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Research Article

Some Remarks on End-Nim

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We reexamine Albert and Nowakowski's variation on the game of Nim, called *End-Nim*, in which the players may only remove coins from the leftmost or rightmost piles. We reformulate Albert and Nowakowski's solution to this game. We examine its misère version and a further variant where the winner is the player who reduces the game to a single pile; we call this *Loop-End-Nim*. We show that the three games, End-Nim, misère-End-Nim, and Loop-End-Nim, all have the same losing positions, except for the positions where all the piles are of equal size. We also give some partial results concerning the higher Sprague-Grundy values of the three games.

1. Introduction

Consider k piles of coins, in a row. In the classic game of *Nim*, the two players move alternately, each removing a nonzero number of coins from a single pile; the winner is the player to remove the last coin [1]. The well-known solution to Nim, using Sprague-Grundy values, is both elegant and complete [1].

In [2], Albert and Nowakowski analysed a variation of Nim; they called *End-Nim* in which the players may only remove coins from either of the end piles. This game had been posed as problem 23 in [3], where it was called *Burning the Candle at Both Ends*. Albert and Nowakowski gave a solution to this game, which we recall below, but the Sprague-Grundy values seem particularly complicated and have not yet been determined. In this paper, we examine the misère version of End-Nim, and a further variant where the winner is the player who reduces the game to a single pile; we call this *Loop-End-Nim*, where "Loop" stands for "leave only one pile". While Loop-End-Nim is not strictly speaking a misère game, it has something of the nature of a misère game. We show that the three games, End-Nim, misère-End-Nim and Loop-End-Nim, all have the same losing positions (i.e., $\mathcal{P}$-positions), except for the positions where all the piles are of equal size. Thus, like the misère form of Nim, the games misère-End-Nim and Loop-End-Nim can be played with the same strategy as End-Nim except that consideration has to be given to the exceptional positions.

The $\mathcal{P}$-positions are the positions with Sprague-Grundy value 0. We also give some partial results concerning the positions of higher Sprague-Grundy values of the three games. It should be mentioned here that the Sprague-Grundy function plays no role in the strategy of playing End-Nim, and it is well known that, in their usual form, they are inappropriate for studying misère games [4]. Our interest here is simply to provide further indications as to the complexity of the Sprague-Grundy function for End-Nim.

The positions in these games will be denoted by the corresponding sequences of coin sizes: $(a_1, \ldots, a_k)$. The assumption is that the pile sizes a_i are all nonempty. By reversing $A = (a_1, \ldots, a_k)$ if necessary, we may assume that $a_1 \leq a_k$.

2. The Losing Positions

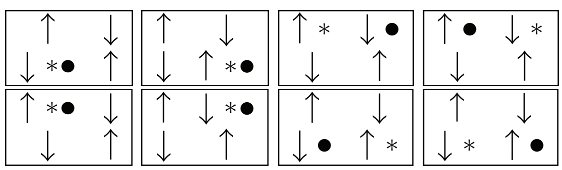
In Albert and Nowakowski's entertaining paper [2], their solution to End-Nim is given in pictorial form, involving a matrix of arrows with asterisks and bullets. We will present it in an alternate form. First, we introduce some notation, employing an idea similar to the one used in [5]. For a position $A = (a_1, a_2, \ldots, a_k)$, we define $l(A)$ to be the largest natural number i for which $a_1 = \cdots = a_{i-1} \leq a_i$. Similarly, for a position $A = (a_1, a_2, \ldots, a_k)$ for which the a_i are not all equal, we define $r(A)$ to be the largest natural number j for which $a_{k-j+1} \geq a_{k-j+2} = \cdots = a_k$. If $a_1 = \cdots = a_k$, we set $r(A) = 0$. Then Albert and Nowakowski's solution can be rephrased as follows.

