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Research Article

Stability of Nonlinear Neutral Stochastic Functional Differential Equations

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Neutral stochastic functional differential equations (NSFDEs) have recently been studied intensively. The well-known conditions imposed for the existence and uniqueness and exponential stability of the global solution are the local Lipschitz condition and the linear growth condition. Therefore, the existing results cannot be applied to many important nonlinear NSFDEs. The main aim of this paper is to remove the linear growth condition and establish a Khasminskii-type test for nonlinear NSFDEs. New criteria not only cover a wide class of highly nonlinear NSFDEs but they can also be verified much more easily than the classical criteria. Finally, several examples are given to illustrate main results.

1. Introduction

Stochastic modelling has played an important role in many areas of science and engineering for a long time. Some of the most frequent and most important stochastic models used when dynamical systems not only depend on present and past states but also involve derivatives with functionals are described by the following neutral stochastic functional differential equation:

$$\mathrm{d}[x(t) - u(x_t)] = f(x_t, t)\mathrm{d}t + g(x_t, t)\mathrm{d}w(t). \tag{1.1}$$

The conditions imposed on their studies are the standard uniform Lipschitz condition and the linear growth condition. The classical result is described by the following well-known Mao's test see [1, page 202, Theorem 2.2].

Theorem 1.1. *Assume that there exist positive constants* K, L, *and* $\kappa \in (0,1)$ *such that*

$$\begin{gathered}\left|f(\varphi,t)-f(\psi,t)\right|^2 \vee \left|g(\varphi,t)-g(\psi,t)\right|^2 \le K\|\varphi-\overline{\varphi}\|^2,\\ \left|f(\varphi,t)\right|^2 \vee \left|g(\varphi,t)\right|^2 \le L\left(1+\|\varphi\|^2\right),\\ \left|u(\varphi)-u(\psi)\right| \le \kappa\|\varphi-\psi\|\end{gathered} \tag{1.2}$$

for all $\varphi, \psi \in C([-\tau,0];R^n)$. *Then there exists a unique solution* $x(t)$ *to* (1.1) *with initial data* $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$ *(i.e.,* ξ *is an* $\mathcal{F}_0$*-measurable* $C([-\tau,0];R^n)$*-valued random variable such that* $E\|\xi\| < \infty$*).*

Theorem 1.1 requires that the coefficients f and g satisfy the Lipschitz condition and the linear growth condition. However, there are many NSFDEs that do not satisfy the linear growth condition. For example, the following nonlinear NSFDE:

$$\begin{gathered}\mathrm{d}[x(t)-u(x_t)] = x(t)\left[a+b\sigma_1(x_t)-x(t)^2\right]\mathrm{d}t + cx(t)\sigma_2(x_t)\mathrm{d}w(t),\\ \left|\sigma_1(\varphi)\right| \vee \left|\sigma_2(\varphi)\right| < \kappa\int_{-\tau}^{0}\left|\varphi(\theta)\right|\mathrm{d}\mu(\theta),\end{gathered} \tag{1.3}$$

where coefficients $f(x,x_t,t) = x(t)[a+b\sigma_1(x_t)-x(t)^2]$ and $g = cx(t)\sigma_2(x_t)$ do not obey the linear growth condition although they are Lipschitz continuous. To the authors' best knowledge, there is so far no result that shows that (1.3) has a unique global solution for any initial data.

On the other hand, we still encounter a new problem when we attempt to deduce the exponential decay of the solution even if there is no problem with the existence of the solution. For example, Mao [2] initiated the following study of exponential stability for NSFDEs employing the Razumikhin technique.

Theorem 1.2. *Let* c_1, c_2, λ, p *be all positive numbers and* $q > (c_2/c_1)(1-\kappa)^{-p}, \kappa \in (0,1)$, *for any* $\varphi \in L^p_{F_{t_0}}([-\tau,0];R^n)$,

$$E\left|u(\varphi)\right|^p \le \kappa^p\|\varphi\|_0^p, \tag{1.4}$$

and assume that there exists a function $V(x,t) \in C^{2,1}(R^n \times [-\tau,\infty);R_+)$ *such that*

$$c_1|x|^p \le V(x,t) \le c_2|x|^p \tag{1.5}$$

for all $(x,t) \in R^n \times [-\tau, \infty)$ *and also for all* $t \geq 0$

$$ELV(\varphi, t) \leq -\lambda EV(\widetilde{\varphi}(0), t) \tag{1.6}$$

provided $\varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\} \in L^p_{F_t}([-\tau, 0]; R^n)$ *satisfying*

$$EV(\varphi(\theta), t+\theta) \leq qEV(\widetilde{\varphi}(0), t) \tag{1.7}$$

for all $-\tau \leq \theta \leq 0$*. Then for all* $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, $\quad t \geq 0$

$$E|x(t;\xi)|^p \leq \frac{c_2}{c_1}\left(\frac{1+\kappa}{1-\kappa_1}\right)^p e^{-\gamma t} E\|\xi\|_0^p, \quad \kappa_1 = \kappa e^{\gamma\tau/p}, \tag{1.8}$$

where $\gamma = \mu \wedge \tau^{-1}\ln[q_1(1+\kappa q_1^{1/p})^{-p}]$, $q_1 = c_1 q / c_2$.

It is very difficult to verify the conditions of Theorem 1.2, and it is clear that $ELV(\varphi, t) \leq -\lambda EV(\widetilde{\varphi}(0), t)$ does not hold for many NSFDEs. In fact, for (1.3), if one chooses $V(x,t) = x^2$, then

$$LV = 2x(x - u(x_t))\left(a + b\sigma_1(x_t) - x^2\right) + c^2 x^2 \sigma_2^2(x_t). \tag{1.9}$$

Here, the polynomial x^4 appears on the right-hand side, and it has an order of 4 which is higher than the order of $V(x) = x^2$. More recently, Mao [3–5], Zhou et al. [6, 7], Yue et al. [8] and Shen et al. [9] provided with some useful criteria on the exponential stability employing the Lyapunov function, but their tests encounters the same problem.

Therefore, we see that there is a necessity to develop new criteria for NSFDEs where the linear growth condition may not hold while the bound on the operator LV may take a much more general form. In the paper, we will establish a Khasminskii-type test for NSFDEs that cover a wide class of highly nonlinear NSFDEs referring to Khasminskii-type theorems [10] and Mao and Rassias [11] results of stochastic delay differential equations. To our best knowledge, there is no such result for NSFDEs and stochastic functional differential equations (SFDEs).

In the next section, we will establish a general existence and uniqueness theorem of the global solution to (1.1) after giving some necessary notations. Boundedness and Moment stability are given under the Khasminskii-type condition in Section 3. Section 4 establishes asymptotic stability theorem by using semimartingale convergence theory. Section 5 gives corresponding criteria for stochastic functional differential equations. Finally, several examples are given to illustrate our results.

2. Global Solution of NSFDEs

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), \ldots, w_m(t))^T$ be an m-dimensional continuous local martingale with $w(0) = 0$ defined on the probability space.

If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$, while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$ (without any confusion with $\|\varphi\|$). $C([-\tau, 0]; R^n)$ denote the family of all continuous functions φ from $[-\tau, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in R^n. Denoted by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ the family of all bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R^n)$-valued random variables.

Consider an n-dimensional neutral stochastic functional differential equation

$$\mathrm{d}[x(t) - u(x_t, t)] = f(x_t, t)\mathrm{d}t + g(x_t, t)\mathrm{d}w(t) \tag{2.1}$$

on $t \ge 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n), u : C([-\tau, 0]; R^n) \rightarrow R^n$, and

$$f : C([-\tau, 0]; R^n) \times R_+ \longrightarrow R^n, \qquad g : C([-\tau, 0]; R^n) \times R_+ \longrightarrow R^{n \times m} \tag{2.2}$$

are Borel measurable. Let $x(t; \xi)$ denote the solution of (2.1) while $x_t = \{x(t + \theta) : -\tau \le \theta \le 0\}$ which is regarded as a $C([-\tau, 0]; R^n)$-valued stochastic process, denoted by $\tilde{x}(t) = x(t) - u(x_t)$.

Let $C^{2,1}(R^n \times R_+; R)$ denote the family of all nonnegative functions $V(x, t)$ on $R^n \times R_+$ which are continuously twice differentiable in x and once differentiable in t. If $V(x, t) \in C^{2,1}(R^n \times R_+; R)$, define an operator $LV : C([-\tau, 0]; R^n) \times R_+$ to R by

$$LV(\varphi, t) = V_t(\tilde{\varphi}(0), t) + V_x(\tilde{\varphi}(0), t) f(\varphi, t) + \frac{1}{2}\text{trace}\Big(g^T(\varphi, t) V_{xx}(\tilde{\varphi}(0), t) g(\varphi, t)\Big), \tag{2.3}$$

where $V_t(x, t) = \partial V(x, t)/\partial t$, $V_x(x, t) = (\partial V(x, t)/\partial x_1, \partial V(x, t)/\partial x_2, \ldots, \partial V(x, t)/\partial x_n)$, $V_{xx}(x, t) = (\partial^2 V(x, t)/\partial x_i x_j)_{n \times n}$.

For the purpose of stability, assume that $f(0, t) = g(0, t) = u(0, t) = 0$. This implies that (2.1) admits a trivial solution, $x(0, t) = 0$. Furthermore, we impose the following assumptions.

(H1) (The local Lipschitz condition). For each integer $R \ge 1$, there is a positive constant K_R such that

$$\big|f(\varphi, t) - f(\psi, t)\big|^2 \vee \big|g(\varphi, t) - g(\psi, t)\big|^2 \le K_R \|\varphi - \psi\|^2 \tag{2.4}$$

for those $\varphi, \psi \in C([-\tau, 0]; R^n)$ with $\|\varphi\| \vee \|\psi\| \le R$ and $t \in R_+$.

(H2) There exists a positive constant $\kappa \in (0, 1)$ and a probability measure ν such that

$$\big|u(\varphi, t) - u(\psi, t)\big| \le \kappa \int_{-\tau}^{0} \big|\varphi - \psi\big| \mathrm{d}\nu(\theta) \tag{2.5}$$

for any $\varphi, \psi \in C([-\tau, 0]; R^n)$.

