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Ulam stability of a generalized reciprocal type functional equation in non-Archimedean fields

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Abstract In this paper, we obtain the solution of a new generalized reciprocal type functional equation in two variables and investigate its generalized Hyers–Ulam stability in non-Archimedean fields. We also present the pertinent stability results of Hyers–Ulam–Rassias stability, Ulam–Gavruta–Rassias stability and J. M. Rassias stability controlled by the mixed product-sum of powers of norms.

Mathematics Subject Classification 39B82 · 39B72

الملخص

نحصل في هذه الورقة على حل لمعادلة دالية جديدة مُعَمَّمة في متغيرين من النوع التعاكسي ونفحص استقرار هيرز – أولام المُعَمَّم الخاص بها في حقول غير أرخميدية. نقدم أيضاً نتائج استقرار وثيقة الصلة لاستقرار هيرز – أولام – راسياس، واستقرار أولام – جافروتا – راسياس، واستقرار ج. م. راسياس التي يتم التحكم بها عبر الجمع – الضرب المختلط لقوى النظيمات.

1 Introduction

A stimulating and famous talk presented by Ulam [31] in 1940, motivated the study of stability problems for various functional equations. He gave wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. Among those was the following question concerning the stability of homomorphisms.

Let G be a group and H be a metric group with metric $d(.,.)$. Given $\epsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with

$$d(f(x), a(x)) < \epsilon$$

for all $x \in G$?

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If the answer is affirmative, we say that the functional equation for homomorphism is stable. In 1941, Hyers [10] was the first mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam for the cases where G and H are assumed to be Banach spaces. The result of Hyers is stated in the following celebrated theorem.

Theorem 1.1 (Hyers [10]) *Assume that E_1 and E_2 are Banach spaces. If a function $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \tag{1.1}$$

for some $\epsilon > 0$ and for all $x, y \in E_1$, then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f\left(2^n x\right) \tag{1.2}$$

exists for each $x \in E_1$ and $A : E_1 \to E_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leq \epsilon \tag{1.3}$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then A is linear.

Taking the above fact into account, the additive functional equation $f(x+y) = f(x) + f(y)$ is said to have **Hyers–Ulam Stability** on (E_1, E_2). In the above Theorem 1.1, an additive function A satisfying (1.3) is constructed directly from the given function f and it is the most powerful tool to study the stability of several functional equations. In course of time, the theorem formulated by Hyers was generalized by Aoki [2] for additive mappings.

There is no reason for the Cauchy difference $f(x+y) - f(x) - f(y)$ to be bounded as in the expression of (1.1). Towards this point, in the year 1978, Rassias [29] tried to weaken the condition for the Cauchy difference and succeeded in proving what is now known to be the Hyers–Ulam–Rassias Stability for the Additive Cauchy Equation. This terminology is justified because the theorem of Th. M. Rassias has strongly influenced mathematicians studying stability problems of functional equation. In fact, Th. M. Rassias proved the following theorem.

Theorem 1.2 (Rassias [29]) *Let X and Y be Banach spaces, let $\theta \in (0, \infty)$ and let $p \in [0, 1)$. If a function $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\left(||x||^p + ||y||^p\right) \tag{1.4}$$

for all $x, y \in X$, then there is a unique additive mapping $A : X \to Y$ such that

$$||f(x) - A(x)|| \leq \frac{2\theta}{2-2^p}||x||^p \tag{1.5}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

The findings of Th. M. Rassias have exercised a delectable influence on the development of what is addressed as the generalized Hyers–Ulam–Rassias stability of functional equations.

In 1982, Rassias [21] gave a further generalization of the result of D.H. Hyers and proved a theorem using weaker conditions controlled by a product of different powers of norms. His theorem is presented as follows:

Theorem 1.3 (Rassias [21]) *Let $f : X \to Y$ be a mapping from a normed vector space X into a Banach space Y subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon||x||^p||y||^p \tag{1.6}$$

for all $x, y \in X$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < \frac{1}{2}$. Then the limit

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right) \tag{1.7}$$

exists for all $x \in X$ and $A : X \to Y$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\epsilon}{2-2^{2p}}||x||^{2p} \tag{1.8}$$

for all $x \in X$. *If* $p < 0$, *then the inequality* (1.6) *holds for* $x, y \neq 0$ *and* (1.8) *for* $x \neq 0$. *If* $p > \frac{1}{2}$, *then the inequality* (1.6) *holds for* $x, y \in X$ *and the limit*

