma(t)} (\Delta s/r^{1/\gamma}(s))}{\int_{\delta_i(a_1)}^{\delta_i(t)} (\Delta s/r^{1/\gamma}(s))} \\ \leq \frac{\int_{\delta_i(a_1)}^{\sigma(t)} (\Delta s/r^{1/\gamma}(s))}{\int_{\delta_i(a_1)}^{\delta_i(t)} (\Delta s/r^{1/\gamma}(s))} = \frac{1}{\xi_{i,1}(t)}. \quad (24)$$

Case 2. $t \in [a_1, b_1]$ and $\delta_i(t) > \sigma(t)$. In this case, we obtain

$$\begin{aligned} x^{\delta_i}(t) - x^\sigma(t) &= \int_{\sigma(t)}^{\delta_i(t)} \frac{(r(s)(x^\Delta(s))^\gamma)^{1/\gamma}}{r^{1/\gamma}(s)} \Delta s \\ &\geq (r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma)^{1/\gamma} \int_{\sigma(t)}^{\delta_i(t)} \frac{\Delta s}{r^{1/\gamma}(s)}. \end{aligned} \tag{25}$$

Therefore,

$$\frac{x^\sigma(t)}{x^{\delta_i}(t)} < 1 - \frac{(r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma)^{1/\gamma}}{x^{\delta_i}(t)} \int_{\sigma(t)}^{\delta_i(t)} \frac{\Delta s}{r^{1/\gamma}(s)}. \tag{26}$$

Also, since $\delta^\Delta(t) \geq 0$, we see $\delta_i(t) \leq \delta_i(b_1)$ for $t \in [a_1, b_1]$, and we have

$$\begin{aligned} -x^{\delta_i}(t) &< x(\delta_i(b_1)) - x(\delta_i(t)) \\ &= \int_{\delta_i(t)}^{\delta_i(b_1)} \frac{(r(s)(x^\Delta(s))^\gamma)^{1/\gamma}}{r^{1/\gamma}(s)} \Delta s \\ &\leq (r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma)^{1/\gamma} \int_{\delta_i(t)}^{\delta_i(b_1)} \frac{\Delta s}{r^{1/\gamma}(s)}, \end{aligned} \tag{27}$$

which implies that

$$-\frac{(r(\delta_i(t))(x^\Delta(\delta_i(t)))^\gamma)^{1/\gamma}}{x^{\delta_i}(t)} < \frac{1}{\int_{\delta_i(t)}^{\delta_i(b_1)} (\Delta s/r^{1/\gamma}(s))}. \tag{28}$$

It follows that

$$\begin{aligned} \frac{x^\sigma(t)}{x^{\delta_i}(t)} &< 1 + \frac{\int_{\sigma(t)}^{\delta_i(t)} (\Delta s/r^{1/\gamma}(s))}{\int_{\delta_i(t)}^{\delta_i(b_1)} (\Delta s/r^{1/\gamma}(s))} \\ &\leq \frac{\int_{\sigma(t)}^{\delta_i(b_1)} (\Delta s/r^{1/\gamma}(s))}{\int_{\delta_i(t)}^{\delta_i(b_1)} (\Delta s/r^{1/\gamma}(s))} = \frac{1}{\theta_{i,1}(t)}. \end{aligned} \tag{29}$$

Case 3. $t \in [a_1, b_1]$ and $\delta_i(t) = \sigma(t)$. It is easy to get

$$\frac{x(\delta_i(t))}{x(\sigma(t))} = 1. \tag{30}$$

Combing (24), (29), and (30), we have

$$x(\delta_i(t)) \geq g_{i,1}(t) x^\sigma(t), \quad i = 0, 1, \dots, n, \quad t \in [a_1, b_1]. \tag{31}$$

When $x(t)$ is an eventually negative solution of (9), its proof follows the similar argument using the interval $[G_1(a_2), G_2(b_2)]$ instead of $[G_1(a_1), G_2(b_1)]$.

The proof is complete. \square

Lemma 5 (see [2]). Let $\alpha_i, i = 1, 2, \dots, n$, be the n -tuple satisfying $\alpha_1 > \alpha_2 > \dots > \alpha_m > \gamma > \alpha_{m+1} > \dots > \alpha_n > 0$.

Then there exists an n -tuple (η_1, \dots, η_n) with $0 < \eta_i < 1$ satisfying

$$\sum_{i=1}^n \alpha_i \eta_i = \gamma \tag{32}$$

and which also satisfies either

$$\sum_{i=1}^n \eta_i < 1 \tag{33}$$

or

$$\sum_{i=1}^n \eta_i = 1. \tag{34}$$

Lemma 6 (see [7]). Let α, β, u, A , and B be positive real numbers and let γ be a quotient of odd positive integers. Then

$$Au^\gamma - Bu^{\gamma-\alpha} \geq -\alpha \left(\left(\frac{\gamma-\alpha}{A} \right)^{\gamma-\alpha} \left(\frac{B}{\gamma} \right)^\gamma \right)^{1/\alpha}, \quad 0 < \alpha < \gamma, \tag{35}$$

$$Au^{\beta-\gamma} + Bu^{-\gamma} \geq \beta \left(\left(\frac{B}{\beta-\gamma} \right)^{\beta-\gamma} \left(\frac{A}{\gamma} \right)^\gamma \right)^{1/\beta}, \quad 0 < \gamma < \beta. \tag{36}$$

Lemma 7 (Yong's Inequality). If $p > 1$ and $q > 1$ are conjugate numbers ($1/p + 1/q = 1$), then

$$\frac{|u|^p}{p} + \frac{|v|^q}{q} \geq |uv|, \quad uv \in \mathbb{R}, \tag{37}$$

and equality holds if and only if $u = |v|^{q-2}v$.