Theorem 2.1. *In End-Nim, a position $A = (a_1, a_2, \ldots, a_k)$ with $a_1 \leq a_k$ is a $\mathcal{P}$-position if and only if one of the following conditions holds:*

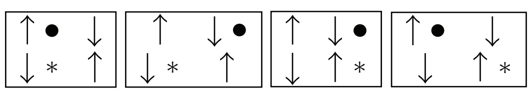
(a) $a_1 = a_k$, *and furthermore,* $l(A) + r(A)$ *is even,*

(b) $a_k = a_1 + 1$ *and, furthermore,* $l(A)$ *is odd and* $r(A)$ *is even.*

Condition (a) corresponds to the following 8 hieroglyphs of Albert and Nowakowski.



Condition (b) corresponds to the following 4 pictures.



In these pictures, if $a_1 = \cdots = a_{i-1} \neq a_i$, then the $*$ symbol lies in the first (resp. second) column if i is even (resp. odd), and it is adjacent to $\uparrow$ (resp. $\downarrow$) if $a_{i-1} < a_i$ (resp. $a_{i-1} > a_i$). The conventions for $\bullet$ are defined analogously, for the right hand end. Clearly, our formulation of the result is more succinct, while Albert and Nowakowski's presentation is more graphic.

To give the solution to misère-End-Nim, we modify slightly the definition of r. We set $r_m(A) = r(A)$ except when $a_1 = \cdots = a_k = 1$, in which case we set $r_m(A) = 1$. Then we have the following.

Theorem 2.2. *In misère-End-Nim, a position $A = (a_1, a_2, \ldots, a_k)$ with $a_1 \leq a_k$ is a $\mathcal{P}$-position if and only if one of the following holds:*

(a) $a_1 = a_k$ *and, furthermore,* $l(A) + r_m(A)$ *is even,*

(b) $a_k = a_1 + 1$ *and, furthermore,* $l(A)$ *is odd and* $r_m(A)$ *is even.*

We postpone the proof until the next section. Let us now describe the solution for Loop-End-Nim. Once again, we modify the r function. We set $r_o(A) = r(A)$ except when $a_1 = \cdots = a_k$, in which case we set $r_o(A) = 1$. Then we have the following.

Theorem 2.3. *In Loop-End-Nim, a position $A = (a_1, a_2, \ldots, a_k)$ with $a_1 \leq a_k$ is a $\mathcal{P}$-position if and only if one of the following holds:*

(a) $a_1 = a_k$ *and, furthermore,* $l(A) + r_o(A)$ *is even,*

(b) $a_k = a_1 + 1$ *and, furthermore,* $l(A)$ *is odd and* $r_o(A)$ *is even.*

As an immediate consequence of the above three results, we have the following.

Corollary 2.4. *The three games, End-Nim, misère-End-Nim, and Loop-End-Nim have the same $\mathcal{P}$-positions of the form $A = (a_1, a_2, \ldots, a_k)$, where the a_i are not all equal. If $A = (a_1, a_2, \ldots, a_k)$ where $a_1 = \cdots = a_k$, then*

(a) *A is a $\mathcal{P}$-position of End-Nim if and only if k is even,*

(b) *A is a $\mathcal{P}$-position of misère-End-Nim if and only if k is even and $a_1 > 1$, or k is odd and $a_1 = 1$,*

(c) *A is a $\mathcal{P}$-position of Loop-End-Nim if and only if k is odd.*

3. Proof of Theorems 2.2 and 2.3

Proof of Theorem 2.2. We say that a position verifying condition (a) or (b) of Theorem 2.2 is a *0-position*. We must prove two facts:

(1) from every position A that is not a 0-position, there is a move to a 0-position,

(2) there is no move from a 0-position to a 0-position.

(1) Let $A = (a_1, a_2, \ldots, a_k)$ be a position that is not a 0-position. First suppose that A is a $\mathcal{P}$-position in End-Nim. By Theorem 2.1, $a_1 = \cdots = a_k = 1$ and k is even. Then we have the move $A \to (a_1, a_2, \ldots, a_{k-1})$, which is a 0-position. Now suppose that A is not a $\mathcal{P}$-position in End-Nim. Then there is a move $A \to B$, where $B = (b_1, \ldots, b_n)$ is a $\mathcal{P}$-position in End-Nim. If B is not a 0-position, then by Theorem 2.1, $b_1 = \cdots = b_n = 1$ and n is even. Hence either $k = n$ and $a_1 = \cdots = a_{k-1} < a_k$, or $k = n + 1$ and $a_1 = \cdots = a_{k-1} \leq a_k$. In the former case, we have the move $A \to (a_1, a_2, \ldots, a_{k-1})$, which is a 0-position. In the latter case, since A is not a 0-position, $a_{k-1} < a_k$ and we have the move $A \to (a_1, a_2, \ldots, a_{k-1}, 1)$, which is a 0-position.

