

Research Article

A Class of Vector Lyapunov Functions for Stability Analysis of Nonlinear Impulsive Differential Systems

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A novel and effective approach to stability of the solutions of nonlinear systems with impulsive effect is considered. The investigations are carried out by means of a class of vector Lyapunov functions and differential inequalities for piecewise continuous functions. Simulation examples are given to illustrate the presented results.

1. Introduction

There has been a significant development in the theory of the impulsive differential systems within the past 20 years (see [1–17]). The efficient applications of impulsive differential systems require the finding of criteria for stability of their solutions.

It is well known that the more complex the considered dynamical system is, the more difficult it is to find a Lyapunov function. This leads to employing several Lyapunov functions, which is called to be a vector Lyapunov function, for which each component provides information about a part of the dynamics. Then some of the complex aspects can be reduced to the connected dynamical systems and less rigid requirements can be satisfied. Hence, the corresponding theory, namely, the method of vector Lyapunov functions, offers a very flexible process (see [1–5, 11, 13, 18]).

In this paper, without assuming Lipschitz conditions on these nonlinear functions, which play important roles in the considered systems, as needed in most other papers, we will use a class of vector Lyapunov functions, by means of the comparison principle (see [1, 2, 13]), to study global exponential stability of the solution of nonlinear impulsive differential systems.

2. Statement of the Problem

Let R^n be the n -dimensional Euclidean space with norm $\|x\| = \sqrt{x^T x} = \sqrt{\langle x, x \rangle} = (\sum_{i=1}^n x_i^2)^{1/2}$, $x = (x_1, x_2, \dots, x_n)^T$, let

Ω be a bounded domain in R^m containing the origin, and $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$, where $u = (u_1, u_2, \dots, u_n)^T$, $v = (v_1, v_2, \dots, v_n)^T$, and $R_+ = [0, +\infty)$.

Let $t_0 \in R_+$. Consider the impulsive functional differential system

$$\dot{x}(t) = f(t, x(t)), \quad (t \geq t_0, t \neq t_k, k = 1, 2, 3, \dots) \quad (1)$$

$$\Delta x = x(t_k + 0) - x(t_k) = I_k x(t_k), \quad (t = t_k > t_0),$$

where $f: (t_0, +\infty) \times \Omega \rightarrow R^n$, $I_k: \Omega \rightarrow \Omega$, $k = 1, 2, 3, \dots$, $t_0 < t_1 < t_2 < \dots$, and $\lim_{k \rightarrow +\infty} t_k = +\infty$.

Definition 1 (see [12, 14, 15, 19]). System (1) is called to be exponentially stable on a neighborhood Ω of the equilibrium point, if there exist constants $\mu > 0$, $M > 0$ such that

$$\|x(t)\| \leq M e^{-\mu t} \|x_0\|, \quad (t \geq 0), \quad (2)$$

where $x(t)$ is any solution of (1) initiated from $x(t_0) = x_0$.

We introduce a kind of partial ordering defined in the following natural way. For $u, v \in R^n$, $u \geq v$ ($u > v$) if and only if $u_j \geq v_j$ ($u_j > v_j$) for any $j = 1, 2, \dots, n$.

Definition 2 (see [1, 2, 13]). The function $\Psi: \Omega \rightarrow R^n$ is called monotonically increasing function in Ω if $\Psi(u) > \Psi(v)$ for $u > v$ and $\Psi(u) \geq \Psi(v)$ for $u \geq v$, $u, v \in \Omega$.

Definition 3 (see [1, 2, 13]). The function $F: R_+ \times \Omega \rightarrow R^n$ is called quasi-monotonically increasing function in $R_+ \times \Omega$

if, for any two points (t, u) and (t, v) of $R_+ \times \Omega$ and for any $i \in 1, 2, \dots, n$, the inequality $F_i(t, u) \geq F_i(t, v)$ holds always when $u_i = v_i$ and $u \geq v$, that is, if, for any $t \in R_+$ fixed and any $i \in 1, 2, \dots, n$, the function $F_i(t, u)$ is nondecreasing with respect to $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$.

In the further considerations we will use the class ν_0 of piecewise continuous auxiliary functions $V : [t_0, +\infty) \times \Omega \rightarrow R_+^m$ which are analogues of Lyapunov functions (see [20]).

We denote by κ the class of all continuous and strictly increasing functions $a : R_+ \rightarrow R_+$ such that $a(0) = 0$.

We are interested in focusing on the possibility of applying the comparison method to the stability of the zero solution of system (1). Together with the system (1), we will also consider the following comparison system:

$$\begin{aligned} \dot{u}(t) &= g(t, u(t)), \quad (t \geq t_0, t \neq t_k, k = 1, 2, 3, \dots) \\ \Delta u &= u(t_k + 0) - u(t_k) = B_k u(t_k), \quad (t = t_k > t_0), \end{aligned} \quad (3)$$

where $g : R_+ \times S(\rho) \rightarrow R^m$, $B_k : S(\rho) \rightarrow R^m$, $k = 1, 2, 3, \dots$, $t_0 < t_1 < t_2 < \dots$, $\lim_{k \rightarrow +\infty} t_k = +\infty$, and $S(\rho) = \{x \in R^n : \|x\| < \rho\}$ is an open subset of R^m .

Lemma 4 (see [1, 2, 13]). *Let the following conditions be fulfilled:*

$$(1) \quad V : R_+ \times S(\rho) \rightarrow R_+^m, \quad V \in \nu_0,$$

$$D^+V(t, x) \leq g(t, V(t, x)), \quad t \neq t_k, \quad (t, x) \in R_+ \times S(\rho), \quad (4)$$

where $g : R_+ \times R_+^m \rightarrow R_+^m$ is continuous, quasimonotonically increasing in $(t_{k-1}, t_k] \times R^m$ and $g(t, 0) = 0$ for $t \in R_+$ and $D^+V(t, x) = \lim_{h \rightarrow 0^+} \sup (1/h)[V(t+h, x+hf(t, x)) - V(t, x)]$;

(2) there exists a ρ_0 which satisfies when $x_0 \in S(\rho_0)$, $u + B_k(u)$ are monotonically increasing in $S(\rho)$ and $B_k(0) = 0$;

(3) there exist the functions $a(\cdot), b(\cdot) \in \kappa$ which satisfy $b(\|x\|) \leq L(t, x) \leq a(\|x\|)$ for all $(t, x) \in R_+ \times S(\rho)$, where $L(t, x) = \sum_{i=1}^m V_i(t, x)$, $V(t, x) = (V_1(t, x), V_2(t, x), \dots, V_m(t, x))^T$.