Let $\beta > \alpha, u = A^{\alpha/\beta} x^\alpha, p = \beta/\alpha$, and $v = (B\alpha)^{1-(\alpha/\beta)} (\beta - \alpha)^{(\alpha/\beta)-1}$. It follows from Lemma 7 that

$$Ax^\beta + B \geq \beta \alpha^{-\alpha/\beta} (\beta - \alpha)^{(\alpha/\beta)-1} A^{\alpha/\beta} B^{1-(\alpha/\beta)} x^\alpha \tag{38}$$

for all $A, B, x \geq 0$. Rewriting the above inequality we also have

$$Cx^\alpha - D \leq \alpha \beta^{-\beta/\alpha} (\beta - \alpha)^{(\beta/\alpha)-1} C^{\beta/\alpha} D^{1-(\beta/\alpha)} x^\beta \tag{39}$$

for all $C, D, x \geq 0$.

3. Main Results

In this section, by employing the Riccati transformation technique we will establish oscillation criteria for (9). Set

$$Q_{1,k}(t) = q_0(t) g_{0,k}^\gamma(t) + \left(\eta_0^{-1} |e(t)| \right)^{\eta_0} \prod_{i=1}^n \left(\eta_i^{-1} q_i(t) g_{i,k}^{\alpha_i}(t) \right)^{\eta_i},$$

$$Q_{2,k}(t) = q_0(t) g_{0,k}^\gamma(t) + \prod_{i=1}^n \left(\eta_i^{-1} q_i(t) g_{i,k}^{\alpha_i}(t) \right)^{\eta_i},$$

$$\begin{aligned}
 Q_{3,k}(t) &= q_0(t) g_{0,k}^\gamma(t) \\
 &+ \sum_{i=1}^m \alpha_i \left(\left(\frac{q_i(t) g_{i,k}^{\alpha_i}(t)}{\gamma} \right)^\gamma \left(\frac{\lambda_i |e(t)|}{\alpha_i - \gamma} \right)^{\alpha_i - \gamma} \right)^{1/\alpha_i} \\
 &- \sum_{i=m+1}^n \alpha_i \left(\left(\frac{(q_i(t) - g_{i,k}^{\alpha_i}(t))}{\gamma} \right)^\gamma \left(\frac{\gamma - \alpha_i}{\lambda_i |e(t)|} \right)^{\gamma - \alpha_i} \right)^{1/\alpha_i}.
 \end{aligned} \tag{40}$$

We define for $a, b \in [t_0, \infty)_{\mathbb{T}}$ with $a < b$ the admissible set

$$A(a, b) := \{u \in C_{rd}^1[a, b] : u(a) = u(b) = 0, u \neq 0\}. \tag{41}$$

By employing (32) and (33) in Lemma 5, we have the following theorem.

Theorem 8. *Let conditions (h_1) and (h_2) hold. Furthermore, assume that, for any $T \geq t_0$, there exist constants $a_k, b_k \in [T, \infty)_{\mathbb{T}}$ ($a_k < b_k, k = 1, 2$) such that (17) holds. If there exists a function $u \in A(a_k, b_k)$ such that*

$$\int_{a_k}^{b_k} \left[(u^\sigma(s))^{\gamma+1} Q_{1,k}(s) - (u^\Delta(s))^{\gamma+1} r(s) \right] \Delta s \geq 0, \quad k=1, 2, \tag{42}$$

then (9) is oscillatory, where $\eta_i > 0$ ($i = 1, 2, \dots, n$) satisfy (32)-(33) and $\eta_0 = 1 - \sum_{i=1}^n \eta_i$.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (9). Without loss of generality, we may assume that $x(t)$ is eventually positive. Then, there exists $T \geq t_0$ sufficiently large such that $x(t) > 0, x(\delta_i(t)) > 0$ $i = 0, 1, 2, \dots, n$, for all $t \geq T$. By assumption, we can choose $b_1 > a_1 > T$, such that $q_i(t) \geq 0$ and $e(t) \leq 0$ on the interval $[G_1(a_1), G_2(b_1)]$. From Lemma 4 and (9), we obtain, for $t \in [a_1, b_1]$,

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + \sum_{i=0}^n q_i(t) g_{i,1}^{\alpha_i}(t) x^{\alpha_i}(\sigma(t)) \leq e(t). \tag{43}$$