(2) Let $A \to B$ be a move between two 0-positions, where $A = (a_1, a_2, \ldots, a_k)$ with $a_1 \leq a_k$ and $B = (b_1, \ldots, b_n)$. First suppose that A is not a $\mathcal{P}$-position in End-Nim. So $a_1 = \cdots = a_k = 1$ and k is odd. But this is a contraction, since necessarily $B = (a_1, a_2, \ldots, a_{k-1})$, which is not a 0-position. So we may suppose that A is a $\mathcal{P}$-position in End-Nim. Therefore, B is not a $\mathcal{P}$-position in End-Nim. Hence $b_1 = \cdots = b_n = 1$ and n is odd. Thus, either $k = n$ and $a_1 = \cdots = a_{k-1} < a_k$, or $k = n+1$ and $a_1 = \cdots = a_{k-1} \leq a_k$. In the former case, $r_m(A) = 1$, which

is impossible as $a_1 < a_k$ and A is a 0-position. In the latter case, $l(A)$ is even and so, as A is a 0-position, $r_m(A)$ is also even. Hence $a_k = 1$. But then $a_1 = \cdots = a_k = 1$, which is impossible as k is even and A is a 0-position. This completes the proof. □

The following argument is modelled closely on the proof we have just given.

Proof of Theorem 2.3. We say that a position verifying condition (a) or (b) of Theorem 2.3 is a 0-position. We must prove two facts:

(1) from every position A that is not a 0-position, there is a move to a 0-position,

(2) there is no move from a 0-position to a 0-position.

(1) Let $A = (a_1, a_2, \ldots, a_k)$ be a position that is not a 0-position. First suppose that A is a $\mathcal{P}$-position in End-Nim. By Theorem 2.1, $a_1 = \cdots = a_k$ and k is even. Then we have the move $A \to (a_1, a_2, \ldots, a_{k-1})$, which is a 0-position. Now suppose that A is not a $\mathcal{P}$-position in End-Nim. Then there is a move $A \to B$, where $B = (b_1, \ldots, b_n)$ is a $\mathcal{P}$-position in End-Nim. If B is not a 0-position, then by Theorem 2.1, $b_1 = \cdots = b_n$ and n is even. Hence either $k = n$ and $a_1 = \cdots = a_{k-1} < a_k$, or $k = n+1$. In the former case, we have the move $A \to (a_1, a_2, \ldots, a_{k-1})$, which is a 0-position. In the latter case, k is odd and at first sight there are two possibilities:

(a) $a_1 < a_2 = a_3 = \cdots = a_k$. Here we have the move $A \to C = (a_1, a_2, \ldots, a_{k-1}, a_1)$, which is a 0-position since $l(C) + r_o(C) = 4$.

(b) $a_1 = a_2 = \cdots = a_{k-1} \le a_k$. Note that as A is not a 0-position, $a_{k-1} < a_k$. Here we have the move $A \to (a_1, a_2, \ldots, a_{k-1}, a_{k-1})$, which is a 0-position.

(2) Let $A \to B$ be a move between two 0-positions, where $A = (a_1, a_2, \ldots, a_k)$ with $a_1 \le a_k$ and $B = (b_1, \ldots, b_n)$. First suppose that A is not a $\mathcal{P}$-position in End-Nim. So $a_1 = \cdots = a_k$ and k is odd. Then either $n = k$, $b_1 < a_1$ and $b_2 = b_3 = \cdots = b_k = a_k$, in which case $l(B) = 2$, or $n = k-1$ and $b_1 = b_2 = \cdots = b_{k-1}$. But in both cases, B is not a 0-position, which gives a contradiction. So we may suppose that A is a $\mathcal{P}$-position in End-Nim. Therefore, B is not a $\mathcal{P}$-position in End-Nim. Hence, $b_1 = \cdots = b_n$ and n is odd. Thus, either $k = n$ and $a_1 = \ldots = a_{k-1} < a_k$, or $k = n+1$. In the former case, $r_o(A) = 1$, which is impossible as $a_1 < a_k$ and A is a 0-position. In the latter case, k is even and at first sight there are two possibilities:

(a) $a_1 < a_2 = a_3 = \cdots = a_k$. Here, $r_o(A) = k - 1 = n$, which is odd, contrary to the assumption that A is a 0-position.

(b) $a_1 = a_2 = \cdots = a_{k-1} \le a_k$. Since A is a 0-position and k is even, we have $a_{k-1} < a_k$. But then $r_o(A) = 1$, again contracting the assumption that A is a 0-position. □

4. Sprague-Grundy Values for Games with Two Piles

Let us denote the Sprague-Grundy function for the games End-Nim, misère-End-Nim, and Loop-End-Nim by $\mathcal{G}, \mathcal{G}_m, \mathcal{G}_o$, respectively. First observe that End-Nim with two piles is the same as Nim with two piles. So we have $\mathcal{G}(a,b) = a \oplus b$, where $\oplus$ denotes Nim addition. The situation regarding Loop-End-Nim is also very simple.

Proposition 4.1. *In Loop-End-Nim, $\mathcal{G}_o(a,b) = ((a-1) \oplus (b-1)) + 1$, for all $a, b > 0$.*

Proof. We will prove that $\mathcal{G}_o(a,b) = \mathcal{G}(a-1,b-1)+1$ by induction on $a+b$. Clearly $\mathcal{G}_o(1,1) = 1 = \mathcal{G}(0,0)+1$. Suppose $a,b > 0$. Then by induction

$$\begin{aligned}
\mathcal{G}_o(a,b) &= \operatorname{Mex}\left(\left\{\mathcal{G}_o(a',b) : 0 \le a' < a\right\} \cup \left\{\mathcal{G}_o(a,b') : 0 \le b' < b\right\}\right)\\
&= \operatorname{Mex}\left(\{0\} \cup \left\{\mathcal{G}_o(a',b) : 1 \le a' < a\right\} \cup \left\{\mathcal{G}_o(a,b') : 1 \le b' < b\right\}\right)\\
&= \operatorname{Mex}\left(\{0\} \cup \left\{\mathcal{G}(a'-1,b-1)+1 \ : \ 1 \le a' < a\right\} \cup \left\{\mathcal{G}(a-1,b'-1)+1 \ : \ 1 \le b' < b\right\}\right)\\
&= \operatorname{Mex}\left(\left\{\mathcal{G}(a'-1,b-1) : 1 \le a' < a\right\} \cup \left\{\mathcal{G}(a-1,b'-1) : 1 \le b' < b\right\}\right)+1\\
&= \mathcal{G}(a-1,b-1)+1.
\end{aligned} \tag{4.1}$$

□

The situation concerning misère-End-Nim seems to be considerably more complicated. Indeed, as far as we are aware, even for two piles, where the game is just misère Nim, the Sprague-Grundy function has not yet been determined! We have only been able to obtain very partial information. From Theorem 2.2, a position $A = (a,b)$ has Sprague-Grundy value 0 if and only if $a = b \neq 1$. We also have the following.

Proposition 4.2. *In the misère-End-Nim, a position $A = (a,b)$ with $a \le b$ has Sprague-Grundy value 1 if and only if either $a = b = 1$ or $a \ge 3$ is odd and $b = a+1$.*

We omit the proof of the above proposition; it is simple and straightforward.

5. Sprague-Grundy Values for Games with Three Piles

As we saw in Section 2, in End-Nim with three piles, a position $A = (a_1,a_2,a_3)$ is a $\mathcal{P}$-position if and only if it is symmetrical but not constant; that is, $a_3 = a_1$ and $a_2 \neq a_1$. The $\mathcal{P}$-positions of misère-End-Nim comprise those of End-Nim, as well as $(1,1,1)$. The $\mathcal{P}$-positions of Loop-End-Nim comprise those of End-Nim, as well as the constant positions (a,a,a), with $a \ge 1$. For the positions of Sprague-Grundy value 1, we have the following three results.