If the zero solution of system (3) is stable, then the zero solution of system (1) is stable with respect to $x(t)$.

3. Theory Results of Impulsive Stability

3.1. Basic Results. Consider the following system

$$\begin{aligned} \dot{x}(t) &= F(t, x(t), y(t)), \\ \dot{y}(t) &= G(t, x(t), y(t)), \end{aligned} \quad (5)$$

where $x(t) \in R^m$, $y(t) \in R^m$, and $F(t, 0, 0) = G(t, 0, 0) = 0$.

We will say that condition (A) is satisfied if the following conditions hold.

(A) The functions $F(t, x(t), y(t))$, $G(t, x(t), y(t))$ are continuous in $(t_{k-1}, t_k] \times R^m \times R^m$. There exists a function $h(t)$, and the condition

$$\begin{aligned} &\frac{(\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)))^T (\alpha x(t) + \beta y(t))}{\|\alpha x(t) + \beta y(t)\|^2} \\ &\leq h(t) \end{aligned} \quad (6)$$

holds for any α, β .

We will say that condition (B) is satisfied if the following conditions hold.

(B) The functions $F_i(t, x(t), y(t))$, $G_i(t, x(t), y(t))$ are continuous in $(t_{k-1}, t_k] \times R^m \times R^m$. There exist functions $h_{ji}(t)$, and the condition

$$\begin{aligned} &\left((\alpha_j F_i(t, x(t), y(t)) + \beta_j G_i(t, x(t), y(t)))^T \right. \\ &\quad \times (\alpha_j x(t) + \beta_j y(t)) \Big) \\ &\quad \times \left(\|\alpha_j x(t) + \beta_j y(t)\| \|\alpha_i x(t) + \beta_i y(t)\| \right)^{-1} \\ &\leq h_{ji}(t) \end{aligned} \quad (7)$$

holds for any $\alpha_j, \beta_j, \alpha_i, \beta_i$.

We will say that condition (C) is satisfied if the following conditions hold.

(C) The functions $F_i(t, x(t), y(t))$, $G_i(t, x(t), y(t))$ are continuous in $(t_{k-1}, t_k] \times R^m \times R^m$. There exist functions $h_{ji}(t)$, and the condition

$$\begin{aligned} &\left((\alpha_j F_i(t, x(t), y(t)) + \beta_j G_i(t, x(t), y(t)))^T \right. \\ &\quad \times (\alpha_j x(t) + \beta_j y(t)) \Big) \\ &\quad \times \left(\|\alpha_j x(t) + \beta_j y(t)\| \|\alpha_i x(t - \tau) + \beta_i y(t - \tau)\| \right)^{-1} \\ &\leq s_{ji}(t) \end{aligned} \quad (8)$$

holds for any $\alpha_j, \beta_j, \alpha_i, \beta_i$.

Theorem 5. *If the operators F, G in system (5) satisfy condition (A) for any $x, y \in \Omega$, then $x(t)$ and $y(t)$ initiated from $x(t_0) = x_0 \in \Omega$, $y(t_0) = y_0 \in \Omega$ satisfy*

$$\left(\|\alpha x(t) + \beta y(t)\| \right)'_t \leq h(t) \|\alpha x(t) + \beta y(t)\| \quad \forall t \geq 0. \quad (9)$$

Proof. For all $t \geq 0$ and $r \geq 0$, we have

$$\begin{aligned} \frac{d\alpha e^{rt} x(t)}{dt} &= \alpha r e^{rt} x(t) + \alpha e^{rt} \dot{x}(t) \\ &= \alpha r e^{rt} x(t) + \alpha e^{rt} F(t, x(t), y(t)), \end{aligned}$$

$$\begin{aligned} \frac{d\beta e^{rt} y(t)}{dt} &= \beta r e^{rt} y(t) + \beta e^{rt} \dot{y}(t) \\ &= \beta r e^{rt} y(t) + \beta e^{rt} G(t, x(t), y(t)), \end{aligned}$$

$$\begin{aligned}
 & \frac{d\alpha e^{rt} x(t)}{dt} + \frac{d\beta e^{rt} y(t)}{dt} \\
 &= \frac{d(\alpha e^{rt} x(t) + \beta e^{rt} y(t))}{dt} = \frac{d[e^{rt}(\alpha x(t) + \beta y(t))]}{dt} \\
 &= e^{rt} [\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)) \\
 &\quad + \alpha r x(t) + \beta r y(t)], \\
 & e^{rt} (\alpha x(t) + \beta y(t)) \\
 &= e^{rs} (\alpha x(s) + \beta y(s)) \\
 &\quad + \int_s^t e^{ru} [\alpha F(u, x(u), y(u)) + \beta G(u, x(u), y(u)) \\
 &\quad + \alpha r x(u) + \beta r y(u)] du, \\
 & e^{rt} \|\alpha x(t) + \beta y(t)\| - e^{rs} \|\alpha x(s) + \beta y(s)\| \\
 &\leq \int_s^t e^{ru} \|\alpha F(u, x(u), y(u)) + \beta G(u, x(u), y(u)) \\
 &\quad + \alpha r x(u) + \beta r y(u)\| du. \tag{10}
 \end{aligned}$$

The derivative with respect to time t is

$$\begin{aligned}
 & e^{rt} (\|\alpha x(t) + \beta y(t)\|)'_t + r e^{rt} \|\alpha x(t) + \beta y(t)\| \\
 &\leq e^{rt} \|\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)) \\
 &\quad + \alpha r x(t) + \beta r y(t)\|. \tag{11}
 \end{aligned}$$