(H3) There are two functions $V \in C^{2,1}(R^n \times [-\tau,+\infty); R_+)$ and $U \in C(R^n \times [-\tau,+\infty)]; R_+)$ as well as positive constants $\lambda_1, \lambda_2, c_1, c_2$ and a probability measure μ on $[-\tau, 0]$ such that

$$c_1|x|^2 \leq V(x,t) \leq c_2|x|^2, \tag{2.6}$$

$$LV(\varphi,t) \leq \lambda_1\left[1 + V(\varphi(0),t) + \int_{-\tau}^{0}(V(\varphi(\theta),t+\theta) + U(\varphi(\theta),t+\theta))\mathrm{d}\mu(\theta)\right] - \lambda_2 U(\varphi(0),t) \tag{2.7}$$

for all $-\tau \leq \theta \leq 0$, $(\varphi,t) \in C([-\tau,0];R^n) \times R_+$.

Remark 2.1. In condition (2.7), we see that the function $U(x,t)$ plays a key role in allowing coefficients f and g to be nonlinear functions.

Theorem 2.2. *Assume that (H1), (H2), and (H3) hold. Then for any initial condition $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$, there exists a unique global solution $x(t)$ to* (2.1) *on $t \in [-\tau,\infty)$. Moreover, the solution has the properties that*

$$EV(x(t),t) < \infty, \qquad E\int_0^t U(x(s),s)\mathrm{d}s < \infty \tag{2.8}$$

for any $t \geq 0$.

Proof. It is clear that for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$, there exists a unique maximal local solution $x(t)$ on $t \in [-\tau,\tau_e)$, where τ_e is the explosion time [1], by applying the standing truncation technique (see Mao [12, 13]) to (2.1). According to (H2), we have

$$|\tilde{x}(0)| \leq |x(0)| + |u(x_0,0)| \leq |x(0)| + \kappa\int_{-\tau}^{0}|x(\theta)|\mathrm{d}\nu(\theta) \leq (1+\kappa)\|\xi\|. \tag{2.9}$$

□

Let $k_0 > (1+\kappa)\|\xi\|$ be sufficiently large such that

$$\frac{1}{k_0} < \min_{-\tau\leq t\leq 0}|\tilde{x}(t)| < \max_{-\tau\leq t\leq 0}|\tilde{x}(t)| < k_0. \tag{2.10}$$

Define the stopping time

$$\tau_k = \inf\{t \in [0,\tau_e] : |\tilde{x}(t)| \overline{\in} I_k\}, \quad I_k \equiv \left(\frac{1}{k}, k\right), \ k \geq k_0, \tag{2.11}$$

where throughout this paper, we set $\inf\emptyset = \infty$ ($\emptyset$ denotes the empty sets). Clearly, τ_k is increasing as $k \to \infty$. Denote $\tau_\infty = \lim_{k\to\infty}\tau_k, \tau_\infty \leq \tau_e$ a.s. We will show that $\tau_e = \infty$ a.s., which implies that $x(t)$ is global.

Itô formula and condition (2.7) yield

$$\begin{aligned} dV(\tilde{x}(t),t) &= LV(x,t)dt + V_x(\tilde{x},t)g(x_t,t)dw(t) \\ &\le \lambda_1\left[1 + V(x(t),t) + \int_{-\tau}^{0} V(x_t,t+\theta)d\mu(\theta) + \int_{-\tau}^{0} U(x_t,t+\theta)d\mu(\theta)\right]dt \\ &\quad - \lambda_2 U(x(t),t)dt + V_x(\tilde{x},t)g(x_t,t)dw(t) \end{aligned} \tag{2.12}$$

for $t \ge 0$. For any $k \ge k_0$ and $t \in [0,\tau]$, we integrate both sides of (2.12) from 0 to $\tau_k \wedge t$ and then take the expectations to get

$$\begin{aligned} &EV(\tilde{x}(\tau_k \wedge t), \tau_k \wedge t) - V(\tilde{x}(0),0) \\ &\quad \le E\int_0^{\tau_k\wedge t} \lambda_1\left[1 + \int_{-\tau}^{0} (V(x_s,s+\theta) + U(x_s,s+\theta))d\mu(\theta)\right]ds \\ &\qquad + E\int_0^{\tau_k\wedge t} [\lambda_1 V(x(s),s) - \lambda_2 U(x(s),s)]ds. \end{aligned} \tag{2.13}$$

According to the integral substitution technique, we estimate

$$\begin{aligned} &\int_0^{\tau_k\wedge t}\int_{-\tau}^{0} V(x_s,s+\theta)d\mu(\theta)ds \\ &\quad = \int_0^{\tau_k\wedge t}\int_{-\tau}^{0} V(x(s+\theta),s+\theta)d\mu(\theta)ds \\ &\quad \le \int_{-\tau}^{0} d\mu(\theta)\int_{\theta}^{\tau_k\wedge t+\theta} V(x(s),s)ds \le \int_{-\tau}^{\tau_k\wedge t} V(x(s),s)ds \\ &\quad \le \int_{-\tau}^{\tau} V(x(s),s)ds < +\infty, \end{aligned} \tag{2.14}$$

Similiarly,

$$\int_0^{\tau_k\wedge t}\int_{-\tau}^{0} U(x_s,s+\theta)d\mu(\theta)ds \le \int_{-\tau}^{\tau} U(x(s),s)ds < +\infty. \tag{2.15}$$

Substituting for (2.14) and (2.15) into (2.13), and by using the Fubini theorem, the result is

$$\begin{aligned} EV(\tilde{x}(\tau_k \wedge t), \tau_k \wedge t) \le V(\tilde{x}(0), 0) + \lambda_1 \tau + \lambda_1 E \int_0^{\tau_k \wedge t} V(x(s), s) \mathrm{d}s \\ - \lambda_2 E \int_0^{\tau_k \wedge t} U(x(s), s) \mathrm{d}s + \lambda_1 E \int_{-\tau}^{\tau} [V(x(s), s) + U(x(s), s)] \mathrm{d}s \\ \le C_1 + \lambda_1 E \int_0^{\tau_k \wedge t} V(x(s), s) \mathrm{d}s \\ \le C_1 + \lambda_1 \int_0^t EV(x(\tau_k \wedge s), \tau_k \wedge s) \mathrm{d}s, \end{aligned} \tag{2.16}$$

where $C_1 = V(\tilde{x}(0), 0) + \lambda_1 \tau + \lambda_1 E \int_{-\tau}^{\tau} [V(x(s), s) + U(x(s), s)] \mathrm{d}s$. Equations (2.6) and (2.9) imply $V(\tilde{x}(0), 0) \le c_2 (1+\kappa)^2 \|\xi\|^2$; thus, C_1 is a finite constant. By using inequality $(a+b)^2 \le (1/(1-\kappa_0)) a^2 + (1/\kappa_0) b^2$, $a, b > 0$, $\kappa_0 \in (0,1)$; thus,

$$E|x(\tau_k \wedge t)|^2 \le (1-\kappa_0)^{-1} E|\tilde{x}(\tau_k \wedge t)|^2 + \kappa_0^{-1} E|u(x_{\tau_k \wedge t}, \tau_k \wedge t)|^2, \tag{2.17}$$

condition (2.6) yields

$$c_2^{-1} V(\tilde{x}(\tau_k \wedge t), \tau_k \wedge t) \le |\tilde{x}(\tau_k \wedge t)|^2 \le c_1^{-1} V(\tilde{x}(\tau_k \wedge t), \tau_k \wedge t). \tag{2.18}$$

(H2) and the Hölder inequality yield

$$E|u(x_{\tau_k \wedge t}, \tau_k \wedge t)|^2 \le \kappa^2 E \left(\int_{-\tau}^0 |\varphi(\theta)| \mathrm{d}\nu(\theta) \right)^2 \le \kappa^2 \int_{-\tau}^0 E|x(\tau_k \wedge t + \theta)|^2 \mathrm{d}\nu(\theta). \tag{2.19}$$

Substituting for (2.16), (2.18), and (2.19) into (2.17), the result is

$$E|x(\tau_k \wedge t)|^2 \le (1-\kappa_0)^{-1} c_1^{-1} \left(C_1 + \lambda_1 \int_0^t EV(x(\tau_k \wedge s), \tau_k \wedge s) \mathrm{d}s \right) + \kappa_0^{-1} \kappa^2 \int_{-\tau}^0 E|x(\tau_k \wedge t + \theta)|^2 \mathrm{d}\nu(\theta). \tag{2.20}$$

For any $t \in [-\tau, \tau]$, (2.20) implies

$$\sup_{-\tau \le s \le t} E|x(\tau_k \wedge s)|^2 \le (1-\kappa_0)^{-1} c_1^{-1} \left(C_1 + \lambda_1 \int_0^t EV(x(\tau_k \wedge s), \tau_k \wedge s) \mathrm{d}s \right) + \kappa_0^{-1} \kappa^2 \sup_{-\tau \le s \le t} E|x(\tau_k \wedge s)|^2. \tag{2.21}$$

Let $\kappa_0 = \kappa$, then

$$\sup_{-\tau\le s\le t} E|x(\tau_k \wedge s)|^2 \le (1-\kappa)^{-2} c_1^{-1} \left(C_1 + \lambda_1 \int_0^t EV(x(\tau_k \wedge s), \tau_k \wedge s) ds \right). \tag{2.22}$$

Therefore, for any $t \in [-\tau, \tau]$,

$$E|x(\tau_k \wedge t)|^2 \le (1-\kappa)^{-2} c_1^{-1} \left(C_1 + \lambda_1 \int_0^t EV(x(\tau_k \wedge s), \tau_k \wedge s) ds \right). \tag{2.23}$$

By (2.6), we may obtain

$$EV(x(\tau_k \wedge t), \tau_k \wedge t) \le c_2 E|x(\tau_k \wedge t)|^2 \le (1-\kappa)^{-2} c_1^{-1} c_2 \left(C_1 + \lambda_1 \int_0^t EV(x(\tau_k \wedge s), \tau_k \wedge s) ds \right). \tag{2.24}$$

For any $k \ge k_0$, $\;t \in [0, \tau]$, the Gronwall inequality implies

$$EV(x(\tau_k \wedge \tau), \tau_k \wedge \tau) \le c_1^{-1} c_2 C_1 (1-\kappa)^{-2} e^{c_1^{-1} c_2 \lambda_1 (1-\kappa)^{-2} t}. \tag{2.25}$$

Thus, for all $k \ge k_0$,

$$EV(x(\tau_k \wedge \tau), \tau_k \wedge \tau) \le c_1^{-1} c_2 C_1 (1-\kappa)^{-2} e^{c_1^{-1} c_2 \lambda_1 (1-\kappa)^{-2} \tau}, \tag{2.26}$$

which implies

$$EV(x(t), t) \le c_1^{-1} c_2 C_1 (1-\kappa)^{-2} e^{c_1^{-1} c_2 \lambda_1 (1-\kappa)^{-2} \tau}, \quad 0 \le t \le \tau. \tag{2.27}$$