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{1.9}$$

exists for all $x \in X$ *and* $A : X \to Y$ *is the unique additive mapping which satisfies*

$$\|f(x) - A(x)\| \leq \frac{\epsilon}{2^{2p} - 2}||x||^{2p} \tag{1.10}$$

for all $x \in X$. *If in addition* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* A *is* $\mathbb{R}$-*linear mapping.*

This type of stability involving a product of powers of norms is called **Ulam–Gavruta–Rassias Stability** by Bouikhalene and Elquorachi [3], Nakmahachalasint [17,18], Park and Najati [19], Pietrzyk [20] and Sibaha et al. [30].

A generalized and modified form of the theorem evolved by Th. M. Rassias was advocated by Gavruta [8] who replaced the unbounded Cauchy difference by driving into study a general control function within the viable approach designed by Th. M. Rassias. The following theorem provides his result.

Theorem 1.4 (Gavruta [8]) *Let* G *and* E *be an abelian group and a Banach space respectively, and let* $\varphi : G^2 \to [0, \infty)$ *be a function satisfying*

$$\Phi(x, y) = \sum_{k=0}^{\infty} 2^{-k-1}\varphi\left(2^k x, 2^k y\right) < \infty \tag{1.11}$$

for all $x, y \in G$. *If a function* $f : G \to E$ *satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y) \tag{1.12}$$

for any $x, y \in G$, *then there exists a unique additive function* $A : G \to E$ *with*

$$\|f(x) - A(x)\| \leq \Phi(x, x)$$

for all $x \in G$. *Moreover, if* $f(tx)$ *is continuous in* t *for each fixed* $x \in G$, *then* A *is a linear function.*

This type of stability is called **Generalized Hyers–Ulam Stability**. The investigation of stability of functional equations involving with the mixed type product-sum of powers of norms is introduced by Rassias [22]. In 2008, Rassias et al. [22] discussed the stability of quadratic functional equation

$$f(mx + y) + f(mx - y) = 2f(x + y) + 2f(x - y) + 2\left(m^2 - 2\right) f(x) - 2f(y)$$

for any arbitrary but fixed real constant m with $m \neq 0$; $m \neq \pm 1$; $m \neq \pm\sqrt{2}$ using mixed product-sum of powers of norms. He proved the following theorem.

Theorem 1.5 *Let* $f : E \to F$ *be a mapping which satisfies the inequality*

$$\begin{aligned}&\left\| f(mx + y) + f(mx - y) - 2f(x + y) - 2f(x - y) - 2\left(m^2 - 2\right) f(x) + 2f(y)\right\|_F \\ &\quad \leq \epsilon \left\{||x||_E^p||y||_E^p + \left(||x||_E^{2p} + ||y||_E^{2p}\right)\right\}\end{aligned} \tag{1.13}$$

for all $x, y \in E$ *with* $x \perp y$, *where* ϵ *and* p *are constants with* $\epsilon, p > 0$ *and either* $m > 1$; $p < 1$ *or* $m < 1$; $p > 1$ *with* $m \neq 0$; $m \neq \pm 1$; $m \neq \pm\sqrt{2}$ *and* $-1 \neq |m|^{p-1} < 1$. *Then the limit*

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}} \tag{1.14}$$

exists for all $x \in E$ *and* $Q : E \to F$ *is the unique orthogonally Euler–Lagrange quadratic mapping such that*

$$\|f(x) - Q(x)\|_F \leq \frac{\epsilon}{2|m^2 - m^{2p}|}||x||_E^{2p} \tag{1.15}$$

for all $x \in E$.

The above-mentioned stability is called **J. M. Rassias Stability involving mixed product-sum of powers of norms** by Ravi et al. ([22,23,26,27]).