Theorem 5.1. *In End-Nim, a position $A = (a_1,a_2,a_3)$ with $a_1 \le a_3$ has Sprague-Grundy value 1 if and only if one of the following three conditions holds*

(a) $A = (1,1,1)$,

(b) $a_3 = a_1 + 1$ *and either*

  (i) a_1 *is even and* $a_2 < a_1$ *or*

  (ii) a_1 *is odd and* $a_2 > a_1$ *and* $a_2 \neq a_1 + 2$,

(c) $a_3 = a_1 + 2$ *and* a_1 *is odd and* $a_2 = a_3$.

Theorem 5.2. *In misère-End-Nim, a position $A = (a_1,a_2,a_3)$ with $a_1 \le a_3$ has Sprague-Grundy value 1 if and only if one of the following three conditions holds*

(a) $A = (1,2,2)$

(b) $a_1 = a_2 = a_3 \ge 3$,

(c) $a_3 = a_1 + 1$ *and either*

(i) a_1 *is even and* $a_2 = 1$ *or*
(ii) a_1 *is odd and either* $2 \leq a_2 < a_1$ *or* $a_2 > a_3$.

Theorem 5.3. *In Loop-End-Nim, a position* $A = (a_1, a_2, a_3)$ *with* $a_1 \leq a_3$ *has Sprague-Grundy value 1 if and only if one of the following two conditions holds*

(a) $a_3 = a_1 + 1$ *and either*

(i) a_1 *is even and* $a_2 < a_1$ *or*
(ii) a_1 *is odd and* $a_2 > a_3$.

(b) $a_3 = a_1 + 2$ *and* a_1 *is odd,* $a_2 = a_1 + 1$.

We provide a proof of Theorem 5.1 in the next section. The proof is simple and straightforward, but rather long. We omit the proofs of Theorems 5.2 and 5.3 which can be established in the same manner. We also provide the following result without proof. It shows that it is unlikely that there is a simple formula for the Sprague-Grundy function for End-Nim.

Theorem 5.4. *In End-Nim, a position* $A = (a_1, a_2, a_3)$ *with* $a_1 \leq a_3$ *has Sprague-Grundy value 2 if and only if one of the following conditions holds, where the congruences are all modulo 4:*

(a) $a_1 = a_2 = a_3 \geq 2,\ a_1 \equiv 1, 2$,

(b) $a_3 = a_1 + 1$ *and either*

(i) $A = (1, 1, 2)$
(ii) $a_1 \equiv 2,\ a_2 = a_3$ *or*
(iii) a_1 *is odd,* $a_1 \geq 7,\ 5 \leq a_2 \leq a_1 - 1,\ a_2 \equiv 1, 2$, *except for* $a_1 \equiv 3$ *where* $a_2 \neq a_1 - 2$.

(c) $a_3 = a_1 + 2$ *and either*

(i) $a_1 \equiv 0,\ a_2 \leq a_1 - 1$ *and if* $a_2 \geq 5$ *then* $a_2 \equiv 0, 3$, *or*
(ii) $a_1 \equiv 1$ *and either*
 (I) $A = (1, 2, 3)$,
 (II) $a_2 \geq a_1 + 3$, *or*
 (III) $a_2 \leq a_1 - 1$ *and if* $a_2 \geq 5$ *then* $a_2 \equiv 0, 3$,
(iii) $a_1 \equiv 2$ *and either* $a_2 = a_1 - 1 \geq 5$ *or* $a_2 \geq a_1 + 2, a_2 \neq a_1 + 4$.

(d) $a_3 = a_1 + 3$, $a_1 \equiv 2$ *and* $a_2 = a_1 + 4$.

6. Proof of Theorem 5.1

We say that a position is a *1-position* if it has the form (a_1, a_2) with Sprague-Grundy value 1, or the form (a_1, a_2, a_3) verifying condition (a), (b), or (c) of Theorem 5.1; in the latter case we say A is of type (a), (b), or (c), respectively. We must prove two properties:

(1) there is no move from a 1-position to a 1-position,

(2) from every position that is not a 0-position or a 1-position, there is a move to a 1-position.