Since $E[I_{\{\tau_k \le \tau\}} V(x(\tau_k \wedge \tau), \tau_k \wedge \tau)] \le EV(x(\tau_k \wedge \tau), \tau_k \wedge \tau)$, and defining $\mu_k = \inf_{|x|\ge k, 0\le t<\infty} V(x,t)$ for $k \ge k_0$, according to (2.26), then

$$\mu_k P(\tau_k \le \tau) \le c_1^{-1} c_2 C_1 (1-\kappa)^{-2} e^{c_1^{-1} c_2 \lambda_1 (1-\kappa)^{-2} \tau}. \tag{2.28}$$

Clearly, condition (2.6) implies $\lim_{k\to\infty} \mu_k = \infty$. Letting $k \to \infty$ in (2.28), then $P(\tau_\infty \le \tau) = 0$, namely,

$$P(\tau_\infty > \tau) = 1. \tag{2.29}$$

Moreover, setting $t = \tau$ in (2.16), we may obtain that

$$\lambda_2 E \int_0^{\tau_k \wedge \tau} U(x(s), s) ds \le C_1 + \lambda_1 E \int_0^{\tau_k \wedge \tau} V(x(s), s) ds \le C_1 + \lambda_1 \tau c_1^{-1} c_2 C_1 (1-\kappa)^{-2} e^{c_1^{-1} c_2 \lambda_1 (1-\kappa)^{-2} \tau}, \tag{2.30}$$

that is,

$$E\int_0^{\tau} U(x(s),s)\mathrm{d}s \le \frac{C_1}{\lambda_2}\left(1+\lambda_1\tau c_1^{-1}c_2C_1(1-\kappa)^{-2}\mathrm{e}^{c_1^{-1}c_2\lambda_1(1-\kappa)^{-2}\tau}\right) < \infty. \tag{2.31}$$

Let us now proceed to prove $\tau_\infty > 2\tau$ a.s. given that we have shown (2.27)–(2.31). For any $k \ge k_0$ and $t \in [0,2\tau]$, we can integrate both sides of (2.12) from 0 to $\tau_k \wedge t$ and then take expectations to get

$$EV(\tilde{x}(\tau_k\wedge t),\tau_k\wedge t) \le C_2 + \lambda_1 E\int_0^{\tau_k\wedge t} V(x(s),s)\mathrm{d}s - \lambda_2 E\int_0^{\tau_k\wedge t} U(x(s),s)\mathrm{d}s, \tag{2.32}$$

where

$$C_2 = V(\tilde{x}(0)) + 2\lambda_1\tau + \lambda_1 E\int_{-\tau}^{2\tau}[V(x(s),s)+U(x(s),s)]\mathrm{d}s < \infty. \tag{2.33}$$

By the Gronwall inequality and (2.32), we have

$$EV(\tilde{x}(\tau_k\wedge t),\tau_k\wedge t) \le c_1^{-1}c_2C_2(1-\kappa)^{-2}\mathrm{e}^{c_1^{-1}c_2\lambda_1(1-\kappa)^{-2}t}, \quad 0\le t\le 2\tau,\ k\ge k_0. \tag{2.34}$$

In particular,

$$EV(\tilde{x}(\tau_k\wedge 2\tau),\tau_k\wedge 2\tau) \le c_1^{-1}c_2C_2(1-\kappa)^{-2}\mathrm{e}^{c_1^{-1}c_2\lambda_1(1-\kappa)^{-2}2\tau}, \quad \forall k\ge k_0. \tag{2.35}$$

This implies

$$\mu_k P(\tau_k \le 2\tau) \le c_1^{-1}c_2C_2(1-\kappa)^{-2}\mathrm{e}^{c_1^{-1}c_2\lambda_1(1-\kappa)^{-2}2\tau}. \tag{2.36}$$

Letting $k \to \infty$, by (2.6), then $P(\tau_\infty \le 2\tau) = 0$, that is,

$$P(\tau_\infty > 2\tau) = 1, EV(x(t),t) \le c_1^{-1}c_2C_2(1-\kappa)^{-2}\mathrm{e}^{c_1^{-1}c_2\lambda_1(1-\kappa)^{-2}2\tau}, \quad 0\le t\le 2\tau. \tag{2.37}$$

By (2.32), we may obtain that

$$\lambda_2 E\int_0^{\tau_k\wedge 2\tau} U(x,t)\mathrm{d}t \le C_2 + \lambda_1 E\int_0^{\tau_k\wedge 2\tau} V(x(t),t)\mathrm{d}t, \tag{2.38}$$

that is,

$$E\int_0^{2\tau} U(x,t)\mathrm{d}t \le \frac{C_2}{\lambda_2}\left(1+2\lambda_1\tau c_1^{-1}c_2(1-\kappa)^{-2}\mathrm{e}^{c_1^{-1}c_2\lambda_1(1-\kappa)^{-2}2\tau}\right) < \infty. \tag{2.39}$$

Repeating this procedure, we can show that, for any integer $i \geq 1, \tau_\infty > i\tau$ a.s. and $EV(x(t), t) \leq c_1^{-1}c_2C_i(1-\kappa)^{-2}e^{c_1^{-1}c_2(1-\kappa)^{-2}\lambda_1 i\tau}, 0 \leq t < i\tau$, and

$$E\int_0^{i\tau} U(x,t)\mathrm{d}t \leq \frac{C_i}{\lambda_2}\Big(1 + i\lambda_1\tau c_1^{-1}c_2(1-\kappa)^{-2}e^{c_1^{-1}c_2(1-\kappa)^{-2}\lambda_1 i\tau}\Big), \tag{2.40}$$

where

$$C_i = V(\tilde{x}(0),0) + \lambda_1 E\int_{-\tau}^{i\tau}[1 + V(x,t) + U(x,t)]\mathrm{d}t < \infty. \tag{2.41}$$

We must therefore have $\tau_\infty = \infty$ a.s. as well as the required assertion.

Note that condition (2.6) may be replaced by more general condition $c_1|x|^p \leq V(x,t) \leq c_2|x|^p, \quad p \geq 2$, which is suitable to the corresponding results below.

3. Boundedness and Moment Stability

In the previous section, we have shown that the solution of (2.1) has the properties that

$$EV(x(t),t) < \infty, \qquad E\int_0^t U(x(s),s)\mathrm{d}s < \infty \tag{3.1}$$

for any $t \geq 0$. In the following, we will give more precise estimations under specified conditions; that is, we will establish the criteria of moment stability and asymptotic stability of the solution to (2.1) under specified conditions.

Theorem 3.1. *Assume that (H1), (H2), and (H3) hold except* (2.7) *which is replaced by*

$$\begin{aligned} LV(\varphi,t) \leq{}& \mu_1 - \mu_2 V(\varphi(0),t) + \mu_3\int_{-\tau}^0 V(\varphi,t+\theta)\mathrm{d}\eta_1(\theta) - \mu_4 U(\varphi(0),t) \\ &+ \mu_5\int_{-\tau}^0 U(\varphi,t+\theta)\mathrm{d}\eta_2(\theta) - \mu_6 V(\tilde{\varphi}(0),t) \end{aligned} \tag{3.2}$$

for all $(\varphi,t) \in C([-\tau,0];R^n)\times R_+,\ -\tau \leq \theta \leq 0$, *where* $\mu_1 \geq 0, \mu_2 > \mu_3 \geq 0, \mu_4 > \mu_5 > 0, \mu_6 > 0$ *are constants and* $\eta_1(\theta)$ *and* $\eta_2(\theta)$ *are probability measures on* $[-\tau,0]$. *Then for any initial data* ξ, *the global solution* $x(t)$ *to* (2.1) *has the property that*

$$\limsup_{t\to\infty} EV(x(t),t) < \frac{c_2\mu_1}{(1-\kappa_0)^2 c_1\varepsilon}, \tag{3.3}$$

where $\varepsilon = \mu_6 \wedge \varepsilon_1 \wedge \varepsilon_2 \wedge \tau^{-1}\log\kappa^{-2}, \kappa_0 = \kappa\sqrt{e^{\varepsilon\tau}}$, *while* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *are the unique roots to the following equations:*

$$\mu_2 = \mu_3 e^{\varepsilon_1\tau}, \qquad \mu_4 = \mu_5 e^{\varepsilon_2\tau}, \tag{3.4}$$

respectively. If $\mu_1 = 0$, *then*

$$\limsup_{t\to\infty} \frac{1}{t} \ln EV(x(t),t) < -\varepsilon, \qquad \int_0^\infty EU(x(t),t)\mathrm{d}t < \infty. \tag{3.5}$$

Proof. We first observe that (3.2) implies (2.7) if we set $\lambda_1 = \mu_1 \vee \mu_3 \vee \mu_5$ and $\lambda_2 = \mu_4$. So, for any initial data, (2.1) has a unique global solution $x(t)$ on $t \geq -\tau$, which has the properties (2.8). Based on these properties, we can apply the Itô formula and condition (3.2) to obtain that for any $t \geq 0$,

$$\begin{aligned}
&\mathrm{d}\left[\mathrm{e}^{\varepsilon t} V(\tilde{x}(t),t)\right] \\
&\quad = \mathrm{e}^{\varepsilon t}\left[\varepsilon V(\tilde{x}(t),t) + LV(x(t),t)\right]\mathrm{d}t + \mathrm{e}^{\varepsilon t} V_x(\tilde{x}(t),t) g(x_t,t)\mathrm{d}w(t) \\
&\quad \leq \mathrm{e}^{\varepsilon t}\left[\mu_1 - \mu_2 V(x,t) + \mu_3 \int_{-\tau}^0 V(x_t,t+\theta)\mathrm{d}\eta_1(\theta) - \mu_4 U(x,t) + \mu_5 \int_{-\tau}^0 U(x_t,t+\theta)\mathrm{d}\eta_2(\theta)\right] \\
&\qquad + \mathrm{e}^{\varepsilon t} V_x(\tilde{x}(t),t) g(x_t,t)\mathrm{d}w(t) - \mathrm{e}^{\varepsilon t}\left(\mu_6 - \varepsilon\right) V(\tilde{x}(t),t),
\end{aligned} \tag{3.6}$$