Defining the function $w(t)$ by the Riccati substitution

$$w(t) := \frac{-r(t)(x^\Delta(t))^\gamma}{x^\gamma(t)} \quad \text{for } t \in [a_1, b_1], \tag{44}$$

and then

$$w^\Delta(t) = -\frac{(r(t)(x^\Delta(t))^\gamma)^\Delta}{x^\gamma(\sigma(t))} - (r(t)(x^\Delta(t))^\gamma) \left(\frac{1}{x^\gamma(t)} \right)^\Delta. \tag{45}$$

By (43), we obtain

$$\begin{aligned}
 &\frac{(r(t)(x^\Delta(t))^\gamma)^\Delta}{x^\gamma(\sigma(t))} \\
 &\leq -\sum_{i=0}^n q_i(t) g_{i,1}^{\alpha_i}(t) x^{\alpha_i - \gamma}(\sigma(t)) + e(t) x^{-\gamma}(\sigma(t)) \\
 &= -q_0(t) g_{0,1}^\gamma(t) \\
 &\quad - \sum_{i=1}^n q_i(t) g_{i,1}^{\alpha_i}(t) x^{\alpha_i - \gamma}(\sigma(t)) - |e(t)| x^{-\gamma}(\sigma(t)).
 \end{aligned} \tag{46}$$

Corresponding to the exponents α_i ($i = 1, 2, \dots, n$) in (9), let η_i ($i = 1, 2, \dots, n$) be chosen to satisfy (32) and (33) in Lemma 5 and set $\eta_0 = 1 - \sum_{i=1}^n \eta_i$.

Setting

$$a_0 = \eta_0^{-1} |e(t)|, \quad a_i = \eta_i^{-1} q_i(t) g_{i,1}^{\alpha_i}(t) x^{\alpha_i}(\sigma(t)) \tag{47}$$

and by the arithmetic geometric mean inequality in [21],

$$\sum_{i=0}^n \eta_i a_i \geq \prod_{i=0}^n a_i^{\eta_i}, \quad a_i \geq 0, \tag{48}$$

we get for $t \in [a_1, b_1]$

$$\begin{aligned}
 &\sum_{i=1}^n q_i(t) g_{i,1}^{\alpha_i}(t) x^{\alpha_i - \gamma}(\sigma(t)) + |e(t)| x^{-\gamma}(\sigma(t)) \\
 &= x^{-\gamma}(\sigma(t)) \left[(\eta_0 \eta_0^{-1} |e(t)|) \right. \\
 &\quad \left. + \sum_{i=1}^n \eta_i \eta_i^{-1} q_i(t) g_{i,1}^{\alpha_i}(t) x^{\alpha_i}(\sigma(t)) \right] \\
 &\geq x^{-\gamma}(\sigma(t)) (\eta_0^{-1} |e(t)|)^{\eta_0} \\
 &\quad \times \prod_{i=1}^n (\eta_i^{-1} q_i(t) g_{i,1}^{\alpha_i}(t) x^{\alpha_i}(\sigma(t)))^{\eta_i} \\
 &= x^{-\gamma}(\sigma(t)) (\eta_0^{-1} |e(t)|)^{\eta_0} \\
 &\quad \times \prod_{i=1}^n (\eta_i^{-1} q_i(t) g_{i,1}^{\alpha_i}(t))^{\eta_i} (x(\sigma(t)))^{\sum_{i=1}^n \alpha_i \eta_i}.
 \end{aligned} \tag{49}$$

For (32), we have

$$\begin{aligned}
 &\frac{(r(t)(x^\Delta(t))^\gamma)^\Delta}{x^\gamma(\sigma(t))} \\
 &\leq -q_0(t) g_{0,1}^\gamma(t) - (\eta_0^{-1} |e(t)|)^{\eta_0} \\
 &\quad \times \prod_{i=1}^n (\eta_i^{-1} q_i(t) g_{i,1}^{\alpha_i}(t))^{\eta_i}.
 \end{aligned} \tag{50}$$

By (45)–(50), we get

$$w^\Delta(t) > Q_{1,1}(t) + \frac{r(t)(x^\Delta(t))^\gamma(x^\gamma(t))^\Delta}{x^\gamma(t)x^\gamma(\sigma(t))}. \quad (51)$$

Set $\Phi_\gamma(y) = |y|^{\gamma-1}y$. Since $\gamma > 0$ is a quotient of odd positive integers, it is easy to prove $\Phi_\gamma(y) = |y|^{\gamma-1}y = y^\gamma$. Multiplying by $(u^\sigma(t))^{\gamma+1}$ on (51) and then using the identity

$$(u^{\gamma+1}(t)w(t))^\Delta = (u^\sigma(t))^{\gamma+1}w^\Delta(t) + (u^{\gamma+1}(t))^\Delta w(t), \quad (52)$$