To establish the first property, we suppose that $A = (a_1, a_2, a_3)$ is a 1-position. If (x, y) has Sprague-Grundy value 1, and $x < y$, then x is even and $y = x + 1$. It follows that as $A = (a_1, a_2, a_3)$ is a 1-position, neither (a_1, a_2) nor (a_2, a_3) has Sprague-Grundy value 1. Indeed, if A is a 1-position, then $|a_3 - a_2| \neq 1$, and if $|a_1 - a_2| = 1$, then A is necessarily of type (b). But in this case, if a_1 is even, $a_2 < a_1$, while if a_1 is odd, $a_2 > a_1$, and both cases are impossible if (a_1, a_2) has Sprague-Grundy value 1. Hence it suffices to consider moves $A \to B = (b_1, b_2, b_3)$. First suppose that A is of type (c), that is, A has the form $(a_1, a_1 + 2, a_1 + 2)$, where a_1 is odd. There is obviously no move from A to $(1, 1, 1)$. So, since the 1-positions (b_1, b_2, b_3) have $|b_3 - b_1| \leq 2$, we need only consider the following moves:

$$\begin{aligned}
(a_1, a_1 + 2, a_1 + 2) &\longrightarrow B_1 = (a_1, a_1 + 2, a_1 + 1), \\
(a_1, a_1 + 2, a_1 + 2) &\longrightarrow B_2 = (a_1, a_1 + 2, a_1 - 1), \\
(a_1, a_1 + 2, a_1 + 2) &\longrightarrow B_3 = (a_1, a_1 + 2, a_1 - 2).
\end{aligned} \tag{6.1}$$

Firstly, B_1 is not a 1-position, since here $b_1 = a_1$ is odd and $b_2 = b_1 + 2$.

Secondly, B_2 is not a 1-position. Indeed, $b_3 = a_1 - 1$ is even and $b_2 > b_3$.

Thirdly, B_3 is not a 1-position, as here $b_1 = b_3 + 2$ but $b_2 \neq b_1$.

Now suppose that A is of type (b), that is A has the form $(a_1, a_2, a_1 + 1)$. We need only to consider the following moves:

$$\begin{aligned}
(a_1, a_2, a_1 + 1) &\longrightarrow B_4 = (a_1 - 1, a_2, a_1 + 1), \\
(a_1, a_2, a_1 + 1) &\longrightarrow B_5 = (a_1, a_2, a_1 - 2), \\
(a_1, a_2, a_1 + 1) &\longrightarrow B_6 = (a_1, a_2, a_1 - 1), \\
(a_1, a_2, a_1 + 1) &\longrightarrow B_7 = (a_1, a_2, a_1).
\end{aligned} \tag{6.2}$$

If B_4 is a 1-position, then it is of type (c) and so a_1 must be even and $a_2 = a_1 + 1$, contradicting the assumption that A is a 1-position. Similarly, if B_5 is a 1-position, then it is of type (c) and so a_1 must be odd and $a_2 = a_1$, again contradicting the assumption that A is a 1-position. If B_6 is a 1-position, then it is of type (b) and either a_1 is odd and $a_2 < a_1 - 1$ or a_1 is even and $a_2 > a_1 - 1$, and both cases contradict the assumption that A is a 1-position. If B_7 is a 1-position, then it is of type (a) and thus $a_1 = 1$, but then $A = (1, 1, 2)$, which is not a 1-position.

Finally, if A is of type (a), then $A = (1, 1, 1)$ and there is only one move, to $(1, 1)$, which is a 0-position. This completes the proof of property 1.