We integrate both sides of the above inequality from 0 to t and take expectations to get

$$\begin{aligned}
&\mathrm{e}^{\varepsilon t} EV(\tilde{x}(t),t) \\
&\quad \leq V(\tilde{x}(0),0) + \frac{\mu_1 \mathrm{e}^{\varepsilon t}}{\varepsilon} - \mu_2 E\int_0^t \mathrm{e}^{\varepsilon s} V(x(s),s)\mathrm{d}s + \mu_3 E\int_0^t \int_{-\tau}^0 \mathrm{e}^{\varepsilon s} V(x_s,s+\theta)\mathrm{d}\eta_1(\theta)\mathrm{d}s \\
&\qquad - \mu_4 E\int_0^t \mathrm{e}^{\varepsilon s} U(x(s),s)\mathrm{d}s + \mu_5 E\int_0^t \int_{-\tau}^0 \mathrm{e}^{\varepsilon s} U(x_s,s+\theta)\mathrm{d}\eta_2(\theta)\mathrm{d}s,
\end{aligned} \tag{3.7}$$

by using of $\varepsilon = \mu_6 \wedge \varepsilon_1 \wedge \varepsilon_2 < \mu_6$. Compute

$$\begin{aligned}
E\int_0^t \int_{-\tau}^0 \mathrm{e}^{\varepsilon s} V(x_s,s+\theta)\mathrm{d}\eta_1(\theta)\mathrm{d}s &= E\int_0^t \int_{-\tau}^0 \mathrm{e}^{\varepsilon s} V(x(s+\theta),s+\theta)\mathrm{d}\eta_1(\theta)\mathrm{d}s \\
&\leq \mathrm{e}^{\varepsilon\tau} E\int_0^t \int_{-\tau}^0 \mathrm{e}^{\varepsilon(s+\theta)} V(x(s+\theta),s+\theta)\mathrm{d}\eta_1(s+\theta)\mathrm{d}s \\
&\leq \mathrm{e}^{\varepsilon\tau} E\int_{-\tau}^0 \mathrm{d}\eta_1(\theta)\int_\theta^{t+\theta} \mathrm{e}^{\varepsilon s} V(x(s),s)\mathrm{d}\mathrm{d}s \\
&\leq \mathrm{e}^{\varepsilon\tau} E\int_{-\tau}^t \mathrm{e}^{\varepsilon s} V(x(s),s)\mathrm{d}s \\
&\leq \mathrm{e}^{\varepsilon\tau} E\int_{-\tau}^0 \mathrm{e}^{\varepsilon s} V(x(s),s)\mathrm{d}s + \mathrm{e}^{\varepsilon\tau} E\int_0^t \mathrm{e}^{\varepsilon s} V(x(s),s)\mathrm{d}s,
\end{aligned} \tag{3.8}$$

Similiarly,

$$E\int_0^t\int_{-\tau}^0 e^{\varepsilon s}U(x_s,s+\theta)d\eta_2(\theta)ds \le e^{\varepsilon\tau}E\int_{-\tau}^0 e^{\varepsilon s}U(x(s),s)ds + e^{\varepsilon\tau}E\int_0^t e^{\varepsilon s}U(x(s),s)ds. \quad (3.9)$$

Substituting for (3.8) and (3.9) into (3.7), the result is

$$\begin{aligned}
&e^{\varepsilon t}EV(\tilde{x}(t),t)\\
&\le V(\tilde{x}(0),0) + \frac{\mu_1 e^{\varepsilon t}}{\varepsilon} - \mu_2 E\int_0^t e^{\varepsilon s}V(x(s),s)ds\\
&\quad + \mu_3 e^{\varepsilon\tau}E\int_{-\tau}^0 e^{\varepsilon s}V(x(s),s)ds + \mu_3 e^{\varepsilon\tau}E\int_0^t e^{\varepsilon s}V(x(s),s)ds\\
&\quad - \mu_4 E\int_0^t e^{\varepsilon s}U(x,s)ds + \mu_5 e^{\varepsilon\tau}E\int_{-\tau}^0 e^{\varepsilon s}U(x(s),s)ds + \mu_5 e^{\varepsilon\tau}E\int_0^t e^{\varepsilon(s)}U(x(s),s)ds\\
&= V(\tilde{x}(0),0) + \frac{\mu_1 e^{\varepsilon t}}{\varepsilon} + e^{\varepsilon\tau}E\int_{-\tau}^0 e^{\varepsilon s}\left(\mu_3 V(x(s),s) + \mu_5 U(x(s),s)\right)ds\\
&\quad - \left(\mu_2 - \mu_3 e^{\varepsilon\tau}\right)E\int_0^t e^{\varepsilon s}V(x(s),s)ds - \left(\mu_4 - \mu_5 e^{\varepsilon\tau}\right)E\int_0^t e^{\varepsilon s}U(x,s)ds\\
&= C + \frac{\mu_1 e^{\varepsilon t}}{\varepsilon} - \left(\mu_2 - \mu_3 e^{\varepsilon\tau}\right)E\int_0^t e^{\varepsilon s}V(x(s),s)ds - \left(\mu_4 - \mu_5 e^{\varepsilon\tau}\right)E\int_0^t e^{\varepsilon s}U(x,s)ds,
\end{aligned} \quad (3.10)$$

where $C = V(\tilde{x}(0),0) + e^{\varepsilon\tau}E\int_{-\tau}^0 e^{\varepsilon s}(\mu_3 V(x(s),s) + \mu_5 U(x(s),s))ds$. It is clear that, for $\varepsilon \le \varepsilon_1 \wedge \varepsilon_2$, we have $\mu_2 - \mu_3 e^{\varepsilon\tau} \ge 0, \mu_4 - \mu_5 e^{\varepsilon\tau} \ge 0$, hence,

$$e^{\varepsilon t}EV(\tilde{x}(t),t) \le C + \frac{\mu_1 e^{\varepsilon t}}{\varepsilon}. \quad (3.11)$$

By (H2) and (H3) and inequality $(a+b)^2 \le (1/(1-\kappa_0))a^2 + (1/\kappa_0)b^2, a, b > 0, \kappa_0 \in (0,1)$, we compute

$$\begin{aligned}
Ee^{\varepsilon t}|x(t)|^2 &\le (1-\kappa_0)^{-1}Ee^{\varepsilon t}|\tilde{x}(t)|^2 + \kappa_0^{-1}Ee^{\varepsilon t}|u(x_t,t)|^2\\
&\le (1-\kappa_0)^{-1}c_1^{-1}e^{\varepsilon t}E|V(\tilde{x}(t),t)|^2 + \kappa_0^{-1}\kappa^2\int_{-\tau}^0 e^{\varepsilon t}E|x(t+\theta)|^2 d\nu(\theta)\\
&\le (1-\kappa_0)^{-1}c_1^{-1}\left(C + \frac{\mu_1}{\varepsilon}e^{\varepsilon t}\right) + \kappa_0^{-1}\kappa^2\int_{-\tau}^0 e^{-\varepsilon\theta}Ee^{\varepsilon(t+\theta)}|x(t+\theta)|^2 d\nu(\theta).
\end{aligned} \quad (3.12)$$

For any $t \geq 0$,

$$\begin{aligned}\sup_{-\tau\leq s\leq t} Ee^{\varepsilon t}|x(t)|^2 &\leq (1-\kappa_0)^{-1}c_1^{-1}\left(C+\frac{\mu_1}{\varepsilon}e^{\varepsilon t}\right)+\kappa_0^{-1}\kappa^2\sup_{-\tau\leq s\leq t} Ee^{\varepsilon t}|x(t)|^2\int_{-\tau}^{0} e^{-\varepsilon\theta}d\nu(\theta)\\ &\leq (1-\kappa_0)^{-1}c_1^{-1}\left(C+\frac{\mu_1}{\varepsilon}e^{\varepsilon t}\right)+\kappa_0^{-1}\kappa^2 e^{\varepsilon\tau}\sup_{-\tau\leq s\leq t} Ee^{\varepsilon t}|x(t)|^2.\end{aligned} \tag{3.13}$$

Leting $\kappa_0 = \kappa\sqrt{e^{\varepsilon\tau}} < 1$, since $\varepsilon < \tau^{-1}\log\kappa^{-2}$, then $\kappa_0 < 1$,

$$\sup_{-\tau\leq s\leq t} Ee^{\varepsilon t}|x(t)|^2 \leq (1-\kappa_0)^{-2}c_1^{-1}\left(C+\frac{\mu_1}{\varepsilon}e^{\varepsilon t}\right), \tag{3.14}$$

and by (H3), $EV(x,t) \leq c_2 E|x(t)|^2$, then

$$\sup_{-\tau\leq s\leq t} e^{\varepsilon t}EV(x,t) \leq c_2\sup_{-\tau\leq s\leq t} e^{\varepsilon t}E|x(t)|^2 \leq (1-\kappa_0)^{-2}c_1^{-1}c_2\left(C+\frac{\mu_1}{\varepsilon}e^{\varepsilon t}\right). \tag{3.15}$$

Therefore,

$$\limsup_{t\to\infty} EV(x(t),t) \leq \frac{c_2\mu_1}{(1-\kappa_0)^2 c_1\varepsilon}. \tag{3.16}$$

When $\mu_1 = 0$, then $EV(x(t),t) \leq c_1^{-1}c_2C(1-\kappa_0)^{-2}e^{-\varepsilon t}$, for all $t \geq 0$, that is,

$$\limsup_{t\to\infty}\frac{1}{t}\log EV(x(t),t) \leq -\varepsilon. \tag{3.17}$$

On the other hand, when $\mu_1 = 0$, by (3.7) and the Itô formula, we may show easily that

$$\begin{aligned}EV(\tilde{x}(t),t) &= V(\tilde{x}(0),0)+E\int_{-\tau}^{0}(\mu_3 V(x(s),s)+\mu_5 U(x(s),s))d(s)\\ &\quad-(\mu_2-\mu_3)E\int_0^t V(x(s),s)ds-(\mu_4-\mu_5)E\int_0^t U(x(s),s)ds.\end{aligned} \tag{3.18}$$

By $\mu_2 = \mu_3 e^{\varepsilon_1\tau} > \mu_3$, $\mu_4 = \mu_5 e^{\varepsilon_2\tau} > \mu_5$, and the Fubini theorem, we obtain

$$\int_0^t EU(x,s) \leq \frac{1}{\mu_4-\mu_5}\left[V(\tilde{x}(0),0)+E\int_{-\tau}^{0}(\mu_3 V(x(s),s)+\mu_5 U(x(s),s))d(s)\right] < \infty. \tag{3.19}$$

The proof is complete. □

4. Asymptotic Stability

In this section, we will establish asymptotic stability of (2.1) without the linear growth condition. It is well known that the linear growth condition is one of the most important conditions to guarantee asymptotic stability. Therefore we introduce the following semitingale convergence thoerem [14, 15], which will play a key role in dealing with nonlinear systems.