That is,

$$\begin{aligned}
 & (\|\alpha x(t) + \beta y(t)\|)'_t \\
 &\leq \|\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)) \\
 &\quad + \alpha r x(t) + \beta r y(t)\| - r \|\alpha x(t) + \beta y(t)\|. \tag{12}
 \end{aligned}$$

It is easy to prove that $\|\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)) + \alpha r x(t) + \beta r y(t)\| - r \|\alpha x(t) + \beta y(t)\|$ is monotonically decreasing on r ; thus the limit

$$\begin{aligned}
 & \lim_{r \rightarrow +\infty} (\|\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)) \\
 &\quad + \alpha r x(t) + \beta r y(t)\| - r \|\alpha x(t) + \beta y(t)\|) \tag{13}
 \end{aligned}$$

exists. From the norm $\|x\| = \sqrt{x^T x} = \sqrt{\langle x, x \rangle} = (\sum_{i=1}^n x_i^2)^{1/2}$ and condition (A), we obtain

$$\begin{aligned}
 & (\|\alpha x(t) + \beta y(t)\|)'_t \\
 &\leq [(\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)) + \alpha r x(t) \\
 &\quad + \beta r y(t))^T (\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)) \\
 &\quad + \alpha r x(t) + \beta r y(t))]^{1/2} \\
 &\quad - r((\alpha x(t) + \beta y(t))^T (\alpha x(t) + \beta y(t)))^{1/2}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{q(t, \alpha, \beta)}{(p(t, \alpha, \beta))^{1/2} + r \|\alpha x(t) + \beta y(t)\|} \\
 &= \frac{q(t, \alpha, \beta) / r}{(p(t, \alpha, \beta))^{1/2} / r + \|\alpha x(t) + \beta y(t)\|} \\
 &\leq \lim_{r \rightarrow +\infty} \frac{q(t, \alpha, \beta) / r}{(p(t, \alpha, \beta))^{1/2} / r + \|\alpha x(t) + \beta y(t)\|} \\
 &= \frac{(\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)))^T (\alpha x(t) + \beta y(t))}{\|\alpha x(t) + \beta y(t)\|} \\
 &= \frac{(\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)))^T (\alpha x(t) + \beta y(t))}{\|\alpha x(t) + \beta y(t)\|^2} \\
 &\quad \times \|\alpha x(t) + \beta y(t)\| \\
 &\leq h(t) \|\alpha x(t) + \beta y(t)\|, \tag{14}
 \end{aligned}$$

where $q(t, \alpha, \beta) = \|\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t))\|^2 + 2r(\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)))^T (\alpha x(t) + \beta y(t))$ and $p(t, \alpha, \beta) = \|\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t))\|^2 + 2r(\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)))^T (\alpha x(t) + \beta y(t)) + r^2 \|\alpha x(t) + \beta y(t)\|^2$.

Namely,

$$(\|\alpha x(t) + \beta y(t)\|)'_t \leq h(t) \|\alpha x(t) + \beta y(t)\|. \tag{15}$$

□

Corollary 6. When $x(t) \in R^n$, $y(t) \in R^m$, and $n > m$, let

$$y(t) \rightarrow \begin{pmatrix} y(t) \\ 0_{n-m} \end{pmatrix} = \tilde{y}(t),$$

$$G(t, x(t), y(t)) \tag{16}$$

$$\rightarrow \begin{pmatrix} G(t, x(t), y(t)) \\ 0_{n-m} \end{pmatrix} = \tilde{G}(t, x(t), \tilde{y}(t)),$$

and $(\alpha F(t, x(t), \tilde{y}(t)) + \beta \tilde{G}(t, x(t), \tilde{y}(t)))^T (\alpha x(t) + \beta \tilde{y}(t)) / \|\alpha x(t) + \beta \tilde{y}(t)\|^2 \leq h(t)$ holds for any α, β , where $0_{n-m} = (0, 0, \dots, 0)^T \in R^{n-m}$. Then one obtains a similar result with Theorem 5:

$$(\|\alpha x(t) + \beta \tilde{y}(t)\|)'_t \leq h(t) \|\alpha x(t) + \beta \tilde{y}(t)\| \quad \forall t \geq 0. \tag{17}$$

Corollary 7. When $x(t) \in R^n$, $y(t) \in R^m$, and $n > m$, let

$$x(t) \rightarrow \begin{pmatrix} x(t) \\ 0_1 \end{pmatrix} = \tilde{x}(t), \quad y(t) \rightarrow \begin{pmatrix} y(t) \\ 0_2 \end{pmatrix} = \tilde{y}(t),$$

$$F(t, x(t), y(t))$$

$$\rightarrow \begin{pmatrix} F(t, x(t), y(t)) \\ 0_1 \end{pmatrix} = \tilde{F}(t, \tilde{x}(t), \tilde{y}(t)),$$

$$G(t, x(t), y(t))$$

$$\rightarrow \begin{pmatrix} G(t, x(t), y(t)) \\ 0_2 \end{pmatrix} = \tilde{G}(t, \tilde{x}(t), \tilde{y}(t)), \tag{18}$$

and $(\alpha\bar{F}(t, \bar{x}(t), \bar{y}(t)) + \beta\bar{G}(t, \bar{x}(t), \bar{y}(t)))^T(\alpha\bar{x}(t) + \beta\bar{y}(t))/\|\alpha\bar{x}(t) + \beta\bar{y}(t)\|^2 \leq h(t)$ holds for any α, β , where $0_1 = (0, 0, \dots, 0)^T \in R^m$, $0_2 = (0, 0, \dots, 0)^T \in R^n$. Then one obtains a similar result with Theorem 5:

$$\|(\alpha\bar{x}(t) + \beta\bar{y}(t))\|'_t \leq h(t) \|\alpha\bar{x}(t) + \beta\bar{y}(t)\| \quad \forall t \geq 0. \quad (19)$$

Remark 8. When $\alpha = 1$, $\beta = -1$, and $F = G$, the constant

$$\sup_{x, y \in \Omega, x \neq y} \frac{(F(x(t)) - F(y(t)))^T(x(t) - y(t))}{\|x(t) - y(t)\|^2} \quad (20)$$

is called the nonlinear measure [21–26]. But the condition

$$\frac{(\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)))^T(\alpha x(t) + \beta y(t))}{\|\alpha x(t) + \beta y(t)\|^2} \leq h(t) \quad (21)$$

is weaker than the nonlinear measure.

