Arab. J. Math. (2016) 5:69–75
DOI 10.1007/s40065-015-0141-x

Arabian Journal of Mathematics

CrossMark

Gopal Datt · Neelima Ohri

Commutativity of slant weighted Toeplitz operators

Received: 2 April 2015 / Accepted: 22 November 2015 / Published online: 17 December 2015

Abstract For a positive integer $k \geq 2$, the kth-order slant weighted Toeplitz operator $U^{\beta}_{k,\phi}$ on $L^2(\beta)$ with $\phi \in L^{\infty}(\beta)$ is defined as $U^{\beta}_{k,\phi} = W_k M^{\beta}_{\phi}$, where $W_k e_n(z) = \frac{\beta_m}{\beta_{km}} e_m(z)$ if $n = km$, $m \in \mathbb{Z}$ and $W_k e_n(z) = 0$ if $n \neq km$. The paper derives relations among the symbols of two kth-order slant weighted Toeplitz operators so that their product is a kth-order slant weighted Toeplitz operator. We also discuss the compactness and the case for two kth-order slant weighted Toeplitz operators to commute essentially.

Mathematics Subject Classification 47B37 · 47B35

الملخص

يعرف مؤثر توبلتز الموزون المائل من الدرجة k، حيث $k \geq 2$ عدد صحيح موجب، على $L^2(\beta)$ حيث $\phi \in L^{\infty}(\beta)$ على أنه $U^{\beta}_{k,\phi} = W_k M^{\beta}_{\phi}$، حيث $W_k e_n(z) = \frac{\beta_m}{\beta_n} e_m(z)$ إذا كانت $n = km, m \in Z$ وَ $W_k e_n(z) = 0$ إذا كانت $n \neq km$. تشتق الورقة علاقة بين رموز مؤثري توبلوتز موزونين مائلين من الدرجة k بحيث يكون حاصل ضربهما مؤثر توبلتز موزونًا مائلًا من الدرجة k. نناقش التراص والحالة التي يكون فيها هذان المؤثران إبداليين بشكل ضروري.

1 Introduction

Toeplitz operators, introduced by Toeplitz [21] in the year 1911, arise in many applications, constitute one of the most important classes of non self-adjoint operators. The ideas and methods prevailing in the field of Toeplitz operators are a fascinating illustration of the fruitful interplay between operator theory, complex analysis and a Banach algebra (see [1,7–10,15,16]). The study of Toeplitz operators becomes more demanding with the inception of the notion of slant Toeplitz operators by Ho [17] in 1995, which has widely appeared in connection with the wavelet theory, having the property that their matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Toeplitz operators. The study in this direction is enhanced with the introduction of various new classes of operators over various function spaces, like, kth-order slant Toeplitz operators, essentially slant Toeplitz operators, λ-Toeplitz and

G. Datt (✉)
Department of Mathematics, PGDAV College, University of Delhi, Delhi 110065, India
E-mail: gopal.d.sati@gmail.com

N. Ohri
Department of Mathematics, University of Delhi, Delhi 110007, India
E-mail: neelimaohri1990@gmail.com

essentially λ-Toeplitz operators, $\lambda \in \mathbb{C}$, the set of all complex numbers (see the references [2–6,11,17] and the references therein). Shields [20], during his study of multiplication operators and the weighted shift operators, discussed weighted sequence spaces which have the tendency to cover Hardy spaces, Bergman spaces and Dirichlet spaces. We begin with the following notational familiarity needed in the paper.

Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1, r \le \frac{\beta_n}{\beta_{n+1}} \le 1$ for $n \ge 0$ and $r \le \frac{\beta_n}{\beta_{n-1}} \le 1$ for $n \le 0$, for some $r > 0$. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be the formal Laurent series (whether or not the series converges for any values of z). Define $\|f\|_\beta$ as

$$\|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2.$$

The space $L^2(\beta)$ consists of all $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ for which $\|f\|_\beta < \infty$. The space $L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n \, \overline{b}_n {\beta_n}^2,$$

for $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $g(z) = \sum_{n=-\infty}^{\infty} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\beta)$.

The collection of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$, is denoted by $H^2(\beta)$. $H^2(\beta)$ is a subspace of $L^2(\beta)$.