Since, the last three decades, many research papers are published on Hyers–Ulam Stability, Hyers–Ulam–Rassias Stability, Ulam–Gavruta–Rassias Stability and Generalized Hyers–Ulam Stability to a number of functional equations and mappings in various spaces, in particular additive functional equations, quadratic functional equations, cubic functional equations, quartic functional equations, quintic functional equations, sextic functional equations, Jensen type functional equations, Pexiderized functional equations, mixed type additive-quadratic, additive-quadratic-cubic, additive-quadratic-cubic-quartic, additive-cubic functional equations and several other functional equations (see [3–5,7,9,14–16,23]). Many research monographs are also available in functional equations, one can see [1,6,11–13].

In the year 2010, Ravi and Senthil Kumar [24] investigated the generalized Hyers–Ulam stability for the reciprocal functional equation

$$g(x+y) = \frac{g(x)g(y)}{g(x)+g(y)} \tag{1.16}$$

where $g : X \to Y$ is a mapping on the spaces of non-zero real numbers. The reciprocal function $g(x) = \frac{c}{x}$ is a solution of the functional equation (1.16). The functional equation (1.16) holds good for the “Reciprocal formula” of any electric circuit with two resistors connected in parallel.

Ravi et al. [25] discussed the generalized Hyers–Ulam stability for the reciprocal functional equation in several variables of the form

$$g\left(\sum_{i=1}^{m} \alpha_i x_i\right) = \frac{\prod_{i=1}^{m} g(x_i)}{\sum_{i=1}^{m}\left[\alpha_i\left(\prod_{j=1, j\neq i}^{m} g(x_j)\right)\right]} \tag{1.17}$$

for arbitrary but fixed real numbers $(\alpha_1, \alpha_2, \ldots, \alpha_m) \neq (0, 0, \ldots, 0)$, so that $0 < \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m = \sum_{i=1}^{m} \alpha_i \neq 1$ and $gX \to Y$ with X and Y are the spaces of non-zero real numbers.

Later, Rassias et.al. [26] introduced the Reciprocal Difference Functional equation

$$r\left(\frac{x+y}{2}\right) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{1.18}$$

and the Reciprocal Adjoint Functional equation

$$r\left(\frac{x+y}{2}\right) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)} \tag{1.19}$$

and investigated the generalized Hyers–Ulam stability of Eqs. (1.18) and (1.19). Soon after, Rassias et al. [27] applied fixed point method to investigate the generalized Hyers–Ulam stability of Eqs. (1.18) and (1.19).

Ravi et al. [28] proved the generalized Hyers–Ulam stability for the reciprocal type functional equations

$$f((k_1-k_2)x + (k_1-k_2)y) = \frac{f(k_1x-k_2y)f(k_1y-k_2x)}{f(k_1x-k_2y)+f(k_1y-k_2x)} \tag{1.20}$$

where k_1 and k_2 are any integers with $k_1 \neq k_2$ and

$$f((k_1+k_2)x + (k_1+k_2)y) = \frac{f(k_1x+k_2y)f(k_1y+k_2x)}{f(k_1x+k_2y)+f(k_1y+k_2x)} \tag{1.21}$$

where k_1 and k_2 are any integers with $k_1 \neq -k_2$.

Wang et al. [32] introduced four Ulam’s type stability concepts and obtained Ulam’s type stability results for impulsive ordinary differential equations by applying the integral inequality of Gronwall type for piecewise continuous functions.

Wang and Zhang [33] established two existence and uniqueness results and investigated Ulam–Hyers–Mittag–Leffler stability on a compact interval for fractional-order delay differential equation with respect to Chebyshev and Bielecki norms.

In this paper, we obtain the solution of a new generalized reciprocal type functional equation in two variables of the form

$$r(x+y) = \frac{kr(x+(k-1)y)r((k-1)x+y)}{r(x+(k-1)y)+r((k-1)x+y)} \tag{1.22}$$

where $k > 2$ is a positive integer and investigate the generalized Hyers–Ulam stability of Eq. (1.22) in non-Archimedean fields. We also present the stability results concerning Hyers–Ulam–Rassias stability, Ulam–Gavruta–Rassias stability and J. M. Rassias stability controlled by the mixed product-sum of powers of norms.

2 Preliminaries

A non-Archimedean field is a filed $\mathbb{K}$ equipped with a function (valuation) $|.|$ from $\mathbb{K}$ into $[0, \infty)$ such that for all $x, y \in \mathbb{K}$,

(i) $|x| = 0$ if and only if $x = 0$
(ii) $|xy| = |x||y|$ and
(iii) $|x + y| \leq max\{|x|, |y|\}$.

Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

An example of a non-Archimedean valuation is the mapping $|.|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial. Another example of a non-Archimedean valuation on a field $\mathbb{K}$ is the mapping

$$\|x\| = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x > 0 \\ -\frac{1}{x} & \text{if } x < 0 \end{cases}$$

for any $x \in \mathbb{K}$.

3 General solution of Eq. (1.22)

Theorem 3.1 *Let $f : \mathbb{R}^* \to \mathbb{R}$ be a mapping. Then f satisfies (1.16) if and only if f satisfies (1.22). Hence (1.22) is also a reciprocal mapping whose solution is $f(x) = \frac{c}{x}$.*

Proof Let f satisfy (1.16). Replacing (x, y) by (x, x) in (1.16), we get $f(2x) = \frac{1}{2} f(x)$, for all $x \in \mathbb{R}^*$. Now, substituting (x, y) by $(x, 2x)$ in (1.16), we obtain $f(3x) = \frac{1}{3} f(x)$, for all $x \in \mathbb{R}^*$. Similarly, it is easy to prove $f(\frac{x}{2}) = 2f(x)$, $f(\frac{x}{3}) = 3f(x)$, for all $x \in \mathbb{R}^*$. Therefore, by induction, we can prove $f(nx) = \frac{1}{n} f(x)$ and $f(\frac{1}{n}x) = nf(x)$, for all $x \in \mathbb{R}^*$ and any positive integer n. Replacing (x, y) by $(x + (k-1)y, (k-1)x + y)$ in (1.16), we arrive (1.22).

Conversely, suppose f satisfy (1.22). Replacing (x, y) by $(\frac{(k-1)y-x}{k^2-2k}, \frac{(k-1)x-y}{k^2-2k})$ in (1.22), we obtain (1.16) which completes the proof. □

4 Generalized Hyers–Ulam stability of (1.22)

Throughout this Section, let us assume $\mathbb{K}$ and $\mathbb{M}$ to be non-Archimedean fields. Also, we assume that $x + (k-1)y \neq 0$, $(k-1)x + y \neq 0$, $f(x + (k-1)y) + f((k-1)x + y) \neq 0$ and $f(x) \neq 0$, for all $x, y \in \mathbb{K}^*$. For the sake of convenience, let us take

$$D_k f(x, y) = f(x + y) - \frac{kf(x + (k-1)y)f((k-1)x + y)}{f(x + (k-1)y) + f((k-1)x + y)}$$

for all $x, y \in \mathbb{K}$.

Theorem 4.1 *Let $\varphi : \mathbb{K}^* \times \mathbb{K}^* \to \mathbb{M}^*$ be a function such that*

$$\lim_{n\to\infty} \left|\frac{k}{2}\right|^n \varphi\left(\frac{k^n}{2^{n+1}}x, \frac{k^n}{2^{n+1}}y\right) = 0 \tag{4.1}$$

for all $x, y \in \mathbb{K}^*$. Suppose that $f : \mathbb{K}^* \to \mathbb{M}$ is a mapping satisfying the inequality

$$\|D_k f(x, y)\| \leq \varphi(x, y) \tag{4.2}$$

for all $x, y \in \mathbb{K}^*$*. Then there exists a unique reciprocal mapping* $R : \mathbb{K}^* \to \mathbb{M}$ *such that*

$$\|f(x) - R(x)\| \leq max \left\{ \left|\frac{k}{2}\right|^j \varphi\left(\frac{k^j}{2^{j+1}}x, \frac{k^j}{2^{j+1}}y\right) : j \in \mathbb{N} \cup \{0\} \right\} \tag{4.3}$$

for all $x \in \mathbb{K}^*$.