When system (5) is written as

$$\begin{aligned} \dot{x}(t) &= F_1(t, x(t), y(t)) + F_2(t, x(t), y(t)), \\ \dot{y}(t) &= G_1(t, x(t), y(t)) + G_2(t, x(t), y(t)), \end{aligned} \quad (22)$$

where $x(t) \in R^m$, $y(t) \in R^n$, and $F_i(t, 0, 0) = G_i(t, 0, 0) = 0$, $i = 1, 2$, we have the following conclusion.

Corollary 9. If the operators F_1, F_2, G_1, G_2 in system (22) satisfy condition (B) for any $x, y \in \Omega$, then $x(t)$ and $y(t)$ initiated from $x(t_0) = x_0 \in \Omega$, $y(t_0) = y_0 \in \Omega$ satisfy

$$\begin{aligned} \|(\alpha_1 x(t) + \beta_1 y(t))\|'_t & \\ & \leq h_{11}(t) \|\alpha_1 x(t) + \beta_1 y(t)\| + h_{12}(t) \|\alpha_2 x(t) + \beta_2 y(t)\| \\ & \quad \forall t \geq 0. \end{aligned} \quad (23)$$

When system (5) is written as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^s F_i(t, x(t), y(t)), \\ \dot{y}(t) &= \sum_{i=1}^s G_i(t, x(t), y(t)), \end{aligned} \quad (24)$$

where $x(t) \in R^m$, $y(t) \in R^n$, and $F_i(t, 0, 0) = G_i(t, 0, 0) = 0$, $i = 1, 2, \dots, s$, we have the following conclusion.

Corollary 10. If the operators F_i, G_i , $i = 1, 2, \dots, s$, in system (24) satisfy condition (B) for any $x, y \in \Omega$, then $x(t)$ and $y(t)$ initiated from $x(t_0) = x_0 \in \Omega$, $y(t_0) = y_0 \in \Omega$ satisfy

$$\|(\alpha_j x(t) + \beta_j y(t))\|'_t \leq \sum_{i=1}^s h_{ji}(t) \|\alpha_i x(t) + \beta_i y(t)\| \quad \forall t \geq 0. \quad (25)$$

Corollary 11. If the operators F_i, G_i , $i = 1, 2, \dots, s$, in system (24) satisfy condition (B) for any $x, y \in \Omega$, then $x(t)$ and $y(t)$ initiated from $x(t_0) = x_0 \in \Omega$, $y(t_0) = y_0 \in \Omega$ and the constant σ satisfy

$$\begin{aligned} (e^{\sigma t} \|\alpha_j x(t) + \beta_j y(t)\|)'_t & \leq \sum_{i=1}^s h_{ji}(t) (e^{\sigma t} \|\alpha_i x(t) + \beta_i y(t)\|) \\ & \quad \forall t \geq 0. \end{aligned} \quad (26)$$

When system (5) is

$$\begin{aligned} \dot{x}(t) &= F_1(t, x(t), y(t)) + F_2(t, x(t-\tau), y(t-\tau)), \\ \dot{y}(t) &= G_1(t, x(t), y(t)) + G_2(t, x(t-\tau), y(t-\tau)), \end{aligned} \quad (27)$$

where $x(t) \in R^m$, $y(t) \in R^n$, $F_i(t, 0, 0) = G_i(t, 0, 0) = 0$, $i = 1, 2$, and $\tau > 0$ is the time delay, we have the following conclusion.

Corollary 12. If the operators F_1, F_2, G_1, G_2 in system (27) satisfy conditions (B) and (C) for any $x, y \in \Omega$, then $x(t)$ and $y(t)$ initiated from $x(t_0) = x_0 \in \Omega$, $y(t_0) = y_0 \in \Omega$ satisfy

$$\begin{aligned} \|(\alpha_1 x(t) + \beta_1 y(t))\|'_t & \\ & \leq h_{11}(t) \|\alpha_1 x(t) + \beta_1 y(t)\| \\ & \quad + s_{11}(t) \|\alpha_1 x(t-\tau) + \beta_1 y(t-\tau)\| \\ & \quad \forall t \geq 0. \end{aligned} \quad (28)$$

When system (5) is written as

$$\begin{aligned} \dot{x}(t) &= F(t, x(t), y(t)), \\ \dot{y}(t) &= G(t, x(t), y(t)), \\ \dot{z}(t) &= H(t, x(t), y(t)), \end{aligned} \quad (29)$$

where $x(t) \in R^m$, $y(t) \in R^n$, $z(t) \in R^m$, and $F(t, 0, 0) = G(t, 0, 0) = H(t, 0, 0) = 0$, we have the following conclusion.

Corollary 13. If the operators F, G, H in system (29) satisfy the condition

$$\begin{aligned} & \left((\alpha F(t, x(t), y(t)) + \beta G(t, x(t), y(t)) \right. \\ & \quad \left. + \eta H(t, x(t), y(t)))^T(\alpha x(t) + \beta y(t) + \eta z(t)) \right) \\ & \quad \times \left(\|\alpha x(t) + \beta y(t) + \eta z(t)\|^2 \right)^{-1} \\ & \leq l(t) \end{aligned} \quad (30)$$

for any α, β, η and $x, y, z \in \Omega$, then $x(t), y(t)$, and $z(t)$ initiated from $x(t_0) = x_0 \in \Omega$, $y(t_0) = y_0 \in \Omega$, and $z(t_0) = z_0 \in \Omega$ satisfy

$$\begin{aligned} \|(\alpha x(t) + \beta y(t) + \eta z(t))\|'_t & \leq l(t) \|\alpha x(t) + \beta y(t) + \eta z(t)\| \\ & \quad \forall t \geq 0. \end{aligned} \quad (31)$$