Lemma 4.1. *Let $M(t)$ be a real-valued local martingale with $M(0) = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. If $X(t)$ is a nonnegative continuous $\mathcal{F}_t$-adapted process and satisfies $X(t) \leq \zeta + M(t)$ for $t \geq 0$, then $X(t)$ is almost surely bounded, namely, $\lim_{t\to\infty} X(t) < \infty$, a.s.*

Theorem 4.2. *Assume that (H1), (H2), and (H3) hold except* (2.7) *which is replaced by*

$$\begin{aligned} LV(\varphi,t) \leq{}& -\mu_2 V(\varphi(0),t) + \mu_3\int_{-\tau}^{0} V(\varphi,t+\theta)\mathrm{d}\eta_1(\theta) - \mu_4 U(\varphi(0),t) \\ &+ \mu_5 \int_{-\tau}^{0} U(\varphi,t+\theta)\mathrm{d}\eta_2(\theta) - \mu_6 V(\tilde{\varphi}(0),t) \end{aligned} \tag{4.1}$$

for all $(\varphi,t) \in R^n \times R_+, -\tau \leq \theta \leq 0$, where $\mu_2 > \mu_3 \geq 0, \mu_4 > \mu_5 > 0, \mu_6 > 0$. Then, for any initial data, the unique global solution $x(t)$ of (2.1) *has the property that*

$$\limsup_{t\to\infty} \frac{1}{t} \ln V(x(t),t) \leq -\varepsilon, \qquad \int_0^\infty U(x(t),t)\mathrm{d}t < \infty, \tag{4.2}$$

where $\varepsilon = \mu_6 \wedge \varepsilon_1 \wedge \varepsilon_2 \wedge \tau^{-1}\log\kappa^{-2}$, while $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are the unique roots to the following equations:

$$\mu_2 = \mu_3 \mathrm{e}^{\varepsilon_1\tau}, \qquad \mu_4 = \mu_5 \mathrm{e}^{\varepsilon_2\tau}, \tag{4.3}$$

respectively.

Proof. We first observe that (4.1) implies (2.7) if we set $\lambda_1 = \mu_1 \vee \mu_3 \vee \mu_5$ and $\lambda_2 = \mu_4$. So, for any initial data, (2.1) has a unique global solution $x(t)$ on $t \geq -\tau$, which has the properties (2.8). Similar to the proof of Theorem 3.1, applying the Itô formula and condition (4.1), for any $t \geq 0$, we may obtain that

$$\begin{aligned} &\mathrm{d}\left[\mathrm{e}^{\varepsilon t} V(\tilde{x}(t),t)\right] \\ &= \mathrm{e}^{\varepsilon t}[\varepsilon V(\tilde{x}(t),t) + LV(x(t),t)]\mathrm{d}t + \mathrm{e}^{\varepsilon t} V_x(\tilde{x}(t),t) g(x_t,t)\mathrm{d}w(t) \\ &\leq \mathrm{e}^{\varepsilon t}\left[-\mu_2 V(x(t),t) + \mu_3 \int_{-\tau}^{0} V(x_t,t+\theta)\mathrm{d}\eta_1(\theta) - \mu_4 U(x,t) + \mu_5\int_{-\tau}^{0} U(x_t,t+\theta)\mathrm{d}\eta_2(\theta)\right] \\ &\quad + \mathrm{e}^{\varepsilon t} V_x(\tilde{x}(t),t) g(x_t,t)\mathrm{d}w(t) - (\mu_6 - \varepsilon) V(\tilde{x}(t),t). \end{aligned} \tag{4.4}$$

For $t > 0$, we can integrate both sides of the above inequality from 0 to t and take expectations to get

$$\begin{aligned} e^{\varepsilon t} EV(\tilde{x}(t), t) \leq {} & V(\tilde{x}(0), 0) - \mu_2 \int_0^t e^{\varepsilon s} V(x(s), s) ds + \mu_3 \int_0^t \int_{-\tau}^0 e^{\varepsilon s} V(x_s, s + \theta) d\eta_1(\theta) ds \\ & - \mu_4 \int_0^t e^{\varepsilon s} U(x, s) ds + \mu_5 \int_0^t \int_{-\tau}^0 e^{\varepsilon s} U(x_s, s + \theta) d\eta_2(\theta) ds + M(t), \end{aligned} \tag{4.5}$$

where $M(t) = \int_0^t e^{\varepsilon s} V_x(\tilde{x}(s), s) g(x_s, s) dw(s) ds$ is a real-valued continuous local martingale with $M(0) = 0$. Similar to Theorem 3.1, we have

$$\begin{aligned} e^{\varepsilon t} V(\tilde{x}(t), t) \leq {} & V(\tilde{x}(0), 0) + e^{\varepsilon \tau} \int_{-\tau}^0 \left(\mu_3 V(x(s), s) + \mu_5 U(x(s), s) \right) ds \\ & - \left(\mu_2 - \mu_3 e^{\varepsilon \tau} \right) \int_0^t e^{\varepsilon s} V(x(s), s) ds - \left(\mu_4 - \mu_5 e^{\varepsilon \tau} \right) \int_0^t e^{\varepsilon s} U(x, s) ds + M(t) \\ \leq {} & \text{const} + M(t). \end{aligned} \tag{4.6}$$

Lemma 4.1 implies

$$\limsup_{t \to \infty} e^{\varepsilon t} V(\tilde{x}(t), t) < \infty \quad \text{a.s.} \tag{4.7}$$

Since $c_1 |x|^2 \leq V(x, t) \leq c_2 |x|^2$, then

$$\limsup_{t \to \infty} e^{\varepsilon t} |\tilde{x}(t)|^2 < \infty \quad \text{a.s.} \tag{4.8}$$

According to the definition of $\tilde{x}(t)$, we compute

$$\begin{aligned} e^{\varepsilon t} |x(t)|^2 &= e^{\varepsilon t} |\tilde{x}(t) + u(x_t, t)|^2 \\ &\leq (1 - \kappa_0)^{-1} e^{\varepsilon t} |\tilde{x}(t)|^2 + \kappa_0^{-1} e^{\varepsilon t} |u(x_t, t)|^2 \\ &\leq (1 - \kappa_0)^{-1} e^{\varepsilon t} |\tilde{x}(t)|^2 + \kappa_0^{-1} \kappa^2 \int_{-\tau}^0 e^{-\varepsilon \theta} e^{\varepsilon (t + \theta)} |x(t + \theta)|^2 d\nu(\theta). \end{aligned} \tag{4.9}$$

Therefore, we may also compute

$$\begin{aligned}
\sup_{-\tau\le s\le t} e^{\varepsilon s}|x(s)|^2 &\le \|\xi\|^2 + \sup_{0\le s\le t} e^{\varepsilon s}|x(s)|^2 \\
&\le \|\xi\|^2 + (1-\kappa_0)^{-1}\sup_{0\le s\le t}\left[e^{\varepsilon s}|\tilde{x}(s)|^2 + \kappa_0^{-1}\kappa^2\int_{-\tau}^{0} e^{-\varepsilon\theta}e^{\varepsilon(s+\theta)}|x(s+\theta)|^2 d\nu(\theta)\right] \\
&\le \|\xi\|^2 + (1-\kappa_0)^{-1}\sup_{0\le s\le t} e^{\varepsilon s}|\tilde{x}(s)|^2 + \kappa_0^{-1}\kappa^2 \sup_{-\tau\le s\le t} e^{\varepsilon s}|x(s)|^2 \int_{-\tau}^{0} e^{-\varepsilon\theta} d\nu(\theta) \\
&\le \|\xi\|^2 + (1-\kappa_0)^{-1}\sup_{0\le s\le t} e^{\varepsilon s}|\tilde{x}(s)|^2 + \kappa_0^{-1}\kappa^2 e^{\varepsilon\tau} \sup_{-\tau\le s\le t} e^{\varepsilon s}|x(s)|^2.
\end{aligned} \tag{4.10}$$

Noting that $\varepsilon < \tau^{-1}\log\kappa^{-2}$, choose $\kappa_0 = \kappa\sqrt{e^{\varepsilon\tau}}$. Then $\kappa_0 < 1$, and we obtain

$$\sup_{-\tau\le s\le t} e^{\varepsilon s}|x(s)|^2 \le (1-\kappa_0)^{-1}\|\xi\|^2 + (1-\kappa_0)^{-2}\sup_{0\le s\le t} e^{\varepsilon s}|\tilde{x}(s)|^2. \tag{4.11}$$

(4.8) and (4.11) yield

$$\sup_{-\tau\le s\le t} e^{\varepsilon s}|x(s)|^2 < +\infty \quad \text{a.s.} \tag{4.12}$$

Recall the condition $c_1|x|^2 \le V(x,t) \le c_2|x|^2$, which implies $\limsup_{t\to\infty} e^{\varepsilon t}V(x(t),t) < C$ a.s. The required result is obtained. □

Remark 4.3. From the processes of the proof of Theorems 3.1 and 4.2, we see that condition (2.6) plays an important role in dealing with the neutral term. Moreover, applying condition (2.6), we can also obtain more precise results

$$\limsup_{t\to\infty}\frac{1}{t}\log E|x(t)| \le -\frac{\varepsilon}{2}, \qquad \limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \le -\frac{\varepsilon}{2}. \tag{4.13}$$

In the next section, condition (2.6) will be replaced by a more general condition for stochastic functional differential equation.

5. Stochastic Functional Differential Equation

Let $u(x_t) = 0$. Then (2.1) reduces to

$$dx(t) = f(x_t,t)dt + g(x_t,t)dw(t). \tag{5.1}$$

This is a stochastic functional differential equation. In this section, we will give the corresponding results for stochastic functional differential equation. We will also see that the conditions are more general.

Define an operator LV from $C([-\tau,0];R^n)\times R_+$ to R by

$$LV(\varphi,t)=V_t(\varphi(0),t)+V_x(\varphi(0),t)f(\varphi,t)+\frac{1}{2}\mathrm{trace}\Big(g^T(\varphi,t)V_{xx}(\varphi(0),t)g(\varphi,t)\Big). \tag{5.2}$$

We impose the following assumption which is more general than (H3).

(H3′) There are two functions $V\in C^{2,1}(R^n\times[-\tau,+\infty);R_+)$ and $U\in C(R^n\times[-\tau,+\infty);R_+)$ as well as two positive constants λ_1,λ_2 and a probability measure μ on $[-\tau,0]$ such that

$$\lim_{|x|\to\infty}\inf_{0\le t<\infty}V(x,t)=\infty, \tag{5.3}$$

$$LV(\varphi,t)\le\lambda_1\left[1+V(\varphi(0),t)+\int_{-\tau}^{0}(V(\varphi(\theta),t+\theta)+U(\varphi(\theta),t+\theta))\mathrm{d}\mu(\theta)\right]-\lambda_2U(\varphi(0),t) \tag{5.4}$$

for all $-\tau\le\theta\le0,(\varphi,t)\in R^n\times R_+$.