Let $L^\infty(\beta)$ denote the set of formal Laurent series $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \le c\|f\|_\beta$ for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \le c\|f\|_\beta \quad \text{for each } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$. $H^\infty(\beta)$ denotes the set of formal power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$. In [20], Shields discussed various properties of the operator $M_\phi^\beta (f \mapsto \phi f)$ on $L^2(\beta)$ with the symbol $\phi \in L^\infty(\beta)$, which we call as weighted Laurent operator. We refer [20] as well as the references therein, for the details of the spaces $L^2(\beta)$, $H^2(\beta)$, $L^\infty(\beta)$ and $H^\infty(\beta)$. If $\phi \in L^\infty(\beta)$ is given by $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ then for an integer $k \ge 2$, we define $\phi(z^k)$ as $\phi(z^k) = \sum_{n=-\infty}^{\infty} a_n z^{kn}$. If the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded, then $\phi(z^k) \in L^\infty(\beta)$ [14].

Let $P^\beta : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. Lauric [18], in the year 2005, discussed the notion of weighted Toeplitz operator $T_\phi^\beta = P^\beta M_\phi^\beta$ on $H^2(\beta)$. The operators of the kind $S_\phi^\beta = W M_\phi^\beta$ on $L^2(\beta)$, where W is the operator on $L^2(\beta)$ given by $W e_{2n} = \frac{\beta_n}{\beta_{2n}} e_n$ and $W e_{2n-1} = 0$ for $n \in \mathbb{Z}$, are discussed in [4] and are named as slant weighted Toeplitz operators. For a positive integer $k \ge 2$, let W_k be the operator on $L^2(\beta)$ given by

$$W_k e_n(z) = \begin{cases} \frac{\beta_m}{\beta_{km}} e_m(z) & \text{if } n = km \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

A kth-order slant weighted Toeplitz operator $U_{k,\phi}^\beta$ on $L^2(\beta)$ with $\phi \in L^\infty(\beta)$ is defined as $U_{k,\phi}^\beta = W_k M_\phi^\beta$ (see [5]). The second-order slant weighted Toeplitz operators $U_{2,\phi}^\beta$ are nothing but the slant weighted Toeplitz operators (denoted by S_ϕ^β).

Liu and Lu [19] discussed the following questions about the slant Toeplitz operators.

(1) What is the product of slant Toeplitz operators?
(2) When do slant Toeplitz operators with different orders commute?
(3) When do slant Toeplitz operators with different orders essentially commute?

Motivated by the direction of study made by Liu and Lu [19] along with mathematicians whose papers are referred in the paper, we discuss the above-mentioned questions for slant weighted Toeplitz operators on $L^2(\beta)$. If we assume $\beta_n = 1$ for each n, then our results provide results for kth-order slant Toeplitz operators, quite of which are proved in [19]. We derive a relation among the symbols of two kth-order slant weighted Toeplitz operators on $L^2(\beta)$ so that their product is a kth-order slant weighted Toeplitz operator on $L^2(\beta)$.

The algebra of all bounded operators on the Hilbert space $L^2(\beta)$ is denoted by $\mathfrak{B}(L^2(\beta))$. We say that operators A and B essentially commute if $AB - BA$ is a compact operator.

2 Commutativity

We recall the following definitions.

Definition 2.1 [4] A slant weighted Toeplitz operator S_ϕ^β on $L^2(\beta)$ is defined as $S_\phi^\beta = WM_\phi^\beta$, where M_ϕ^β is the weighted Laurent operator on $L^2(\beta)$ induced by $\phi \in L^\infty(\beta)$ and $We_{2n} = \frac{\beta_n}{\beta_{2n}} e_n$ and $We_{2n-1} = 0$ for $n \in \mathbb{Z}$.

Definition 2.2 [5] A kth-order slant weighted Toeplitz operator $U_{k,\phi}^\beta$, $k \geq 2$ induced by $\phi \in L^\infty(\beta)$, is an operator on $L^2(\beta)$ defined as $U_{k,\phi}^\beta = W_k M_\phi^\beta$, where W_k is the operator on $L^2(\beta)$ given by

$$W_k e_n(z) = \begin{cases} \frac{\beta_m}{\beta_{km}} e_m(z) & \text{if } n = km \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Then, if $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, for each integer j,

$$U_{k,\phi}^\beta e_j = \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{nk-j} \beta_n e_n \quad \text{and} \quad U_{k,\phi}^{\beta *} e_j = \beta_j \sum_{n=-\infty}^{\infty} \overline{a}_{jk-n} \frac{e_n}{\beta_n}.$$