we obtain

$$\begin{aligned} (u^{\gamma+1}(t)w(t))^\Delta &> (u^\sigma(t))^{\gamma+1}Q_{1,1}(t) + (u^{\gamma+1}(t))^\Delta w(t) \\ &\quad + (u^\sigma(t))^{\gamma+1} \frac{r(t)(x^\Delta(t))^\gamma(x^\gamma(t))^\Delta}{x^\gamma(t)x^\gamma(\sigma(t))} \\ &= (u^\sigma(t))^{\gamma+1}Q_{1,1}(t) - (u^\Delta(t))^{\gamma+1}r(t) \\ &\quad + G(u, w), \end{aligned} \quad (53)$$

where

$$\begin{aligned} G(u, w) &= (u^\Delta(t))^{\gamma+1}r(t) + (u^{\gamma+1}(t))^\Delta w(t) \\ &\quad + (u^\sigma(t))^{\gamma+1} \frac{r(t)(x^\Delta(t))^\gamma(x^\gamma(t))^\Delta}{x^\gamma(t)x^\gamma(\sigma(t))} \\ &= (u^\Delta(t))^{\gamma+1}r(t) + (u^{\gamma+1}(t))^\Delta w(t) \\ &\quad + (u^\sigma(t))^{\gamma+1} \frac{r(t)\Phi_\gamma(x^\Delta(t))(\Phi_\gamma(x(t)))^\Delta}{\Phi_\gamma(x(t))\Phi_\gamma(x(\sigma(t)))}. \end{aligned} \quad (54)$$

As demonstrated in [10], we know that $G(u, w) \geq 0$ and that $G(u, w) = 0$ if and only if

$$\Phi_\gamma^{-1}\left(-\frac{w(t)}{r(t)}\right)u(t) = u^\Delta(t), \quad (55)$$

where Φ_γ^{-1} stands for the inverse function. In our case, since $1 + \mu\Phi_\gamma^{-1}(-w(t)/r(t)) = x^\sigma(t)/x(t) > 0$, and dynamic equation (55) has a unique solution satisfying $u(a_1) = 0$. Clearly, the unique solution is $u \equiv 0$. Therefore, $G(u, w) > 0$ on $[a_1, b_1]$. So, we get

$$(u^{\gamma+1}(t)w(t))^\Delta > (u^\sigma(t))^{\gamma+1}Q_{1,1}(t) - (u^\Delta(t))^{\gamma+1}r(t). \quad (56)$$

Integrating from a_1 to b_1 and using $u(a_1) = u(b_1) = 0$, we find

$$\int_{a_1}^{b_1} \left[(u^\sigma(s))^{\gamma+1}Q_{1,1}(s) - (u^\Delta(s))^{\gamma+1}r(s) \right] \Delta s < 0, \quad (57)$$

which leads to a contradiction to (42).

The proof when $x(t)$ is eventually negative follows the similar arguments using the interval $[G_1(a_2), G_2(b_2)]$ instead of $[G_1(a_1), G_2(b_2)]$

The proof is complete. \square

By employing (32) and (34) in Lemma 5, we have the following theorem.

Theorem 9. *Let conditions (h_1) and (h_2) hold. Furthermore, assume that, for any $T \geq t_0$, there exist constants $a_k, b_k \in [T, \infty)_{\mathbb{T}}$ ($a_k < b_k, k = 1, 2$) such that (17) holds. If there exists a function $u \in A(a_k, b_k)$ such that*

$$\int_{a_k}^{b_k} \left[(u^\sigma(s))^{\gamma+1}Q_{2,k}(s) - (u^\Delta(s))^{\gamma+1}r(s) \right] \Delta s \geq 0, \quad k = 1, 2, \quad (58)$$

then (9) is oscillatory, where $\eta_i > 0$ ($i = 1, 2, \dots, n$) satisfy (32) and (34).

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (9). Without loss of generality, we may assume that $x(t)$ is eventually positive. Then, there exists $T \geq t_0$ sufficiently large such that $x(t) > 0, x(\delta_i(t)) > 0, i = 0, 1, 2, \dots, n$, for all $t \geq T$. By assumption, we can choose $b_1 > a_1 > T$, such that $q_i(t) \geq 0$ and $e(t) \leq 0$ on the interval $[G_1(a_1), G_2(b_1)]$. Define the Riccati substitution $w(t)$ as (44). Let η_i ($i = 1, 2, \dots, n$) be chosen to satisfy (32) and (34) in Lemma 5. By (46), we can get

$$\begin{aligned} &\frac{(r(t)(x^\Delta(t))^\gamma)^\Delta}{x^\gamma(\sigma(t))} \\ &\leq -q_0(t)g_{0,1}^\gamma(t) \\ &\quad - \sum_{i=1}^n q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(\sigma(t)) - |e(t)|x^{-\gamma}(\sigma(t)) \quad (59) \\ &\leq -q_0(t)g_{0,1}^\gamma(t) - x^{-\gamma}(\sigma(t)) \\ &\quad \times \sum_{i=1}^n \eta_i \eta_i^{-1} q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i}(\sigma(t)). \end{aligned}$$