Proof Replacing (x, y) by (x, x) in (4.2), we get

$$\left\| f(2x) - \frac{k}{2} f(kx) \right\| \leq \varphi(x, x) \tag{4.4}$$

for all $x \in \mathbb{K}^*$. Now, replacing x by $\frac{x}{2}$ in (4.4), we obtain

$$\left\| f(x) - \frac{k}{2} f\left(\frac{k}{2}x\right) \right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \tag{4.5}$$

for all $x \in \mathbb{K}^*$. Substituting x by $(\frac{k}{2})^n x$ in (4.5) and multiplying by $|\frac{k}{2}|^n$, we have

$$\left\| \frac{k^n}{2^n} f\left(\frac{k^n}{2^n}x\right) - \frac{k^{n+1}}{2^{n+1}} f\left(\frac{k^{n+1}}{2^{n+1}}x\right) \right\| \leq \left|\frac{k}{2}\right|^n \varphi\left(\frac{k^n}{2^{n+1}}x, \frac{k^n}{2^{n+1}}y\right) \tag{4.6}$$

for all $x \in \mathbb{K}^*$ and all non-negative integers k. Thus, the sequence $\{\frac{k^j}{2^j} f(\frac{k^j}{2^j}x)\}$ is Cauchy by (4.1) and (4.6). Completeness of the non-Archimedean space $\mathbb{M}$ allows us to assume that there exists a mapping R so that

$$\lim_{n\to\infty} \frac{k^n}{2^n} f\left(\frac{k^n}{2^n}x\right) = R(x). \tag{4.7}$$

For each $x \in \mathbb{K}^*$ and non-negative integers n, we have

$$\begin{aligned}
\left\| \frac{k^n}{2^n} f\left(\frac{k^n}{2^n}x\right) - f(x) \right\| &= \left\| \sum_{j=0}^{n-1} \left\{ \frac{k^{j+1}}{2^{j+1}} f\left(\frac{k^{j+1}}{2^{j+1}}x\right) - \frac{k^j}{2^j} f\left(\frac{k^j}{2^j}x\right) \right\} \right\| \\
&\leq \max \left\{ \left\| \frac{k^{j+1}}{2^{j+1}} f\left(\frac{k^{j+1}}{2^{j+1}}x\right) - \frac{k^j}{2^j} f\left(\frac{k^j}{2^j}x\right) \right\| : 0 \leq j < n \right\} \\
&\leq \max \left\{ \left|\frac{k}{2}\right|^j \varphi\left(\frac{k^j}{2^{j+1}}x, \frac{k^j}{2^{j+1}}x\right) : 0 \leq j < n \right\}.
\end{aligned} \tag{4.8}$$

Applying (4.7) and letting n to infinity, we find that the inequality (4.3) holds. From (4.1), (4.2) and (4.7), we have for all $x, y \in \mathbb{K}^*$

$$\begin{aligned}
\|D_k R(x, y)\| &= \lim_{n\to\infty} \left|\frac{k}{2}\right|^n \left\| D_k f\left(\frac{k^n}{2^n}x, \frac{k^n}{2^n}y\right) \right\| \\
&\leq \lim_{n\to\infty} \left|\frac{k}{2}\right|^n \varphi\left(\frac{k^n}{2^n}x, \frac{k^n}{2^n}y\right) = 0.
\end{aligned}$$

Hence, the mapping R satisfies (1.22). By Theorem 3.1, the mapping R is reciprocal. Now, let $r : \mathbb{K}^* \to \mathbb{M}$ be another reciprocal mapping satisfying (4.3). Then we have

$$\begin{aligned}\|R(x)-r(x)\| &= \lim_{m\to\infty}\left|\frac{k}{2}\right|^m \left\|R\left(\frac{k^m}{2^m}x\right)-r\left(\frac{k^m}{2^m}x\right)\right\| \\ &\le \lim_{m\to\infty}\left|\frac{k}{2}\right|^m max\left\{R\left(\frac{k^m}{2^m}x\right)-f\left(\frac{k^m}{2^m}x\right),\right. \\ &\qquad \left.\left\|f\left(\frac{k^m}{2^m}x\right)-r\left(\frac{k^m}{2^m}x\right)\right\|\right\} \\ &\le \lim_{m\to\infty}\lim_{n\to\infty} max\left\{max\left\{\left|\frac{k}{2}\right|^{j+m}\varphi\left(\frac{k^{j+m}}{2^{j+m+1}}x, \frac{k^{j+m}}{2^{j+m+1}}x\right):\right.\right. \\ &\qquad \left.\left. m\le j\le n+mmax\left|\frac{k}{2}\right|^{j+m}\varphi\left(\frac{k^{j+m}}{2^{j+m+1}}x, \frac{k^{j+m}}{2^{j+m+1}}x\right)\right\}\right\} \\ &= 0\end{aligned}$$