It is evident to see that $\|W_k\| = \sup_n \frac{\beta_n}{\beta_{kn}} \leq 1$ and the adjoint W_k^* of W_k is given by $W_k^* e_n(z) = \frac{\beta_n}{\beta_{kn}} e_{kn}(z)$ for $n \in \mathbb{Z}$. As a consequence, $\|U_{k,\phi}^\beta\| \leq \|\phi\|_\infty$. It is apparent to see that the doubly infinite matrix $[\lambda_{i,j}]_{i,j\in\mathbb{Z}}$ of the kth-order slant weighted Toeplitz operator $U_{k,\phi}^\beta$ with respect to the standard orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ of $L^2(\beta)$ satisfies

$$\frac{\beta_j}{\beta_i} \lambda_{i,j} = \frac{\beta_{j+k}}{\beta_{i+1}} \lambda_{i+1,j+k},$$

where $\lambda_{i,j} = \langle U_{k,\phi}^\beta e_j, e_i \rangle$, for all $i, j \in \mathbb{Z}$. If k_1 and k_2 are integers satisfying $2 \leq k_1 \leq k_2$ then $\|W_{k_2}\| = \sup_n \frac{\beta_n}{\beta_{nk_2}} \leq \sup_n \frac{\beta_n}{\beta_{nk_1}} = \|W_{k_1}\| \leq 1$. Further, it is easy to prove the following.

Proposition 2.3 *Let k_1 and $k_2(\geq 2)$ be two integers. Then $W_{k_1} W_{k_2} = W_{k_1 k_2}$.*

Proof It is sufficient to prove that $(W_{k_1} W_{k_2})^* = W_{k_1 k_2}^*$. This follows as for each integer n,

$$\begin{aligned} (W_{k_1} W_{k_2})^*(e_n(z)) &= W_{k_2}^* W_{k_1}^*(e_n(z)) \\ &= \frac{\beta_n}{\beta_{nk_1}} \frac{\beta_{nk_1}}{\beta_{nk_1 k_2}} e_{nk_1 k_2}(z) \\ &= \frac{\beta_n}{\beta_{nk_1 k_2}} e_{nk_1 k_2}(z) = (W_{k_1 k_2})^*(e_n(z)). \end{aligned}$$

□

The following is obvious from here.

Corollary 2.4 *Let k_1 and k_2 be integers with $min\{k_1, k_2\} \geq 2$. Then $W_{k_1} W_{k_2} = W_{k_2} W_{k_1}$.*

We utilize the above results and the fact that if $\phi \in L^\infty(\beta)$ is such that $\phi(z^k) \in L^\infty(\beta)$ then $M_\phi^\beta W_k = W_k M_{\phi(z^k)}^\beta$ (see [6]) to conclude that the product of two slant weighted Toeplitz operators of different order is again a slant weighted Toeplitz operator.

In [14], it is shown that, in case $S_k(\beta) = \{f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^2(\beta) : f(z^k) = \sum_{n=-\infty}^{\infty} a_n z^{kn} \in L^2(\beta)\}$ is a closed subspace of $L^2(\beta)$ then $\phi(z^k) \in L^\infty(\beta)$ for each $\phi \in L^\infty(\beta)$. This helps to provide the following.

Theorem 2.5 *Let $k_1, k_2(\geq 2)$ be integers such that the space $S_{k_2}(\beta)$ is closed. Then $U_{k_1,\phi}^\beta U_{k_2,\psi}^\beta = U_{k_1k_2,\phi(z^{k_2})\psi}^\beta$.*

Proof Result follows as

$$\begin{aligned} U_{k_1,\phi}^\beta U_{k_2,\psi}^\beta &= W_{k_1} M_\phi^\beta W_{k_2} M_\psi^\beta \\ &= W_{k_1} W_{k_2} M_{\phi(z^{k_2})}^\beta M_\psi^\beta \\ &= W_{k_1k_2} M_{\phi(z^{k_2})\psi}^\beta = U_{k_1k_2,\phi(z^{k_2})\psi}^\beta. \end{aligned}$$

□

An immediate observation using Theorem 2.5 is the following.