Setting

$$a_i = \frac{1}{\eta_i} q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i}(\sigma(t)) \quad (i = 1, 2, \dots, n), \quad (60)$$

using again the arithmetic-geometric mean inequality in [21],

$$\sum_{i=1}^n \eta_i a_i \geq \prod_{i=1}^n a_i^{\eta_i}, \quad a_i \geq 0, \quad (61)$$

and similar to (50), we have

$$\frac{(r(t)(x^\Delta(t))^\gamma)^\Delta}{x^\gamma(\sigma(t))} \leq -q_0(t)g_{0,1}^\gamma(t) - \prod_{i=1}^n (\eta_i^{-1} q_i(t)g_{i,1}^{\alpha_i}(t))^{\eta_i}. \quad (62)$$

By (45) and (62), we get

$$w^\Delta(t) > Q_{2,1}(t) + \frac{r(t)(x^\Delta(t))^\gamma(x^\gamma(t))^\Delta}{x^\gamma(t)x^\gamma(\sigma(t))}. \quad (63)$$

The remainder of the proof is similar to that of Theorem 8. The proof is complete. \square

By employing (35) and (36) in Lemma 6, we have the following theorem.

Theorem 10. *Let conditions (h_1) and (h_2) hold. Furthermore, assume that, for any $T \geq t_0$, there exist constants $a_k, b_k \in [T, \infty)_\mathbb{T}$ ($a_k < b_k, k = 1, 2$) such that (17) holds. If there exists a function $u \in A(a_k, b_k)$ such that*

$$\int_{a_k}^{b_k} \left[(u^\sigma(s))^{\gamma+1} Q_{3,k}(s) - (u^\Delta(s))^{\gamma+1} r(s) \right] \Delta s \geq 0, \quad k = 1, 2, \quad (64)$$

then (9) is oscillatory, where λ_i ($i = 1, 2, \dots, n$) are positive numbers with $\sum_{i=1}^n \lambda_i = 1$ and $(q_i(t))_- = \max\{-q_i(t), 0\}, i = 1, 2, \dots, n$.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (9). Without loss of generality, we may assume that $x(t)$ is eventually positive. Then, there exists $T \geq t_0$ sufficiently large such that $x(t) > 0, x(\delta_i(t)) > 0, i = 0, 1, 2, \dots, n$, for all $t \geq T$. By assumption, we can choose $b_1 > a_1 > T$, such that $q_i(t) \geq 0$ and $e(t) \leq 0$ on the interval $[G_1(a_1), G_2(b_1)]$. Define the Riccati substitution $w(t)$ as (44). Let λ_i ($i = 1, 2, \dots, n$) be chosen to satisfy $\sum_{i=1}^n \lambda_i = 1$. Similar to the proof of Theorem 8, we can get

$$\begin{aligned} & \frac{(r(t)(x^\Delta(t))^\gamma)^\Delta}{x^\gamma(\sigma(t))} \\ & \leq -q_0(t)g_{0,1}^\gamma(t) - \sum_{i=1}^m [q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(\sigma(t)) \\ & \quad + \lambda_i|e(t)|x^{-\gamma}(\sigma(t))] \\ & \quad - \sum_{i=m+1}^n [q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(\sigma(t)) + \lambda_i|e(t)|x^{-\gamma}(\sigma(t))]. \end{aligned} \quad (65)$$

From (36), we get, for $t \in (a_1, b_1)$ and $\alpha_i > \gamma$ ($i = 1, 2, \dots, m$),

$$\begin{aligned} & q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(\sigma(t)) + \lambda_i|e(t)|x^{-\gamma}(\sigma(t)) \\ & \geq \alpha_i \left(\left(\frac{q_i(t)g_{i,1}^{\alpha_i}(t)}{\gamma} \right)^\gamma \left(\frac{\lambda_i|e(t)|}{\alpha_i - \gamma} \right)^{\alpha_i-\gamma} \right)^{1/\alpha_i}. \end{aligned} \quad (66)$$

From (35), we get, for $t \in (a_1, b_1)$ and $\alpha_i < \gamma$ ($i = m + 1, m + 2, \dots, n$),

$$\begin{aligned} & q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(\sigma(t)) + \lambda_i|e(t)|x^{-\gamma}(\sigma(t)) \\ & \geq \lambda_i|e(t)|x^{-\gamma}(\sigma(t)) - (q_i(t))_-g_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(\sigma(t)) \\ & \geq -\alpha_i \left(\left(\frac{(q_i(t))_-g_{i,1}^{\alpha_i}(t)}{\gamma} \right)^\gamma \left(\frac{\gamma - \alpha_i}{\lambda_i|e(t)|} \right)^{\gamma-\alpha_i} \right)^{1/\alpha_i}. \end{aligned} \quad (67)$$

By (45), (65), (66), and (67), we get

$$w^\Delta(t) > Q_{3,1}(t) + \frac{r(t)(x^\Delta(t))^\gamma(x^\gamma(t))^\Delta}{x^\gamma(t)x^\gamma(\sigma(t))}. \quad (68)$$

The remainder of the proof is similar to that of Theorem 8. The proof is complete. \square

By employing (38) and (39), we have the following theorem.