To prove property 2, consider a position $B = (b_1, b_2, b_3)$, with $b_1 \leq b_3$, that is not a 0-position or a 1-position. There are 7 cases to consider;

(a) $b_3 > b_1 + 2$,

(b) $b_3 = b_1 + 2$ and b_1 is even,

(c) $b_3 = b_1 + 2$ and b_1 is odd and $b_2 \neq b_3$,

(d) $b_3 = b_1 + 1$ and b_1 is even and $b_2 \geq b_1$,

(e) $b_3 = b_1 + 1$ and b_1 is odd and $b_2 \leq b_1$,

(f) $b_3 = b_1 + 1$ and b_1 is odd and $b_2 = b_1 + 2$,

(g) $b_3 = b_2 = b_1$ and $b_1 \neq 1$.

In each case, we must exhibit moves to 1-positions.

Case (a). We divide this further into subcases;

(i) If b_1 is even and $b_2 < b_1$, consider the move $B \to (b_1, b_2, b_1 + 1)$.

(ii) If b_1 is even and $b_2 = b_1$ or $b_2 > b_1 + 1$, consider the move $B \to (b_1, b_2, b_1 - 1)$.

(iii) If b_1 is even and $b_2 = b_1 + 1$, consider the move $B \to (b_1, b_2)$.

(iv) If b_1 is odd and $b_2 = b_1 + 1$ or $b_2 > b_1 + 2$, consider the move $B \to (b_1, b_2, b_1 + 1)$.

(v) If b_1 is odd and $b_2 = b_1 + 2$, consider the move $B \to (b_1, b_2, b_2)$.

(vi) If b_1 is odd and $b_1 > 1$ and $b_2 < b_1 - 1$, consider the move $B \to (b_1, b_2, b_1 - 1)$.

(vii) If b_1 is odd and $b_1 > 1$ and $b_2 = b_1$, consider the move $B \to (b_1, b_1, b_1 - 2)$.

(viii) If b_1 is odd and $b_1 > 1$ and $b_2 = b_1 - 1$, consider the move $B \to (b_1, b_1 - 1)$.

(ix) If $b_1 = b_2 = 1$, consider the move $B \to (1, 1, 1)$.

Case (b). We have subcases;

(i) If $b_2 < b_1$, consider the move $B \to (b_1, b_2, b_1 + 1)$.

(ii) If $b_2 = b_1$ or $b_2 > b_1 + 1$, consider the move $B \to (b_1, b_2, b_1 - 1)$.

(iii) If $b_2 = b_1 + 1$, consider the move $B \to (b_1, b_2)$.

Case (c). We have subcases;

(i) If $b_2 = b_1 + 1$ or $b_2 > b_1 + 2$, consider the move $B \to (b_1, b_2, b_1 + 1)$.

(ii) If $b_1 > 1$ and $b_2 < b_1 - 1$, consider the move $B \to (b_1, b_2, b_1 - 1)$.

(iii) If $b_1 > 1$ and $b_2 = b_1$, consider the move $B \to (b_1, b_1, b_1 - 2)$.

(iv) If $b_1 > 1$ and $b_2 = b_1 - 1$, consider the move $B \to (b_1, b_1 - 1)$.

(v) If $b_1 = b_2 = 1$, consider the move $B \to (1, 1, 1)$.

Case (d). We have subcases;

(i) If $b_2 = b_1$ or $b_2 > b_1 + 1$, consider the move $B \to (b_1, b_2, b_1 - 1)$.

(ii) If $b_2 = b_1 + 1$, consider the move $B \to (b_1, b_2)$.

Case (e). We have subcases;

(i) If $b_1 > 1$ and $b_2 < b_1 - 1$, consider the move $B \to (b_1, b_2, b_1 - 1)$.

(ii) If $b_1 > 1$ and $b_2 = b_1$, consider the move $B \to (b_1, b_1, b_1 - 2)$.

(iii) If $b_1 > 1$ and $b_2 = b_1 - 1$, consider the move $B \to (b_1, b_1 - 1)$.

(iv) If $b_1 = b_2 = 1$, consider the move $B \to (1, 1, 1)$.

Case (f). Consider the move $B \to (b_2, b_3) = (b_1 + 2, b_1 + 1)$.

Case (g). We have subcases;

(i) If b_1 is odd, consider the move $B \to (b_1 - 2, b_1, b_1)$.

(ii) If b_1 is even, consider the move $B \to (b_1, b_1, b_1 - 1)$.

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