for all $x \in \mathbb{K}^*$, which proves that R is unique. □

Theorem 4.2 *Let $\varphi : \mathbb{K}^* \times \mathbb{K}^* \to \mathbb{M}^*$ be a function such that*

$$\lim_{n\to\infty}\left|\frac{2}{k}\right|^n \varphi\left(\frac{2^n}{k^{n+1}}x, \frac{2^n}{k^{n+1}}y\right) = 0 \tag{4.9}$$

for all $x, y \in \mathbb{K}^*$. Suppose that $f : \mathbb{K}^* \to \mathbb{M}$ is a mapping satisfying the inequality (4.2) for all $x, y \in \mathbb{K}^*$. Then there exists a unique reciprocal mapping $R : \mathbb{K}^* \to \mathbb{M}$ such that

$$\|f(x)-R(x)\| \le max\left\{\left|\frac{2}{k}\right|^{j+1}\varphi\left(\frac{2^j}{k^{j+1}}x, \frac{2^j}{k^{j+1}}y\right) : j \in \mathbb{N}\cup\{0\}\right\} \tag{4.10}$$

for all $x \in \mathbb{K}^*$.

Proof Replacing (x, y) by $(\frac{x}{k}, \frac{x}{k})$ in (4.2) and multiplying by $|\frac{2}{k}|$, we get

$$\left\|\frac{2}{k}f\left(\frac{2}{k}x\right)-f(x)\right\| \le \left|\frac{2}{k}\right|\varphi\left(\frac{x}{k}, \frac{x}{k}\right) \tag{4.11}$$

for all $x \in \mathbb{K}^*$. Substituting x by $(\frac{2}{k})^n x$ in (4.11) and multiplying by $|\frac{2}{k}|^n$, we have

$$\left\|\frac{2^n}{k^n}f\left(\frac{2^n}{k^n}x\right)-\frac{2^{n+1}}{k^{n+1}}f\left(\frac{2^{n+1}}{k^{n+1}}x\right)\right\| \le \left|\frac{2}{k}\right|^{n+1}\varphi\left(\frac{2^n}{k^{n+1}}x, \frac{2^n}{k^{n+1}}y\right) \tag{4.12}$$

for all $x \in \mathbb{K}^*$ and all non-negative integers k. As $n \to \infty$ in (4.12) and using (4.9), we see that the sequence $\{\frac{2^j}{k^j}f(\frac{2^j}{k^j}x)\}$ is a Cauchy sequence. Since $\mathbb{M}$ is complete, this Cauchy sequence converges to a mapping $R : \mathbb{K}^* \to \mathbb{M}$ defined by

$$\lim_{n\to\infty}\frac{2^n}{k^n}f\left(\frac{2^n}{k^n}x\right) = R(x). \tag{4.13}$$

For each $x \in \mathbb{K}^*$ and non-negative integers n, we have

$$\begin{aligned}\left\|\frac{2^n}{k^n}f\left(\frac{2^n}{k^n}x\right)-f(x)\right\| &= \left\|\sum_{j=0}^{n-1}\frac{2^{j+1}}{k^{j+1}}f\left(\frac{2^{j+1}}{k^{j+1}}x\right)-\frac{2^j}{k^j}f\left(\frac{2^j}{k^j}x\right)\right\| \\ &\le \max\left\{\left\|\frac{2^{j+1}}{k^{j+1}}f\left(\frac{2^{j+1}}{k^{j+1}}x\right)-\frac{2^j}{k^j}f\left(\frac{2^j}{k^j}x\right)\right\| : 0\le j<n\right\} \\ &\le \max\left\{\left|\frac{2}{k}\right|^{j+1}\varphi\left(\frac{2^j}{k^{j+1}}x, \frac{2^j}{k^{j+1}}x\right) : 0\le j<n\right\}.\end{aligned} \tag{4.14}$$