Theorem 11. *Let conditions (h_1) and (h_2) hold. Furthermore, assume that, for any $T \geq t_0$, there exist constants $a_k, b_k \in [T, \infty)_\mathbb{T}$ ($a_k < b_k, k = 1, 2$) such that (17) holds. If there exists a function $u \in A(a_k, b_k)$ such that*

$$\int_{a_k}^{b_k} \left[(u^\sigma(s))^{\gamma+1} Q_{4,k}(s) - (u^\Delta(s))^{\gamma+1} r(s) \right] \Delta s \geq 0, \quad k = 1, 2, \quad (69)$$

then (9) is oscillatory, where λ_i ($i = 1, 2, \dots, n$) are positive numbers with $\sum_{i=1}^n \lambda_i = 1, (q_i(t))_- = \max\{-q_i(t), 0\}, i = 1, 2, \dots, n$,

$$\begin{aligned} P_{i,k}(t) &= \alpha_i \gamma^{-\gamma/\alpha_i} (\alpha_i - \gamma)^{(\gamma/\alpha_i)-1} (q_i(t)g_{i,k}^{\alpha_i}(t))^{\gamma/\alpha_i} \\ & \quad \times (\lambda_i|e(t)|)^{1-(\gamma/\alpha_i)}, \\ R_{i,k}(t) &= \alpha_i \gamma^{-\gamma/\alpha_i} (\gamma - \alpha_i)^{(\gamma/\alpha_i)-1} ((q_i(t))_-g_{i,k}^{\alpha_i}(t))^{\gamma/\alpha_i} \\ & \quad \times (\lambda_i|e(t)|)^{1-(\gamma/\alpha_i)}, \\ Q_{4,k}(t) &= q_0(t)g_{0,k}^\gamma(t) + \sum_{i=1}^m P_{i,k}(t) - \sum_{i=m+1}^n R_{i,k}(t). \end{aligned} \quad (70)$$

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (9). Without loss of generality, we may assume that $x(t)$ is eventually positive. Then, there exists $T \geq t_0$ sufficiently large such that $x(t) > 0, x(\delta_i(t)) > 0, i = 0, 1, 2, \dots, n$, for all $t \geq T$. By assumption, we can choose $b_1 > a_1 > T$, such that $q_i(t) \geq 0$ and $e(t) \leq 0$ on the interval

$[G_1(a_1), G_2(b_1)]$. Define the Riccati substitution $w(t)$ as (44). By (43), we have

$$\begin{aligned} & (r(t)(x^\Delta(t))^\gamma)^\Delta + q_0(t)g_{0,1}^\gamma(t)x^\gamma(\sigma(t)) \\ & + \sum_{i=1}^m [q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i}(\sigma(t)) + \lambda_i|e(t)|] \\ & - \sum_{i=m+1}^n [-q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i}(\sigma(t)) - \lambda_i|e(t)|] \leq 0. \end{aligned} \tag{71}$$

Applying (38) and setting

$$\begin{aligned} A &= q_i(t)g_{i,1}^{\alpha_i}(t), & B &= \lambda_i|e(t)|, \\ \beta &= \alpha_i, & \alpha &= \gamma (\alpha_i > \gamma), \end{aligned} \tag{72}$$

we have

$$\sum_{i=1}^m [q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i}(\sigma(t)) + \lambda_i|e(t)|] \geq \sum_{i=1}^m P_{i,1}(t)x^\gamma(\sigma(t)). \tag{73}$$

On the other hand, we can get

$$\begin{aligned} & \sum_{i=m+1}^n [-q_i(t)g_{i,1}^{\alpha_i}(t)x^{\alpha_i}(t) - \lambda_i|e(t)|] \\ & \leq \sum_{i=m+1}^n [(q_i(t))_- g_{i,1}^{\alpha_i}(t)x^{\alpha_i}(\sigma(t)) - \lambda_i|e(t)|]. \end{aligned} \tag{74}$$

Applying (39) and setting

$$\begin{aligned} C &= (q_i(t))_- g_{i,1}^{\alpha_i}(t), & D &= \lambda_i|e(t)|, \\ \beta &= \gamma, & \alpha &= \alpha_i (\gamma > \alpha_i), \end{aligned} \tag{75}$$

we obtain that

$$\begin{aligned} & \sum_{i=m+1}^n [(q_i(t))_- g_{i,1}^{\alpha_i}(t)x^{\alpha_i}(\sigma(t)) + \lambda_i|e(t)|] \\ & \leq \sum_{i=m+1}^n R_{i,1}(t)x^\gamma(\sigma(t)). \end{aligned} \tag{76}$$

From (71), (73), and (76), we have

$$\begin{aligned} & (r(t)(x^\Delta(t))^\gamma)^\Delta \leq -q_0(t)g_{0,1}^\gamma(t)x^\gamma(\sigma(t)) \\ & - \sum_{i=1}^m P_{i,1}(t)x^\gamma(\sigma(t)) \\ & + \sum_{i=m+1}^n R_{i,1}(t)x^\gamma(\sigma(t)). \end{aligned} \tag{77}$$