3.2. *Impulsive Stability.* Consider the impulsive differential system

$$\begin{aligned} \dot{x}(t) &= F(t, x(t), y(t)), \quad (t \geq t_0, t \neq t_k, k = 1, 2, 3, \dots) \\ \dot{y}(t) &= G(t, x(t), y(t)), \quad (t \geq t_0, t \neq t_k, k = 1, 2, 3, \dots) \\ \Delta(x(t_k), y(t_k))^T & \\ &= (x(t_k + 0) - x(t_k), y(t_k + 0) - y(t_k))^T \\ &= I_k(x(t_k), y(t_k)), \quad (t = t_k > t_0), \end{aligned} \quad (32)$$

where $F, G : (t_0, +\infty) \times \Omega \times \Omega \rightarrow R^m, I_k : \Omega \times \Omega \rightarrow \Omega \times \Omega, k = 1, 2, 3, \dots, t_0 < t_1 < t_2 < \dots,$ and $\lim_{k \rightarrow +\infty} t_k = +\infty.$

Theorem 14. *If the operators F, G in system (32) and the constants $\alpha_i, \beta_i, i = 1, 2,$ satisfy condition (B) for any $x, y \in \Omega$ and the matrix $\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$ is invertible,*

$$\begin{pmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{pmatrix} \begin{pmatrix} \|\alpha_1 x(t) + \beta_1 y(t)\| \\ \|\alpha_2 x(t) + \beta_2 y(t)\| \end{pmatrix} \quad (33)$$

is continuous, quasi-monotonically increasing in $(t_{k-1}, t_k] \times R^m,$

$$\left\| \begin{pmatrix} \alpha_i & \beta_i \\ \alpha_i & \beta_i \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + I_k(x(t_k), y(t_k)) \right\| \quad (34)$$

are monotonically increasing, $i = 1, 2,$ and $I_k(0, 0) = (0, 0)^T.$ If the zero solution of system (3) is stable, $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, u_i(t) = \|\alpha_i x(t) + \beta_i y(t)\|,$ and $i = 1, 2, B_k u(t) = (\|\alpha_1 x(t) + \beta_1 y(t) + I_k(x(t_k), y(t_k))\|, \|\alpha_2 x(t) + \beta_2 y(t) + I_k(x(t_k), y(t_k))\|)^T - u(t),$ then the zero solution of system (32) is also stable.

Proof. From Corollary 10, we can obtain

$$\begin{aligned} &(\|\alpha_1 x(t) + \beta_1 y(t)\|)'_t \\ &\leq h_{11}(t) \|\alpha_1 x(t) + \beta_1 y(t)\| + h_{12}(t) \|\alpha_2 x(t) + \beta_2 y(t)\| \\ &\quad \forall t \geq 0, \\ &(\|\alpha_2 x(t) + \beta_2 y(t)\|)'_t \\ &\leq h_{21}(t) \|\alpha_1 x(t) + \beta_1 y(t)\| + h_{22}(t) \|\alpha_2 x(t) + \beta_2 y(t)\| \\ &\quad \forall t \geq 0. \end{aligned} \quad (35)$$

It is easy to verify that all the conditions in the lemma are satisfied when $V = (u_1(t), u_2(t))^T.$ So it is true that the zero solution of system (32) is stable. \square

Remark 15. Imitating Theorem 14 and combining the above 7 corollaries, respectively, we can obtain similar results with Theorem 14, respectively.

4. Example

Example 1. We consider the system [3, 13]

$$\begin{aligned} \frac{dx}{dt} &= e^{-t}x + y \sin t - (x^3 + xy^2) \sin^2 t, \quad t \neq t_k, \\ \frac{dy}{dt} &= x \sin t + e^{-t}y - (x^2y + y^3) \sin^2 t, \quad t \neq t_k, \\ \Delta x &= ax + by, \quad \Delta y = bx + ay, \quad t = t_k, \end{aligned} \quad (36)$$

where $2a = \sqrt{1+c_1} + \sqrt{1+c_2} - 2, 2b = \sqrt{1+c_1} - \sqrt{1+c_2}, -1 < c_1 \leq 0,$ and $-1 < c_2 \leq 0.$ Consider

$$\begin{aligned} &\alpha F(t, x, y) + \beta G(t, x, y) \\ &= \alpha e^{-t}x + \alpha y \sin t - \alpha(x^3 + xy^2) \sin^2 t + \beta x \sin t \\ &\quad + \beta e^{-t}y - \beta(x^2y + y^3) \sin^2 t \\ &= e^{-t}(\alpha x + \beta y) + (\alpha y + \beta x) \sin t \\ &\quad - (\alpha x + \beta y)(x^2 + y^2) \sin^2 t, \\ &\langle \alpha F(t, x, y) + \beta G(t, x, y), \alpha x + \beta y \rangle \\ &= e^{-t}(\alpha x + \beta y)^2 + (\alpha y + \beta x)(\alpha x + \beta y) \sin t \\ &\quad - (\alpha x + \beta y)^2(x^2 + y^2) \sin^2 t, \\ &\frac{(\alpha F(t, x, y) + \beta G(t, x, y))^T(\alpha x + \beta y)}{\|\alpha x + \beta y\|} \end{aligned} \quad (37)$$

$$\begin{aligned} &\leq \frac{e^{-t}(\alpha x + \beta y)^2}{\|\alpha x + \beta y\|^2} \|\alpha x + \beta y\| \\ &\quad + \frac{(\alpha y + \beta x)(\alpha x + \beta y) \sin t}{\|\alpha x + \beta y\| \|\alpha y + \beta x\|} \|\alpha y + \beta x\| \\ &\leq e^{-t} \|\alpha x + \beta y\| + \|\alpha y + \beta x\| |\sin t|. \end{aligned}$$

So $(\|\alpha x + \beta y\|)'_t \leq e^{-t} \|\alpha x + \beta y\| + \|\alpha y + \beta x\| |\sin t|, t \neq t_k.$

Similarly, $(\|\alpha y + \beta x\|)'_t \leq e^{-t} \|\alpha y + \beta x\| + \|\alpha x + \beta y\| |\sin t|, t \neq t_k.$