Corollary 2.6 *Let $k_2(\geq 2)$ be such that $\{\frac{\beta_{nk_2}}{\beta_n}\}$ is bounded. Then $U_{k_1,\phi}^\beta U_{k_2,\psi}^\beta = U_{k_1k_2,\phi(z^{k_2})\psi}^\beta$.*

Proof The assumption of $\{\frac{\beta_{nk_2}}{\beta_n}\}$ is bounded ensures that the space $S_{k_2}(\beta)$ is closed (see [14]). Rest of the proof is immediate using Theorem 2.5. □

Now we can conclude from Theorem 2.5 that two kth-order slant weighted Toeplitz operators, in general, do not commute. In fact, we have $U_{k,\phi}^\beta U_{k,\psi}^\beta = U_{k^2,\phi(z^k)\psi}^\beta$ and $U_{k,\psi}^\beta U_{k,\phi}^\beta = U_{k^2,\psi(z^k)\phi}^\beta$.

It can be seen through a straightforward computation that if $U_{k,\phi}^\beta = U_{m,\psi}^\beta$ for $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n \in L^\infty(\beta)$ then for each $j \in \mathbb{Z}$, $\langle \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{nk-j}\beta_n e_n, e_0\rangle = \langle \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} b_{nm-j}\beta_n e_n, e_0\rangle$, which provides that $a_j = b_j$ for each $j \in \mathbb{Z}$. This implies that $\phi = \psi$. As a consequence, the mapping $\phi \mapsto U_{k,\phi}^\beta$ is one–one between $L^\infty(\beta)$ and $\mathfrak{B}(L^2(\beta))$, which is also shown in [6]. Linearity of this mapping implies that, in this case, $\phi = \psi = 0$. We extend this result to obtain the following.

Theorem 2.7 *Let $\phi \in L^\infty(\beta)$ and $m \neq k$. Then $U_{k,\phi}^\beta$ is a mth-order slant weighted Toeplitz operator on $L^2(\beta)$ if and only if $\phi = 0$.*

Proof Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ be such that $U_{k,\phi}^\beta$ is a mth-order slant weighted Toeplitz operator. Then for all $i, j \in \mathbb{Z}$, we have

$$\frac{\beta_j}{\beta_i}\langle U_{k,\phi}^\beta e_j, e_i\rangle = \frac{\beta_{j+m}}{\beta_{i+1}}\langle U_{k,\phi}^\beta e_{j+m}, e_{i+1}\rangle \tag{2.1}$$

As $U_{k,\phi}^\beta$ is a kth-order slant weighted Toeplitz operator, we have $\langle U_{k,\phi}^\beta e_j, e_i\rangle = \frac{\beta_i}{\beta_j}\frac{\beta_{j+k}}{\beta_{i+1}}\langle U_{k,\phi}^\beta e_{j+k}, e_{i+1}\rangle$ and $U_{k,\phi}^\beta e_j = \frac{1}{\beta_j}\sum_{n=-\infty}^{\infty} a_{nk-j}\beta_n e_n$. Now (2.1) gives that

$$\langle U_{k,\phi}^\beta e_j, e_i\rangle = \frac{\beta_i}{\beta_j}\frac{\beta_{j+mk}}{\beta_{i+k}}\langle U_{k,\phi}^\beta e_{j+mk}, e_{i+k}\rangle,$$

equivalently,

$$\frac{\beta_i}{\beta_j} a_{ki-j} = \frac{\beta_i}{\beta_j}\frac{\beta_{j+mk}}{\beta_{i+k}}\frac{\beta_{i+k}}{\beta_{j+mk}} a_{k(i+k)-j-mk}$$

for each $i, j \in \mathbb{Z}$. This yields that

$$a_{ki-j} = a_{k(i+k)-j-mk} \tag{2.2}$$

for each $i, j \in \mathbb{Z}$. From (2.2), we get that $a_0 = a_{tk|k-m|}, a_1 = a_{tk|k-m|+1}, \ldots, a_{k|k-m|-1} = a_{tk|k-m|+k|k-m|-1}$. But $\phi \in L^\infty(\beta) \subseteq L^2(\beta)$, so we have $\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n{}^2 < \infty$. Thus, $a_n \to 0$ as $n \to \infty$, and this helps us to conclude that $a_0 = a_1 = \ldots, a_{k|k-m|-1} = 0$. As a consequence of this (2.2) helps to provide that $a_n = 0$ for each $n \in \mathbb{Z}$, which gives that $\phi = 0$.