By (45) and (77), we have

$$w^\Delta(t) > Q_{4,1}(t) + \frac{r(t)(x^\Delta(t))^\gamma(x^\gamma(t))^\Delta}{x^\gamma(t)x^\gamma(\sigma(t))}. \tag{78}$$

The remainder of the proof is similar to that of Theorem 8. The proof is complete. \square

4. Forms Related to (9)

Related to (9) are the dynamic equations with mixed delta and nabla derivatives:

$$(r(t)(x^\Delta(t))^\gamma)^\nabla + \sum_{i=0}^n q_i(t)|x(\delta_i(t))|^{\alpha_i} \operatorname{sgn} x(\delta_i(t)) = e(t), \tag{79}$$

$$(r(t)(x^\nabla(t))^\gamma)^\Delta + \sum_{i=0}^n q_i(t)|x(\delta_i(t))|^{\alpha_i} \operatorname{sgn} x(\delta_i(t)) = e(t). \tag{80}$$

It is not difficult to see that time scale modifications of the previous arguments give rise to completely parallel results for the above dynamic equations. For an illustrative example we provide below the version of Theorem 8 for (79), for which a lemma analogous to Lemma 4 can be stated easily. The other theorems for (79) and (80) can be obtained by employing arguments developed for (9).

Set

$$\begin{aligned} \zeta_{i,k}(t) &= \frac{\int_{\delta_i(a_k)}^{\delta_i(t)} r^{-1/\gamma}(s) \Delta s}{\int_{\delta_i(a_k)}^{\rho(t)} r^{-1/\gamma}(s) \Delta s}, \\ \eta_{i,k}(t) &= \frac{\int_{\delta_i(t)}^{\delta_i(b_k)} r^{-1/\gamma}(s) \Delta s}{\int_{\delta_i(b_k)}^{\rho(t)} r^{-1/\gamma}(s) \Delta s}, \\ p_{i,k}(t) &= \begin{cases} \zeta_{i,k}, & \delta_i(t) < \rho(t), \\ 1, & \delta_i(t) = \rho(t), \\ \eta_{i,k}, & \delta_i(t) > \rho(t), \end{cases} \end{aligned} \tag{81}$$

$$\begin{aligned} Q_{5,k}(t) &= q_0(t)P_{0,k}^\gamma(t) + (\eta_0^{-1}|e(t)|)^{\eta_0} \\ & \times \prod_{i=1}^n (\eta_i^{-1}q_i(t)p_{i,k}^{\alpha_i}(t))^{\eta_i}. \end{aligned}$$

Theorem 12. *Let conditions (h_1) and (h_2) hold. Furthermore, assume that, for any $T \geq t_0$, there exist constants $a_k, b_k \in [T, \infty)_\mathbb{T}$ ($a_k < b_k, k = 1, 2$) such that (17) holds. If there exists a function $u \in A_*(a_k, b_k) := \{u \in C_{\text{id}}^1[a_k, b_k] : u(a_k) = u(b_k) = 0, u(t) \neq 0\}$ such that*

$$\int_{a_k}^{b_k} [(u^\rho(s))^{\gamma+1}Q_{5,k}(s) - (u^\nabla(s))^{\gamma+1}r(s)] \nabla s \geq 0, \quad k = 1, 2, \tag{82}$$

then (79) is oscillatory, where ρ denotes the backward jump operator, and $\eta_i > 0$ ($i = 1, 2, \dots, n$) satisfy (32)-(33) and $\eta_0 = 1 - \sum_{i=1}^n \eta_i$.

5. Examples

In this section, we give two examples to illustrate our main results. We first give an example to show Theorems 8.

Example 1. Consider on \mathbb{R} the differential equation:

$$\begin{aligned} & ((\sin 8t + 2)(x'(t))^\gamma)' + c_0 t \sin^2 8t x^\gamma(t) \\ & + 4c_1 t^2 \cos 2t |x(t)|^{5\gamma/2} \operatorname{sgn} x(t) \\ & + c_2 t \sin 2t \operatorname{sgn} |x(t)|^{\gamma/2} = -\cos 4t, \quad t \geq 1. \end{aligned} \tag{83}$$

We set

$$\begin{aligned} r(t) &= \sin 8t + 2, \quad q_0(t) = c_0 t \sin^2 8t, \quad q_1(t) = 4c_1 t^2 \cos 2t, \\ q_2(t) &= c_2 t \sin 2t, \quad e(t) = \cos 4t, \quad \delta_i(t) = t \ (i = 0, 1, 2), \\ \alpha_0 &= \gamma, \quad \alpha_1 = \frac{5\gamma}{2}, \quad \alpha_2 = \frac{\gamma}{2}, \\ g_{0,1}^\gamma(t) &= g_{1,1}^\gamma(t) = g_{2,1}^\gamma(t) = 1. \end{aligned} \tag{84}$$