When $\alpha x + \beta y = \varphi(\alpha y + \beta x)$ for all $x, y,$ we can obtain $\varphi = \pm 1.$ That is, $\alpha = \beta \neq 0$ and $\alpha = -\beta \neq 0,$ respectively. Then we get the comparison system

$$\begin{aligned} \frac{du_1}{dt} &= e^{-t}u_1 + |\sin t|u_2, \quad t \neq t_k, \\ \frac{du_2}{dt} &= |\sin t|u_1 + e^{-t}u_2, \quad t \neq t_k, \\ u_1(t_k^+) &= |(x + ax + by) + (y + bx + ay)| \\ &= \sqrt{1+c_1}u_1, \quad t = t_k, \\ u_2(t_k^+) &= |(y + bx + ay) - (x + ax + by)| \\ &= \sqrt{1+c_2}u_2, \quad t = t_k. \end{aligned} \quad (38)$$

When $\alpha x + \beta y \neq \varphi(\alpha y + \beta x)$ for all x, y , we can obtain the comparison system

$$\begin{aligned} \frac{du_1}{dt} &= e^{-t} u_1 + |\sin t| u_2, & t \neq t_k, \\ \frac{du_2}{dt} &= |\sin t| u_1 + e^{-t} u_2, & t \neq t_k, \\ u_1(t_k^+) &= |1+a| u_1 + |b| u_2, & t = t_k, \\ u_2(t_k^+) &= |b| u_1 + |1+a| u_2, & t = t_k. \end{aligned} \quad (39)$$

We know that when the constants α, β satisfy the condition in Theorem 14, u_1, u_2 in the systems (38) and (39) are stable by stability theory of impulsive differential systems. So it is true that the zero solution of system (36) is stable. Taking $c_1 = -0.2, c_2 = -0.5$, and initial conditions $(x, y)^T = (1, 2)^T$, the stability simulations of the states $x(t), y(t)$ of system (36) are as shown in Figures 1 and 2, respectively.

Example 2. We consider the system [3]

$$\begin{aligned} \frac{dx}{dt} &= -2x(y^2 - (2 + \sin t)x)^2 - \frac{x \cos t}{2 + \sin t}, & t \neq t_k, \\ \frac{dy}{dt} &= -y(y^2 - (2 + \sin t)x)^2, & t \neq t_k, \\ \Delta x &= \left(\frac{-1}{2}\right)x, & \Delta y = \left(\frac{1}{\sqrt{2}} - 1\right), & t = t_k. \end{aligned} \quad (40)$$

Combining Corollary 11 and imitating Example 1, we can obtain the comparison system

$$\begin{aligned} \frac{du_1}{dt} &= \left(\sigma + \left|\frac{\cos t}{2 + \sin t}\right|\right) u_1, & t \neq t_k, \\ \frac{du_2}{dt} &= \sigma u_2, & t \neq t_k, \\ u_1(t_k^+) &= \left(\frac{1}{2}\right) u_1, & u_2(t_k^+) = \left(\frac{1}{\sqrt{2}}\right) u_2, & t = t_k, \end{aligned} \quad (41)$$

where $u_1 = e^{\sigma t} |\alpha x|, u_2 = e^{\sigma t} |\beta y|$, and $0 \leq \sigma < 1$. When the constants α, β satisfy the condition in Theorem 14, it is true that the zero solution of system (40) is stable. Taking initial conditions $(x, y)^T = (1, 2)^T$, the stability simulations of the states $x(t), y(t)$ of system (40) are as shown in Figures 3 and 4, respectively.

Remark 16. Comparing Examples 1 and 2 in this paper with Examples 2 and 4 in the literature [3, 13] or [3], the vector Lyapunov functions $V(t, x, y) = \begin{pmatrix} |\alpha x + \beta y| \\ |\beta x + \alpha y| \end{pmatrix}, V(t, x, y) = \begin{pmatrix} e^{\sigma t} |\alpha x| \\ e^{\sigma t} |\beta y| \end{pmatrix}$ in this paper are simpler than the vector Lyapunov functions $V(t, x, y) = \begin{pmatrix} (x+y)^2 \\ (x-y)^2 \end{pmatrix}, V(t, x, y) = (y^2 - (2 + \sin t)x)^2$ in [3, 13] or [3], respectively, and the approach is easy to operate.

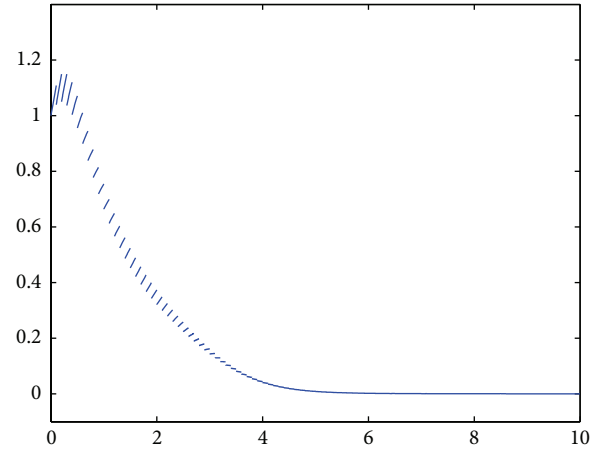


FIGURE 1: Stability of $x(t)$ of system (36).

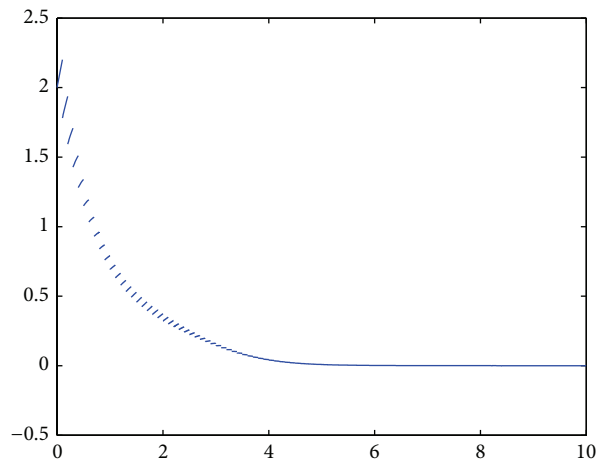


FIGURE 2: Stability of $y(t)$ of system (36).