The converse is straightforward. □

Theorems 2.5 and 2.7 suggest the following without any extra effort.

Corollary 2.8 *If $U^\beta_{k,\phi} = U^\beta_{m,\psi}$ for $\phi, \psi \in L^\infty(\beta)$ and $m \neq k$, then $\phi = \psi = 0$.*

Corollary 2.9 *Let $\phi, \psi \in L^\infty(\beta)$. Then $U^\beta_{k_1,\phi} U^\beta_{k_2,\psi}$ is a kth-order slant weighted Toeplitz operator if and only if one of the following holds:*

(1) $k = k_1 k_2$.
(2) $\phi(z^{k_2})\psi = 0$, *when* $k \neq k_1 k_2$.

The following information about the kth-order slant weighted Toeplitz operator, which is shown in [3], can be justified from here on replacing k_1 and k_2 by k.

(1) Product of two kth-order slant weighted Toeplitz operators can not be a non-zero kth-order slant weighted Toeplitz operator.
(2) No non-zero kth-order slant weighted Toeplitz operator can be an idempotent.

Corollary 2.10 *Let $\phi, \psi \in L^\infty(\beta)$ be such that $\phi(z^{k_2}), \psi(z^{k_1}) \in L^\infty(\beta)$. Then we have the following.*

(1) $U^\beta_{k_1,\phi} U^\beta_{k_2,\psi} = 0$ *if and only if* $\phi(z^{k_2})\psi = 0$.
(2) $U^\beta_{k_1,\phi} U^\beta_{k_2,\psi} = U^\beta_{k_2,\psi} U^\beta_{k_1,\phi}$ *if and only* $\phi(z^{k_2})\psi - \psi(z^{k_1})\phi = 0$.

We need the following result [13, Lemma3.6] to proceed ahead.

Lemma 2.11 *If $\phi \in L^\infty(\beta)$ is such that $\phi(z^m) = \phi(z)$ for any integer $m \neq 1$, then ϕ is constant.*

Proposition 2.12 *Let $\{\frac{\beta_{nk}}{\beta_n}\}$ be bounded. Then for $\phi, \psi \in L^\infty(\beta)$, $M^\beta_\phi U^\beta_{k,\psi} = U^\beta_{k,\psi} M^\beta_\phi$ if and only if $\psi(z)\phi(z) = \psi(z)\phi(z^k)$. In particular, if ψ is invertible then $M^\beta_\phi U^\beta_{k,\psi} = U^\beta_{k,\psi} M^\beta_\phi$ if and only if ϕ is constant.*

Proof Under the hypothesis, $M^\beta_\phi U^\beta_{k,\psi} = U^\beta_{k,\phi(z^k)\psi}$ and $U^\beta_{k,\psi} M^\beta_\phi = U^\beta_{k,\phi\psi}$. Now result follows as the mapping $\phi \mapsto U^\beta_{k,\phi}$ is one–one. The forward part of the particular case follows using Lemma 2.11 and its converse is trivial. □

Continuing on in the same vein, we state another result, which provides a result analogous to Corollary 2.9.

Theorem 2.13 *The operator $W_k U^\beta_{k,\phi}$, $\phi \in L^\infty(\beta)$, is a kth-order slant weighted Toeplitz operator if and only if $\phi = 0$.*

Proof Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ be such that $W_k U^\beta_{k,\phi} = U^\beta_{k,\psi}$ for some $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n \in L^\infty(\beta)$. Then for all $i \in \mathbb{Z}$, $W_k U^\beta_{k,\phi} e_i = U^\beta_{k,\psi} e_i$. It gives that $a_{k^2 n-i} = b_{kn-i}$ for each $n, i \in \mathbb{Z}$. This on taking $n = 0$ provides $a_i = b_i$ and hence for each $i \in \mathbb{Z}$, we have $a_{k-i} = a_{k^2-i} = a_{k^3-i} = \cdots$. As a_n tends to zero as $n \to \infty$, we get that $\phi = 0$. The converse is apparent. □

If the sequence $\beta = \{\beta_n\}$ is such that $\{\frac{\beta_{kn}}{\beta_n}\}$ is bounded then for $\phi, \psi \in L^\infty(\beta)$, $U^\beta_{k,\phi} U^\beta_{k,\psi} = W_k U^\beta_{k,\phi(z^k)\psi}$ and hence Theorem 2.13 delivers the following.