Theorem 5.1. *Assume that (H1) and (H3′) hold. Then for any initial condition $\xi\in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$, there exists a unique global solution $x(t)$ of* (5.1) *on $t\in[-\tau,\infty)$. Moreover, the solution has the properties that*

$$EV(x(t),t)<\infty,\qquad E\int_0^t U(x(s),s)\mathrm{d}s<\infty \tag{5.5}$$

for any $t\ge0$.

Proof. Since the proof is similar to Theorem 2.2, we will only outline the proof. It is clear that for any initial data $\xi\in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$, there is a unique maximal local solution $x(t)$ on $t\in[-\tau,\tau_e)$, where τ_e is the explosion time [1]. Let $k_0>0$ be sufficiently large for

$$\frac{1}{k_0}<\min_{-\tau\le t\le0}|x(t)|<\max_{-\tau\le t\le0}|x(t)|<k_0. \tag{5.6}$$

Define the stopping time

$$\tau_k=\inf\{t\in[0,\tau_e]:|x(t)|\,\overline{\in}\,I_k\},\quad I_k\equiv\left(\frac{1}{k},k\right),\ k\ge k_0, \tag{5.7}$$

where throughout this paper, we set $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty sets). Clearly, τ_k is increasing as $k \to \infty$. Denote $\tau_\infty = \lim_{k\to\infty}\tau_k$, $\tau_\infty \le \tau_e$ a.s. We will show that $\tau_\infty = \infty$ a.s., which implies $\tau_e = \infty$ a.s. By Itô formula and (5.4), for any $k \ge k_0$ and $t \in [0,\tau]$, we obtain

$$EV(x(\tau_k \wedge t), \tau_k \wedge t) \le C_1 + \lambda_1 \int_0^t EV(x(\tau_k \wedge s), \tau_k \wedge s)\mathrm{d}s - \lambda_2 \int_0^t EU(x(\tau_k \wedge s), \tau_k \wedge s)\mathrm{d}s, \tag{5.8}$$

where $C_1 = V(x(0),0) + \lambda_1\tau + \lambda_1 E\int_{-\tau}^{\tau}[V(x(s),s) + U(x(s),s)]\mathrm{d}s$. For any $k \ge k_0$, the Gronwall inequality yields

$$E\left[I_{\{\tau_k \le \tau\}} V(x(\tau_k \wedge \tau), \tau_k \wedge \tau)\right] \le EV(x(\tau_k \wedge \tau), \tau_k \wedge \tau) \le C_1 \mathrm{e}^{\lambda_1 \tau}, \quad \forall k \ge k_0, \tag{5.9}$$

which implies

$$EV(x(t),t) \le C_1 \mathrm{e}^{\lambda_1\tau}, \quad 0 \le t \le \tau. \tag{5.10}$$

Defining $\mu_k = \inf_{|x|\ge k,\ 0\le t<\infty} V(x,t)$ for $k \ge k_0$, according to (5.3), then

$$\mu_k P(\tau_k \le \tau) \le C_1 \mathrm{e}^{\lambda_1\tau}. \tag{5.11}$$

Condition (5.3) implies $\lim_{k\to\infty}\mu_k = \infty$. Letting $k \to \infty$ in (5.11), then $P(\tau_\infty \le \tau) = 0$, namely,

$$P(\tau_\infty > \tau) = 1. \tag{5.12}$$

Moreover, setting $t = \tau$ in (5.8), we may obtain that

$$\lambda_2 E\int_0^{\tau_k\wedge\tau} U(x,s)\mathrm{d}s \le C_1 + \lambda_1 E\int_0^{\tau_k\wedge\tau} V(x(s),s)\mathrm{d}s \le C_1 + \lambda_1 C_1 \tau \mathrm{e}^{\lambda_1\tau}, \tag{5.13}$$

that is,

$$E\int_0^{\tau} U(x,s)\mathrm{d}s \le \frac{C_1}{\lambda_2}\left(1 + \lambda_1\tau\mathrm{e}^{\lambda_1\tau}\right) < \infty. \tag{5.14}$$

Let us now proceed to prove $\tau_\infty > 2\tau$ a.s. given that we have shown (5.10)–(5.14). For any $k \ge k_0$ and $t \in [0, 2\tau]$, we get

$$EV(x(\tau_k \wedge t), \tau_k \wedge t) \le C_2 + \lambda_1 E\int_0^{\tau_k\wedge t} V(x(s),s)\mathrm{d}s - \lambda_2 E\int_0^{\tau_k\wedge t} U(x(s),s)\mathrm{d}s, \tag{5.15}$$

where

$$C_2 = V(x(0)) + 2\lambda_1\tau + \lambda_1 E\int_{-\tau}^{2\tau}[V(x(s),s) + U(x(s),s)]\mathrm{d}s < \infty. \tag{5.16}$$

By the Gronwall inequality and (5.8), we have

$$EV(x(\tau_k \wedge t), \tau_k \wedge t) \le C_2 e^{2\lambda_1 \tau}, \quad 0 \le t \le 2\tau,\ k \ge k_0. \tag{5.17}$$

In particular,

$$EV(x(\tau_k \wedge 2\tau), \tau_k \wedge 2\tau) \le C_2 e^{2\lambda_1 \tau}, \quad \forall k \ge k_0. \tag{5.18}$$

This implies

$$\mu_k P(\tau_k \le 2\tau) \le C_2 e^{2\lambda_1 \tau}. \tag{5.19}$$

Letting $k \to \infty$, by (5.3), then $P(\tau_\infty \le 2\tau) = 0$, that is,

$$P(\tau_\infty > 2\tau) = 1. EV(x(t), t) \le C_2 e^{2\lambda_1 \tau}, \quad 0 \le t \le 2\tau. \tag{5.20}$$

By (5.8), we may also obtain that

$$\lambda_2 E \int_0^{\tau_k \wedge 2\tau} U(x,t)\mathrm{d}t \le C_2 + \lambda_1 E \int_0^{\tau_k \wedge 2\tau} V(x(t),t)\mathrm{d}t, \tag{5.21}$$

that is,

$$E \int_0^{2\tau} U(x,t)\mathrm{d}t \le \frac{C_2}{\lambda_2}\left(1 + 2\lambda_1 \tau e^{2\lambda_1 \tau}\right) < \infty. \tag{5.22}$$

Repeating this procedure, we can show that, for any integer $i \ge 1, \tau_\infty > i\tau$ a.s and $EV(x) \le C_i e^{i\lambda_1 \tau},\ 0 \le t < i\tau$, and

$$E \int_0^{i\tau} U(x,t)\mathrm{d}t \le \frac{C_i}{\lambda_2}\left(1 + i\lambda_1 \tau e^{i\lambda_1 \tau}\right), \tag{5.23}$$

where

$$C_i = V(x(0),0) + \lambda_1 E \int_{-\tau}^{i\tau} [1 + V(x,t) + U(x,t)]\mathrm{d}t < \infty. \tag{5.24}$$

We must therefore have $\tau_\infty = \infty$ a.s. as well as the required assertion (5.5). □

Theorem 5.2. *Assume that (H1) and (H3′) hold except* (5.4) *which is replaced by*

$$\begin{aligned} LV(\varphi,t) \le{} & \mu_1 - \mu_2 V(\varphi(0),t) + \mu_3 \int_{-\tau}^{0} V(\varphi, t+\theta)\mathrm{d}\eta_1(\theta) - \mu_4 U(\varphi(0),t) \\ & + \mu_5 \int_{-\tau}^{0} U(\varphi, t+\theta)\mathrm{d}\eta_2(\theta) \end{aligned} \tag{5.25}$$

for all $(\varphi, t) \in R^n \times R_+, -\tau \le \theta \le 0$, *where* $\mu_1 \ge 0, \mu_2 > \mu_3 \ge 0, \mu_4 > \mu_5 > 0$. *Then for any initial data, the global solution* $x(t)$ *to* (5.1) *has the property that*

$$\limsup_{t \to \infty} EV(x(t), t) \le \frac{\mu_1}{\varepsilon}, \tag{5.26}$$

where $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$, *while* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *are the unique roots to the following equations:*

$$\mu_2 = \mu_3 e^{\varepsilon_1 \tau}, \qquad \mu_4 = \mu_5 e^{\varepsilon_2 \tau}, \tag{5.27}$$

respectively. If $\mu_1 = 0$, *then*

$$\limsup_{t \to \infty} \frac{1}{t} \ln EV(x(t), t) \le -\varepsilon, \qquad \int_0^\infty EU(x(t), t) dt < \infty. \tag{5.28}$$

Proof. Since the proof is similar to Theorem 3.1, we will only outline the proof. We first observe that (5.25) implies (5.4) if we set $\lambda_1 = \mu_1 \vee \mu_3 \vee \mu_5$ and $\lambda_2 = \mu_4$. So for any initial data, (5.1) has a unique global solution $x(t)$ on $t \ge -\tau$, which has the properties (5.5). Based on these properties, we can apply the Itô formula and condition (5.4) to obtain that for any $t \ge 0$,

$$\begin{aligned} e^{\varepsilon t} EV(x(t), t) \le{}& V(x(0), 0) + \frac{\mu_1 e^{\varepsilon t}}{\varepsilon} - \mu_2 E \int_0^t e^{\varepsilon s} V(x(s), s) ds + \mu_3 E \int_0^t \int_{-\tau}^0 e^{\varepsilon s} V(x_s, s + \theta) d\eta_1(\theta) ds \\ & - \mu_4 E \int_0^t e^{\varepsilon s} U(x, s) ds + \mu_5 E \int_0^t \int_{-\tau}^0 e^{\varepsilon s} U(x_s, s + \theta) d\eta_2(\theta) ds. \end{aligned} \tag{5.29}$$