Example 3. We consider the system

$$\begin{aligned} \frac{dx}{dt} &= \sin(\log(t+1)) y + e^{-t} x \\ &+ y(t-\tau) \sin t - (x^3 + xy^2) \sin^2 t, & t \neq t_k, \\ \frac{dy}{dt} &= \sin(\log(t+1)) x + e^{-t} y + x(t-\tau) \sin t \\ &- (x^2 y + y^3) \sin^2 t, & t \neq t_k, \\ \Delta x &= ax + by, & \Delta y = bx + ay, & t = t_k, \end{aligned} \quad (42)$$

where $2a = \sqrt{1+c_1} + \sqrt{1+c_2} - 2, 2b = \sqrt{1+c_1} - \sqrt{1+c_2}, -1 < c_1 \leq 0$, and $-1 < c_2 \leq 0$.

Combining Corollary 12 and imitating Example 1, we can obtain the comparison system

$$\begin{aligned} \frac{du_1}{dt} &= |\sin(\log(t+1))| u_2 + e^{-t} u_1 + |\sin t| u_2(t-\tau), \\ & & t \neq t_k, \end{aligned}$$

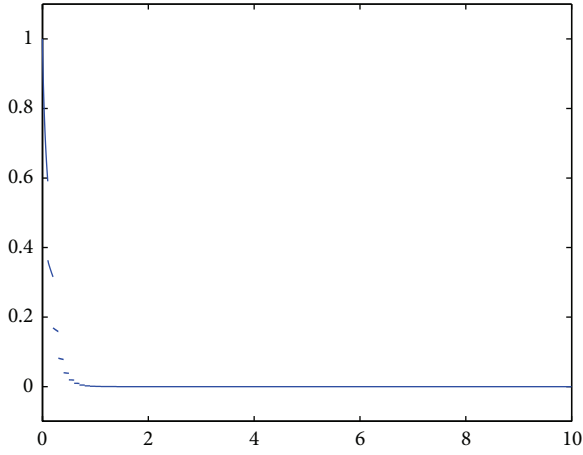


FIGURE 3: Stability of $x(t)$ of system (40).

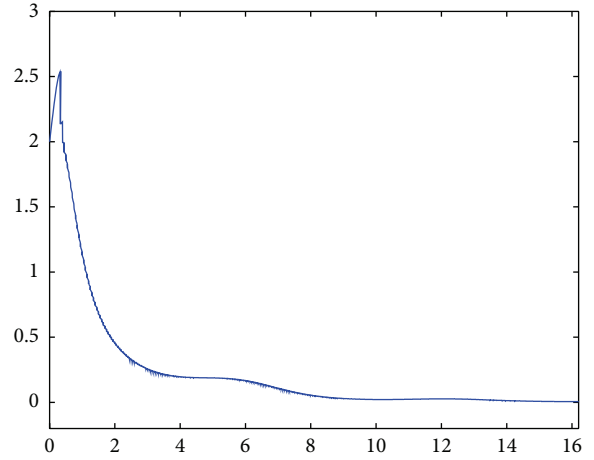


FIGURE 6: Stability of $y(t)$ of system (42).

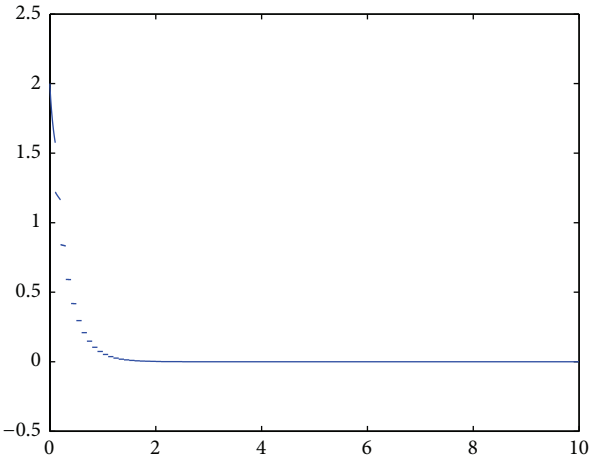


FIGURE 4: Stability of $y(t)$ of system (40).

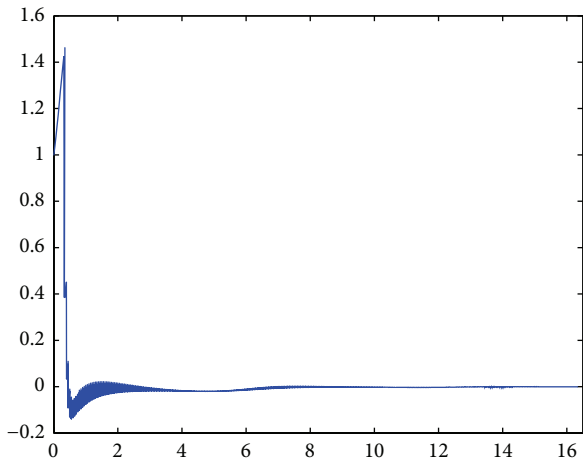


FIGURE 5: Stability of $x(t)$ of system (42).

$$\frac{du_2}{dt} = |\sin(\log(t+1))| u_1 + e^{-t} u_2 + |\sin t| u_1(t-\tau),$$

$$t \neq t_k,$$

$$u_1(t_k^+) = |1+a| u_1 + |b| u_2, \quad u_2(t_k^+) = |b| u_1 + |1+a| u_2,$$

$$t = t_k,$$

$$(43)$$

where $u_1 = |\alpha x + \beta y|$, $u_2 = |\alpha y + \beta x|$. When the constants α, β satisfy the condition in Theorem 14, it is true that the zero solution of system (42) is stable. Taking $c_1 = -0.8$, $c_2 = -0.9$, $\tau = 1$, and initial conditions $(x, y)^T = (1, 2)^T$, the stability simulations of the states $x(t), y(t)$ of system (42) are as shown in Figures 5 and 6, respectively.