Corollary 2.14 *Let $\{\frac{\beta_{nk}}{\beta_n}\}$ be bounded. A necessary and sufficient condition for the product $U^\beta_{k,\phi} U^\beta_{k,\psi}$ of two kth-order slant weighted Toeplitz operators $U^\beta_{k,\phi}$ and $U^\beta_{k,\psi}$ on $L^2(\beta)$ to be a kth-order slant weighted Toeplitz operator is that $\phi(z^k)\psi = 0$.*

3 Compact and essentially commuting operators

The aim of this section is to investigate the compact kth-order slant weighted Toeplitz operators on $L^2(\beta)$ and conditions for two kth-order slant weighted Toeplitz operators on $L^2(\beta)$ to essentially commute. To discuss the compactness of the operators, we consider the transformation V_k on $L^2(\beta)$ given by $V_k e_n = \frac{\beta_{kn}}{\beta_n} e_{kn}$ for each $n \in \mathbb{Z}$. It is a bounded operator if the sequence $\beta = \{\beta_n\}$ is such that $\{\frac{\beta_{kn}}{\beta_n}\}$ is bounded. For more properties and applications of this operator, we refer [13,14]. A simple computation using the definition of the symbols gives the following.

Proposition 3.1 *Let $\beta = \{\beta_n\}$ be such that $\{\frac{\beta_{kn}}{\beta_n}\}$ is bounded and $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$. Then the following hold.*

(1) *$U^\beta_{k,\phi} V_k$ is a weighted Laurent operator. In fact, $U^\beta_{k,\phi} V_k = M^\beta_\psi$, where $\psi(z) = \sum_{n=-\infty}^{\infty} a_{kn} z^n$.*

(2) *$U^\beta_{k,\phi(z^k)} V_k$ is a weighted Laurent operator. In fact, $U^\beta_{k,\phi(z^k)} V_k = M^\beta_\phi$.*

(3) *For each $1 \le i \le k-1$, $U^\beta_{k,\phi} M^\beta_{z^i} V_k$ is the weighted Laurent operator $M^\beta_{\psi_i}$, where $\psi_i(z) = \sum_{n=-\infty}^{\infty} a_{kn-i} z^n$.*

It is shown in [12, Lemma3.1] that a weighted Laurent operator M^β_ϕ is compact if and only if the inducing symbol is zero. This fact along with Proposition 3.1 immediately provides the following.

Theorem 3.2 *Let $\beta = \{\beta_n\}$ be such that $\{\frac{\beta_{kn}}{\beta_n}\}$ is bounded. The kth-order slant weighted Toeplitz operator $U^\beta_{k,\phi}$ on $L^2(\beta)$ is compact if only if $\phi = 0$.*

Proof Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be such that $U^\beta_{k,\phi}$ is compact. Then each $U^\beta_{k,\phi} V_k$ and $U^\beta_{k,\phi} M^\beta_{z^i} V_k$, $1 \le i \le k-1$ is compact. Using Proposition 3.1 and [12, Lemma3.1], we get that $\phi = 0$. Converse is obvious. □

Another result that we get immediately from here in the light of Theorem 2.5 (or Corollary 2.6) is the following.

Corollary 3.3 *Let $\beta = \{\beta_n\}$ be such that $\{\frac{\beta_{k_2 n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Let $\phi, \psi \in L^\infty(\beta)$. Then the following are equivalent.*

(1) *$U^\beta_{k_1,\phi} U^\beta_{k_2,\psi}$ is compact.*

(2) *$U^\beta_{k_1,\phi} U^\beta_{k_2,\psi} = 0$.*

(3) *$\phi(z^{k_2})\psi = 0$.*

We conclude this study with our next results which again follow using Theorem 3.2 and Corollary 2.10 and prove that the notion of essentially commuting and commuting coincides for kth-order slant weighted Toeplitz operator on $L^2(\beta)$.