Applying for (3.8) and (3.9), similarly, we have

$$\begin{aligned} e^{\varepsilon t} EV(x(t), t) \le{}& V(x(0), 0) + \frac{\mu_1 e^{\varepsilon t}}{\varepsilon} - \mu_2 E \int_0^t e^{\varepsilon s} V(x(s), s) ds \\ & + \mu_3 e^{\varepsilon \tau} E \int_{-\tau}^0 e^{\varepsilon s} V(x(s), s) ds + \mu_3 e^{\varepsilon \tau} E \int_0^t e^{\varepsilon s} V(x(s), s) ds \\ & - \mu_4 E \int_0^t e^{\varepsilon s} U(x, s) ds + \mu_5 e^{\varepsilon \tau} E \int_{-\tau}^0 e^{\varepsilon s} U(x(s), s) ds + \mu_5 e^{\varepsilon \tau} E \int_0^t e^{\varepsilon (s)} U(x(s), s) ds \\ ={}& \overline{C} + \frac{\mu_1 e^{\varepsilon t}}{\varepsilon} - \left(\mu_2 - \mu_3 e^{\varepsilon \tau}\right) E \int_0^t e^{\varepsilon s} V(x(s), s) ds - \left(\mu_4 - \mu_5 e^{\varepsilon \tau}\right) E \int_0^t e^{\varepsilon s} U(x, s) ds, \end{aligned} \tag{5.30}$$

where $\overline{C} = V(x(0),0)+e^{\varepsilon\tau}E\int_{-\tau}^{0} e^{\varepsilon s}(\mu_3 V(x(s),s)+\mu_5 U(x(s),s))ds$. It is clear that, for $\varepsilon \le \varepsilon_1 \wedge \varepsilon_2$, we have $\mu_2 - \mu_3 e^{\varepsilon\tau} \ge 0, \mu_4 - \mu_5 e^{\varepsilon\tau} \ge 0$; hence,

$$e^{\varepsilon t} EV(x(t),t) \le \overline{C} + \frac{\mu_1 e^{\varepsilon t}}{\varepsilon}, \tag{5.31}$$

that is,

$$EV(x(t),t) \le \overline{C}e^{-\varepsilon t} + \frac{\mu_1}{\varepsilon}, \quad \forall t \ge 0. \tag{5.32}$$

Therefore

$$\limsup_{t\to\infty} EV(x(t),t) \le \frac{\mu_1}{\varepsilon}. \tag{5.33}$$

When $\mu_1 = 0$, then $EV(x(t),t) \le \overline{C}e^{-\varepsilon t}$, for all $t \ge 0$, that is,

$$\limsup_{t\to\infty} \frac{1}{t} \log EV(x(t),t) \le -\varepsilon. \tag{5.34}$$

On the other hand, when $\mu_1 = 0$, we may show easily that

$$\begin{aligned} EV(x(t),t) &= V(x(0),0) + E\int_{-\tau}^{0} (\mu_3 V(x(s),s) + \mu_5 U(x(s),s))ds \\ &\quad - (\mu_2 - \mu_3)E\int_0^t V(x(s),s)ds - (\mu_4 - \mu_5)E\int_0^t U(x(s),s)ds. \end{aligned} \tag{5.35}$$

Recalling that $\mu_2 = \mu_3 e^{\varepsilon_1\tau} > \mu_3, \mu_4 = \mu_5 e^{\varepsilon_2\tau} > \mu_5$, the Fubini theorem yields

$$\int_0^t EU(x,s) \le \frac{1}{\mu_4 - \mu_5}\left[V(x(0),0) + E\int_{-\tau}^{0} (\mu_3 V(x(s),s) + \mu_5 U(x(s),s))ds\right] < \infty. \tag{5.36}$$

The proof is complete. □

Theorem 5.3. *Assume that (H1) and (H3′) hold except* (5.3) *which is replaced by*

$$\begin{aligned} LV(\varphi,t) &\le -\mu_2 V(\varphi(0),t) + \mu_3 \int_{-\tau}^{0} V(\varphi,t+\theta)d\eta_1(\theta) - \mu_4 U(\varphi(0),t) \\ &\quad + \mu_5 \int_{-\tau}^{0} U(\varphi,t+\theta)d\eta_2(\theta) \end{aligned} \tag{5.37}$$

for all $(\varphi, t) \in R^n \times R_+, -\tau \le \theta \le 0$, *where* $\mu_2 > \mu_3 \ge 0, \mu_4 > \mu_5 > 0$. *Then for any initial data, the unique global solution* $x(t)$ *to* (5.1) *has the property that*

$$\limsup_{t\to\infty} \frac{1}{t} \ln V(x(t),t) \le -\varepsilon, \qquad \int_0^\infty U(x(t),t)\mathrm{d}t < \infty, \tag{5.38}$$

where $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$, *while* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *are the unique roots to the following equations*

$$\mu_2 = \mu_3 e^{\varepsilon_1 \tau}, \qquad \mu_4 = \mu_5 e^{\varepsilon_2 \tau}, \tag{5.39}$$

respectively.

Proof. It is clear that (5.1) has a unique global solution $x(t)$ on $t \ge -\tau$, which has the properties (2.8). For any $t \ge 0$, we can obtain

$$\begin{aligned} \mathrm{e}^{\varepsilon t} EV(x(t),t) \le V(x(0),0) &- \mu_2 \int_0^t \mathrm{e}^{\varepsilon s} V(x(s),s)\mathrm{d}s + \mu_3 \int_0^t \int_{-\tau}^0 \mathrm{e}^{\varepsilon s} V(x_s, s+\theta)\mathrm{d}\eta_1(\theta)\mathrm{d}s \\ &- \mu_4 \int_0^t \mathrm{e}^{\varepsilon s} U(x,s)\mathrm{d}s + \mu_5 \int_0^t \int_{-\tau}^0 \mathrm{e}^{\varepsilon s} U(x_s, s+\theta)\mathrm{d}\eta_2(\theta)\mathrm{d}s + M(t), \end{aligned} \tag{5.40}$$

where $M(t) = \int_0^t \mathrm{e}^{\varepsilon s} V_x(\tilde{x}(s),s) g(x, x_s, s)\mathrm{d}w(s)\mathrm{d}s$ is a real-valued continuous local martingale with $M(0) = 0$. Similar to Theorem 4.2,

$$\begin{aligned} \mathrm{e}^{\varepsilon t} V(x(t),t) &\le C - \left(\mu_2 - \mu_3 \mathrm{e}^{\varepsilon\tau}\right) \int_0^t \mathrm{e}^{\varepsilon s} V(x(s),s)\mathrm{d}s - \left(\mu_4 - \mu_5 \mathrm{e}^{\varepsilon\tau}\right) \int_0^t \mathrm{e}^{\varepsilon s} U(x,s)\mathrm{d}s + M(t) \\ &\le \text{const} + M(t). \end{aligned} \tag{5.41}$$

By Lemma 4.1, we have

$$\limsup_{t\to\infty} \mathrm{e}^{\varepsilon t} V(x(t),t) \le \infty \quad \text{a.s.} \tag{5.42}$$

The required result is obtained. □

6. Example

In the following, we will consider several examples to illustrates our ideas.

Example 6.1. Consider a one-dimensional SFDE

$$\mathrm{d}x(t) = x(t)\left[\left(a + b\sigma_1(x_t) - x^2(t)\right)\mathrm{d}t + c\sigma_2(x_t)\mathrm{d}w(t)\right], \tag{6.1}$$

where $w(t)$ is a one-dimensional Brownian motion, $a, b, c(b, c > 0)$ are bounded real numbers, and the functions $\sigma_1, \sigma_2 \in C([-\tau, 0]; R)$ having the property of

$$|\sigma_1(\varphi)| \vee |\sigma_2(\varphi)| \leq \kappa \int_{-\tau}^{0} |\varphi(\theta)| \mathrm{d}\mu(\theta) \quad \kappa \in (0,1). \tag{6.2}$$

Let $V(x) = x^2$. Then the corresponding operator $LV : R \times R \times R_+$ has the form

$$\begin{aligned} LV(x_t, t) &= 2x^2(t)\Big(a + b\sigma_1(x_t) - x^2(t)\Big) + c^2 x^2(t)\sigma_2^2(x_t) \\ &= 2ax^2(t) + 2bx^2(t)\sigma_1(x_t) - 2x^4(t) + c^2x^2(t)\sigma_2^2(x_t) \\ &\leq 2ax^2(t) + bx^4(t) + b\sigma_1^2(x_t) - 2x^4(t) + 0.5c^2x^4(t) + 0.5c^2\sigma_2^4(x_t) \\ &\leq 2ax^2(t) + b\sigma_1^2(x_t) + 0.5c^2\sigma_2^4(x_t) - \Big(2 - b - 0.5c^2\Big)x^4(t) \\ &\leq 2ax^2(t) + b\kappa^2 \int_{-\tau}^{0} x_t^2(\theta)\mathrm{d}\mu(\theta) + 0.5c^2\kappa^4 \int_{-\tau}^{0} x_t^4(\theta)\mathrm{d}\mu(\theta) - \Big(2 - b - 0.5c^2\Big)x^4(t), \end{aligned} \tag{6.3}$$

where $\lambda_1 = \max\{2a, b\kappa^2, 0.5c^2\kappa^4\}, \lambda_2 = 2 - b - 0.5c^2, U(x) = x^4$. If $2 - b - 0.5c^2 > 0$, then by Theorem 5.1, we can conclude that for any initial data $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R)$, there is a unique global solution $x(t)$ to (6.1) on $t \in [-\tau, \infty)$. Moreover, the solution has the properties that

$$E|x|^2 < \infty, \qquad E\int_0^t |x(s)|^4 \mathrm{d}s < \infty \tag{6.4}$$

for any $t \geq 0$. If $a < 0,\ 2 - b - 0.5c^2 > 0.5c^2\kappa^4 > 0, -2a > b\kappa^2 \geq 0, U(x) = x^4, \varepsilon_1 > 0$ and $\varepsilon_2 > 0$ will be the unique roots to the following equations:

$$-2a = b\kappa^2 e^{\varepsilon_1 \tau}, \qquad 2 - b - 0.5c^2 = 0.5c^2\kappa^4 e^{\varepsilon_2 \tau}, \tag{6.5}$$

respectively. Set $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$, by Theorem 5.2, we can conclude that the unique global solution of (6.1) has the property that

$$\limsup_{t\to\infty} \frac{1}{t} \ln E|x(t)|^2 \leq -\varepsilon, \qquad \limsup_{t\to\infty} \frac{1}{t} \ln |x(t)|^2 \leq -\varepsilon, \qquad \int_0^\infty Ex^4(t)\mathrm{d}t < +\infty. \tag{6.6}$$