Also, let $\eta_i = 1/3$, and

$$\begin{aligned} a_1 &= 2h\pi, \quad b_i = a_2 = 2h\pi + \frac{\pi}{8}, \\ b_2 &= 2h\pi + \frac{\pi}{4} \ (h = 1, 2, 3 \dots), \end{aligned} \tag{85}$$

such that (32) and (33) hold

$$\begin{aligned} q_i(t) &\geq 0 \quad \text{on} \left[2h\pi, 2h\pi + \frac{\pi}{8} \right) \cup \left[2h\pi + \frac{\pi}{8}, 2h\pi + \frac{\pi}{4} \right), \\ & \quad \quad \quad i = 0, 1, 2, \\ (-1)^k e(t) &\geq 0, \quad t \in \left[2h\pi, 2h\pi + \frac{\pi}{8} \right) \cup \left[2h\pi + \frac{\pi}{8}, 2h\pi + \frac{\pi}{4} \right), \\ & \quad \quad \quad k = 1, 2. \end{aligned} \tag{86}$$

Setting $u(t) = \sin 8t$, we get $Q_{1,1}(t) = t(c_0 \sin^2 8t + 3(2c_1 c_2 \sin 4t |\cos 4t|)^{1/3})$. By (42), we have

$$\begin{aligned} & \int_{a_k}^{b_k} \left[(u^\sigma(s))^{\gamma+1} Q_{1,k}(s) - (u^\Delta(s))^{\gamma+1} r(s) \right] ds \\ &= \int_{2\pi}^{2\pi+\pi/8} \left[\sin^{\gamma+1} 8t \cdot t \left(c_0 \sin^2 8t + 3(2c_1 c_2 \sin 4t |\cos 4t|)^{1/3} \right) \right. \\ & \quad \left. - (8 \cos 8t)^{\gamma+1} (\sin 8t + 2) \right] dt \\ &\geq \int_0^{\pi/8} \left[2\pi \sin^{\gamma+1} 8t \left(c_0 \sin^2 8t + 3(c_1 c_2 \sin 8t)^{1/3} \right) \right. \\ & \quad \left. - 8^\gamma (2 \cos^{\gamma+1} 8t + \sin 8t \cos^{\gamma+1} 8t) \right] dt \end{aligned}$$

$$\begin{aligned} &= \frac{\pi \sqrt{\pi}}{4} \left[\frac{c_0 \gamma (\gamma + 2) \Gamma(\gamma/2)}{(\gamma + 3)(\gamma + 1) \Gamma((\gamma + 1)/2)} \right. \\ & \quad \left. + \frac{3 \sqrt[3]{c_1 c_2} (3\gamma + 1) \Gamma((3\gamma + 1)/6)}{3\gamma + 4 \Gamma((3\gamma + 4)/6)} \right] \\ &\geq 8^\gamma \left[\frac{\gamma \sqrt{\pi} \Gamma(\gamma/2)}{(\gamma + 1) \Gamma((\gamma + 1)/2)} + \frac{1}{\gamma + 2} \right] > 0. \end{aligned} \tag{87}$$

Hence, by Theorems 8, (83) is oscillatory.

Example 2. Consider on \mathbb{Z} the equation:

$$\begin{aligned} & \Delta \left((\Delta x(t))^3 \right) + A |x(t)|^3 \operatorname{sgn} (x(t)) \\ & + B |x(t+2)|^4 \operatorname{sgn} (x(t+2)) \\ & + C |x(t+1)|^2 \operatorname{sgn} (t+1) = e(t), \end{aligned} \tag{88}$$

where $t \in 0, 1, 2, \dots, A, B, C > 0$,

$$e(t) = f(n) = \begin{cases} 0, & t = 8j, 8j + 4, \\ -1, & t = 8j + 1, 8j + 2, 8j + 3 \\ 1 & t = 8j + 5, 8j + 6, 8j + 7. \end{cases} \tag{89}$$

We set $q_0(t) = A, q_1(t) = B, q_2(t) = C, \alpha_0 = \gamma = 3, \alpha_1 = 4, \alpha_2 = 2, \delta_0(t) = t, \delta_1(t) = t + 2, \delta_2(t) = t + 1$.

Setting $a_1 = 8j, b_1 = 8j + 2$, we can get $G_1(a_1) = 8j, G_2(b_1) = 8j + 4$.

So, $e(t)$ satisfies the assumption in Theorem 9. Let $\eta_1 = \eta_2 = 1/2$ such that (32) and (34) hold. Setting $u(t) = t \pmod 2$, we get $Q_{2,1}(8j) = (8/9)\sqrt{BC}$. By (58), we get

$$\begin{aligned} & \int_{a_1}^{b_1} \left[(u^\sigma(s))^{\gamma+1} Q_{2,k}(s) - (u^\Delta(s))^{\gamma+1} r(s) \right] \Delta s \\ &= \sum_{t=8j}^{8j+1} \left(u^4(t+1) Q_{2,1}(t) - (u(t+1) - u(t))^4 \right) \\ &= -2 + \frac{8}{9} \sqrt{BC}. \end{aligned} \tag{90}$$

Consequently, if the constant coefficients $A, B, C > 0$ satisfy the relation

$$\frac{8}{9} \sqrt{BC} \geq 2, \tag{91}$$

then the above sum is nonnegative and hence (88) is oscillatory by Theorem 9.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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