Corollary 3.4 *Let $\phi, \psi \in L^\infty(\beta)$ be such that $\phi(z^{k_2}), \psi(z^{k_1}) \in L^\infty(\beta)$. Then the following are equivalent.*

(1) *$U^\beta_{k_1,\phi}$ and $U^\beta_{k_2,\psi}$ essentially commute.*

(2) *$U^\beta_{k_1,\phi}$ and $U^\beta_{k_2,\psi}$ commute.*

(3) *$\phi(z^{k_2})\psi - \phi\psi(z^{k_1}) = 0$.*

Corollary 3.5 *If $2 \le k_1 \le k_2$ and $\beta = \{\beta_n\}$ is such that $\{\frac{\beta_{k_2 n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Then the following are equivalent for $\phi, \psi \in L^\infty(\beta)$.*

(1) *$U^\beta_{k_1,\phi}$ and $U^\beta_{k_2,\psi}$ essentially commute.*

(2) *$U^\beta_{k_1,\phi}$ and $U^\beta_{k_2,\psi}$ commute.*

(3) *$\phi(z^{k_2})\psi - \phi\psi(z^{k_1}) = 0$.*

Acknowledgments The suggestions of the referee(s) for the improvement of the paper and the support of DST Grant ($DST/INSPIRE/03/2014/000442$) to the second author are gratefully acknowledged.

References

1. Abrahamse, M. B.: Toeplitz operators in multiply connected regions. Bull. Am. Math. Soc., **77**, 441–454 (1971)
2. Arora, S.C., Bhola, J.: Essentially slant Toeplitz operators. Banach J. Math. Anal. Iran **3**(2), 1–8 (2009)
3. Arora, S.C., Batra, R.: On generalized slant Toeplitz operators. Indian J. Math. **45**(2), 121–134 (2003)
4. Arora, S.C., Kathuria, R.: Slant weighted Toeplitz operators. Int. J. Pure Appl. Math. **62**(4), 433–442 (2010)
5. Arora, S.C., Kathuria, R.: Generalized slant weighted Toeplitz operators. World Acad. Sci. Eng. Technol. **75**, 1103–1106 (2011)
6. Arora, S.C., Kathuria, R.: On kth-order slant weighted Toeplitz operators. Sci. World J. **2013**, Article ID. 960853, 1–5 (2013)
7. Barr$\acute{\imath}a$, J, Halmos, P.R.: Asymptotic Toeplitz operators. Trans. Am. Math. Soc. **273**, 621–630 (1982)
8. Brown, A., Halmos, P.R.: Algebraic properties of Toeplitz operators. J. Reigne Angew. Math. **213**, 89–102 (1963)
9. Cohen, L.W., Dunford, N.: Transformations on sequence spaces. Duke Math. J. **3**, 689–701 (1937)
10. Conway, J.B.: A course in Functional analysis, 2nd edn. Springer-Verlag, New York (1990)
11. Datt, G., Aggarwal, R.: On a generalization of slant Toeplitz operators. Gen. Math. **21**(2), 57–69 (2013)
12. Datt, G, Porwal, D.K.: Weighted Hankel operators and matrices. Matematicki Vesnik **65**(3), 353–363 (2013)
13. Datt, G., Porwal, D.K.: On weighted slant Hankel operators. Filomat **27**(2), 227–243 (2013)
14. Datt, G., Porwal, D.K.: On a generalization of weighted slant Hankel operators. To appear in Mathematica Slovaca
15. Devinatz, A.: Toeplitz operators on H^2 spaces. Trans. Am. Math. Soc. **112**, 304–317 (1964)
16. Halmos, P.R.: A Hilbert Space Problem Book. Springer-Verlag, New York (1982)
17. Ho, M.C.: Properties of slant Toeplitz operators. Indiana Univ. Math. J. **45**(3), 843–862 (1996)
18. Lauric, V.: On a weighted Toepliz operator and its commutant. Int. J. Math. Math. Sci. **6**, 823–835 (2005)
19. Liu, C., Lu, Y.: Product and commutativity of slant Toeplitz operators. J. Math. Res. Appl. **33**(1), 122–126 (2013)
20. Shields, A.L.: Weighted shift operators and analytic function theory. Topics in Operator Theory, Math. Surveys, vol. 13, pp. 49–128. American Mathematical Society, Rhode Ireland (1974)
21. Toeplitz, O.: Zur theorie der quadratischen und bilinearan Formen von unendlichvielen, Veranderlichen. Math. Ann. **70**, 351–376 (1911)