If we choose $a = -2,\ b = 1,\ c = 1,\ \kappa = 0.5,\ \tau = 8$, then $\varepsilon_1 = 0.34657, \varepsilon_2 = 0.34657$, which implies

$$\limsup_{t\to\infty} \frac{1}{t} \ln E|x(t)|^2 < -0.34657. \tag{6.7}$$

Example 6.2. Consider a one-dimensional NSFDE

$$d(x(t) - \sigma_1(x_t)) = x(t)\left[\left(a - x^2(t)\right)dt + c\sigma_2(x_t)dw(t)\right], \tag{6.8}$$

where $w(t)$ is a one-dimensional Brownian motion and both a, c are bounded positive real numbers, $\sigma_1, \sigma_2 \in C([-\tau, 0]; R)$ having the property of

$$\left|\sigma_1(\varphi)\right| \vee \left|\sigma_2(\varphi)\right| \leq \kappa \int_{-\tau}^{0} \left|\varphi(\theta)\right| d\mu(\theta) \quad \kappa \in (0,1). \tag{6.9}$$

Let $V(x) = x^2$. Then the corresponding operator $LV : R \times R \times R_+$ has the form

$$\begin{aligned}
LV(x, x_t, t) &= 2(x(t) - \sigma_1(x_t))\left(a - x^2(t)\right)x(t) + c^2x^2(t)\sigma_2^2(x_t) \\
&= 2ax^2(t) - 2ax(t)\sigma_1(x_t) - 2x^4(t) + 2x^3(t)\sigma_1(x_t) + c^2x^2(t)\sigma_2^2(x_t) \\
&\leq 2ax^2(t) + ax^2(t) + a\sigma_1^2(x_t) - 2x^4(t) + 2\left(x^4(t)\right)^{3/4}\left(\sigma_1^4(x_t)\right)^{1/4} + 0.5c^2\left(x^4(t) + \sigma_2^4(x_t)\right) \\
&\leq 2ax^2(t) + ax^2(t) + a\sigma_1^2(x_t) - 2x^4(t) + 1.5x^4(t) + 0.5\sigma_1^4(x_t) + 0.5c^2\left(x^4(t) + \sigma_2^4(x_t)\right) \\
&= 3ax^2(t) + a\kappa^2 \int_{-\tau}^{0} x_t^2(\theta)d\mu(\theta) + \left(0.5 + 0.5c^2\right)\kappa^4 \int_{-\tau}^{0} x_t^4(\theta)d\mu(\theta) - 0.5\left(1 - c^2\right)x^4(t) \\
&= \lambda_1\left(x^2(t) + \int_{-\tau}^{0} x_t^2(\theta)d\mu(\theta) + \int_{-\tau}^{0} x_t^4(\theta)d\mu(\theta)\right) - \lambda_2 x^4(t),
\end{aligned} \tag{6.10}$$

where $\lambda_1 = \max\{3a, a\kappa^2, (0.5 + 0.5c^2)\kappa^4\}, \lambda_2 = 0.5(1 - c^2), U(x) = x^4$. By Theorem 2.2, we can conclude that for any initial data, there is a unique global solution $x(t)$ to (6.8) on $t \in [-\tau, \infty)$. Moreover, the solution has the properties that for any $t \geq 0$

$$E|x(t)|^2 < \infty, \qquad E\int_0^t |x(s)|^4 ds < \infty. \tag{6.11}$$

Example 6.3. Consider a one-dimensional NSFDE

$$d[x(t) - \sigma_1(x_t)] = \left[(a - b)x(t) - a\sigma_1(x_t) - x(t)^3\right]dt + c\sigma_2(x_t)dw(t), \tag{6.12}$$

where $w(t)$ is a one-dimensional Brownian motion, $a, b, c (b, c > 0)$ are real numbers, $\sigma_1, \sigma_2 \in C([-\tau, 0]; R)$ having the property of

$$\left|\sigma_1(\varphi)\right| \vee \left|\sigma_2(\varphi)\right| \leq \kappa \int_{-\tau}^{0} \left|\varphi(\theta)\right| d\mu(\theta) \quad \kappa \in (0,1). \tag{6.13}$$

Then, the corresponding operator LV has the form

$$\begin{aligned}
&LV(x,x_t,t)\\
&= 2(x(t)-\sigma_1(x_t))\left[(a-b)x(t)-a\sigma_1(x_t)-x(t)^3\right]+c^2\sigma_2^2(x_t)\\
&= 2a(x(t)-\sigma_1(x_t))^2-2bx^2(t)+2bx(t)\sigma_1(x_t)-2x^4(t)+2x^3(t)\sigma_1(x_t)+c^2\sigma_2^2(x_t)\\
&\le 2a(x(t)-\sigma_1(x_t))^2-2bx^2(t)+bx^2(t)+b\sigma_1^2(x_t)-2x^4(t)+2\left(x^4(t)\right)^{3/4}\left(\sigma_1^4(x_t)\right)^{1/4}+c^2\sigma_2^2(x_t)\\
&\le 2a(x(t)-\sigma_1(x_t))^2-bx^2(t)+b\sigma_1^2(x_t)-2x^4(t)+1.5x^4(t)+0.5\sigma_1^4(x_t)+c^2\sigma_2^2(x_t)\\
&\le -bx^2(t)+\left(b+c^2\right)\kappa^2\int_{-\tau}^0 x_t^2(\theta)\mathrm{d}\mu(\theta)-0.5x^4(t)+0.5\kappa^4\int_{-\tau}^0 x_t^4(\theta)\mathrm{d}\mu(\theta)+2a(x(t)-\sigma_1(x_t))^2,
\end{aligned} \tag{6.14}$$

where the first and second inequalities using the elementary inequality $u^\alpha v^{1-\alpha} \le (\alpha u + (1-\alpha)v)$. If $a<0, b>(b+c^2)\kappa^2 \ge 0, \varepsilon_1>0$ and $\varepsilon_2>0$ be the unique roots to the following equations,

$$b=\left(b+c^2\right)\kappa^2 e^{\varepsilon_1\tau}, \qquad 0.5=0.5\kappa^4 e^{\varepsilon_2\tau}, \tag{6.15}$$

respectively. And set $\varepsilon=-2a\wedge\varepsilon_1\wedge\varepsilon_2\wedge\tau^{-1}\ln\kappa^{-2}$, by Theorem 3.1, we can conclude that the unique global solution of (6.12) has the property that

$$\limsup_{t\to\infty}\frac{1}{t}\ln E|x(t)|^2\le-\varepsilon, \qquad \limsup_{t\to\infty}\frac{1}{t}\ln|x(t)|^2\le-\varepsilon. \tag{6.16}$$

If we let $a=-2, b=0.5=c, \kappa=0.5, \tau=0.9$, then

$$0.5=2.6666e^{0.9\varepsilon_1}, \qquad 0.5=0.5^5 e^{0.9\varepsilon_2}, \tag{6.17}$$

which give their roots $\varepsilon_1=1.0898, \varepsilon_2=3.08065$, respectively, and $\tau^{-1}\ln\kappa^{-2}=0.10168$,

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left(E|x(t)|^2\right)\le-0.10168, \qquad \limsup_{t\to\infty}\frac{1}{t}\ln|x(t)|^2\le-0.10168. \tag{6.18}$$

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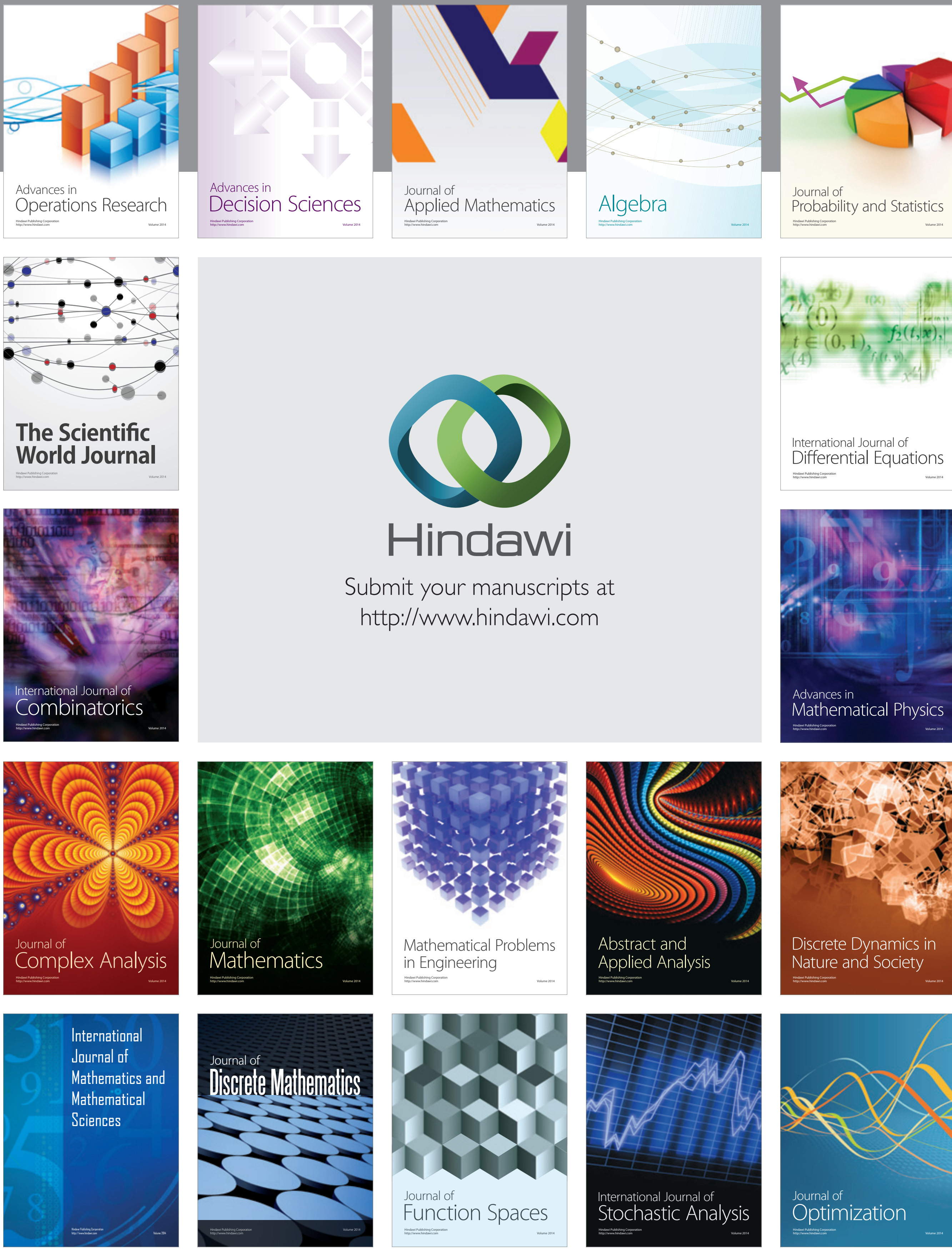
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