Example 4. We consider the system

$$\frac{dx}{dt} = -2x(y^2 - (2 + \sin t)z)^2 - \frac{x \cos t}{2 + \sin t}, \quad t \neq t_k,$$

$$\frac{dy}{dt} = e^{-t} y + z \sin t - (y^3 + yz^2) \sin^2 t, \quad t \neq t_k,$$

$$\frac{dz}{dt} = y \sin t + e^{-t} z - (y^2 z + z^3) \sin^2 t, \quad t \neq t_k,$$

$$\Delta x = \left(\frac{-1}{2}\right) x, \quad \Delta y = ay + bz, \quad \Delta z = by + az,$$

$$t = t_k,$$

$$(44)$$

where $2a = \sqrt{1+c_1} + \sqrt{1+c_2} - 3$, $2b = \sqrt{1+c_1} - \sqrt{1+c_2}$, $-1 < c_1 \leq 0$, and $-1 < c_2 \leq 0$.

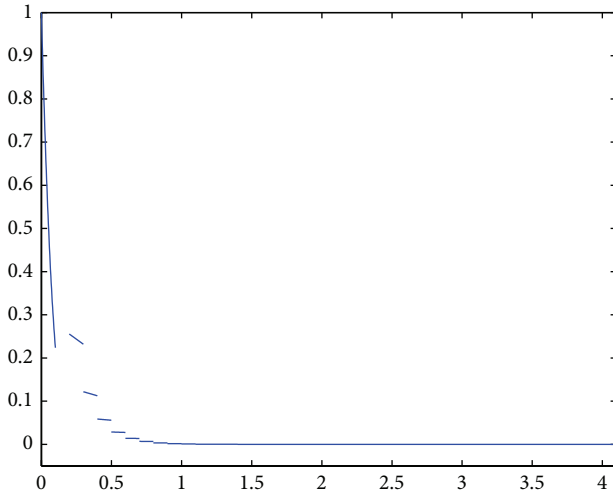


FIGURE 7: Stability of $x(t)$ of system (44).

Combining Corollary 6 or Corollary 13 and imitating Example 1, we can obtain the comparison system

$$\begin{aligned} \frac{du_1}{dt} &= \left| \frac{\cos t}{2 + \sin t} \right| u_1, & t \neq t_k, \\ \frac{du_2}{dt} &= |e^{-t} + \sin t| u_2, & t \neq t_k, \\ u_1(t_k^+) &= \left(\frac{1}{2}\right) |\alpha| u_1, & u_2(t_k^+) = \left\| \begin{pmatrix} 1+a & b \\ b & 1+a \end{pmatrix} \right\| u_2, \\ & & t = t_k, \end{aligned} \tag{45}$$

where $u_1 = \|\alpha \binom{x}{0}\|$, $u_2 = \|\beta \binom{y}{z}\|$, or the comparison system

$$\begin{aligned} \frac{du_1}{dt} &= \left| \frac{\cos t}{2 + \sin t} \right| u_1, & t \neq t_k, \\ \frac{du_2}{dt} &= e^{-t} u_2 + |\sin t| u_3, & t \neq t_k, \\ \frac{du_3}{dt} &= |\sin t| u_2 + e^{-t} u_3, & t \neq t_k, \\ u_1(t_k^+) &= \left(\frac{1}{2}\right) u_1, \\ u_2(t_k^+) &= |1+a| u_2 + |b| u_3, \\ u_3(t_k^+) &= |b| u_2 + |1+a| u_3, \\ & & t = t_k, \end{aligned} \tag{46}$$

where $u_1 = |\gamma x|$, $u_2 = |\alpha y + \beta z|$, and $u_3 = |\alpha z + \beta y|$. When the constants α, β , and γ satisfy the conditions in Theorem 14 or Remark 15, it is true that the zero solution of system (44) is stable. Taking $c_1 = -0.2$, $c_2 = -0.5$, and initial conditions $(x, y, z)^T = (1, 2, 0.8)^T$, the stability simulations of the states $x(t)$, $y(t)$, and $z(t)$ of system (44) are as shown in Figures 7, 8, and 9, respectively.

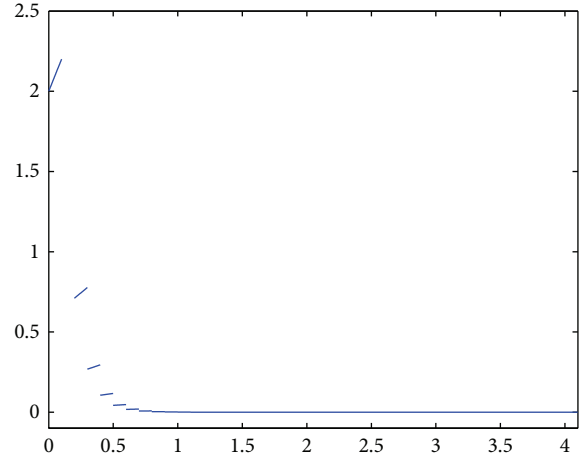


FIGURE 8: Stability of $y(t)$ of system (44).

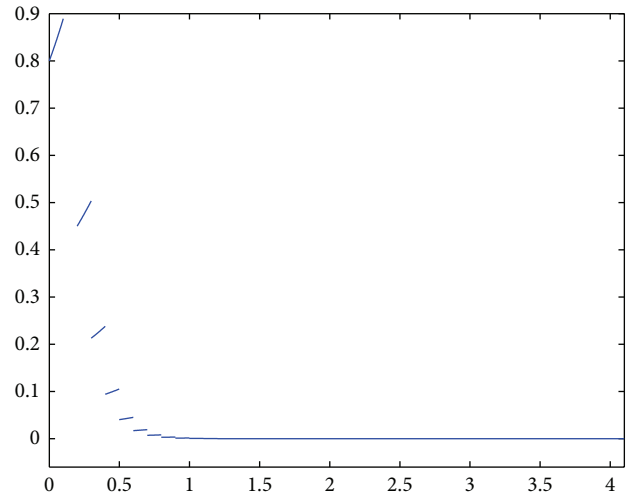


FIGURE 9: Stability of $z(t)$ of system (44).

5. Conclusion

This letter, which uses a class of vector Lyapunov functions, has studied the issue on impulsive stability of nonlinear systems and improves the existing results. Numerical simulations have verified the effectiveness of the method.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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