Applying (4.13) and letting n to infinity, we find that the inequality (4.10) holds. From (4.9), (4.2) and (4.13), we have for all $x, y \in \mathbb{K}^*$

$$\begin{aligned}\|D_k R(x,y)\| &= \lim_{n\to\infty}\left|\frac{2}{k}\right|^n \left\|D_k f\left(\frac{2^n}{k^n}x, \frac{2^n}{k^n}y\right)\right\| \\ &\le \lim_{n\to\infty}\left|\frac{2}{k}\right|^n \varphi\left(\frac{2^n}{k^n}x, \frac{2^n}{k^n}y\right) = 0.\end{aligned}$$

Hence the mapping R satisfies (1.22). By Theorem 3.1, the mapping R is reciprocal. Now, let $r : \mathbb{K}^* \to \mathbb{M}$ be another reciprocal mapping satisfying (4.10). Then we have

$$\begin{aligned}\|R(x) - r(x)\| &= \lim_{m\to\infty}\left|\frac{2}{k}\right|^m \left\|R\left(\frac{2^m}{k^m}x\right) - r\left(\frac{2^m}{k^m}x\right)\right\| \\ &\le \lim_{m\to\infty}\left|\frac{2}{k}\right|^m \max\left\{R\left(\frac{2^m}{k^m}x\right) - f\left(\frac{2^m}{k^m}x\right),\right. \\ &\qquad \left.\left\|f\left(\frac{2^m}{k^m}x\right) - r\left(\frac{2^m}{k^m}x\right)\right\|\right\} \\ &\le \lim_{m\to\infty}\lim_{n\to\infty}\max\left\{\max\left\{\left|\frac{2}{k}\right|^{j+m+1}\varphi\left(\frac{2^{j+m}}{k^{j+m+1}}x, \frac{2^{j+m}}{k^{j+m+1}}x\right) : \right.\right. \\ &\qquad \left.\left. m \le j \le n+m\right\}\right\} \\ &= 0\end{aligned}$$

for all $x \in \mathbb{K}^*$, which proves that R is unique. □

Corollary 4.3 *For any fixed $c_1 \ge 0$ and $p \ne -1$, if $f : \mathbb{K}^* \to \mathbb{M}$ satisfies*

$$\|D_k f(x,y)\| \le c_1\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in \mathbb{K}^*$, then there exists a unique reciprocal mapping $R : \mathbb{K}^* \to \mathbb{M}$ satisfying (1.22) and

$$\|f(x) - R(x)\| \le \begin{cases}\frac{2c_1}{|2|^p}\|x\|^p, & \text{for } p < -1\\ \frac{4c_1}{|k|^{p+1}}\|x\|^p, & \text{for } p > -1\end{cases}$$

for every $x \in \mathbb{K}^*$.

Proof The required results are obtained by choosing $\varphi(x,y) = c_1(\|x\|^p + \|y\|^p)$, for all $x, y \in \mathbb{K}^*$ in Theorem 4.1 with $p < -1$ and in Theorem 4.2 with $p > -1$ and proceeding by similar arguments as in Theorems 4.1 and 4.2. □

Corollary 4.4 *Let $f : \mathbb{K}^* \to \mathbb{M}$ be a mapping and let α and β be real numbers such that $\rho = \alpha + \beta \ne -1$. Assume there exists $c_2 \ge 0$ such that*

$$\|D_k f(x,y)\| \le c_2\|x\|^\alpha\|y\|^\beta$$

for all $x, y \in \mathbb{K}^*$. Then there exists a unique reciprocal mapping $R : \mathbb{K}^* \to \mathbb{M}$ satisfying (1.22) and

$$\|f(x) - R(x)\| \le \begin{cases}\frac{c_2}{|2|^\rho}\|x\|^\rho, & \text{for } \rho < -1\\ \frac{2c_2}{|k|^{\rho+1}}\|x\|^\rho, & \text{for } \rho > -1\end{cases}$$

for every $x \in \mathbb{K}^*$.

Proof Considering $\varphi(x,y) = c_2\|x\|^\alpha\|y\|^\beta$, for all $x, y \in \mathbb{K}^*$ in Theorem 4.1 with $\rho < -1$ and in Theorem 4.2 with $\rho > -1$, the proof of the Corollary is complete. □

Corollary 4.5 *Let $c_3 \geq 0$ and $q \neq -1$ be real numbers, and $f : \mathbb{K}^* \to \mathbb{M}$ be a mapping satisfying the functional inequality*

$$\|D_k f(x, y)\| \leq c_3 \left(\|x\|^{\frac{q}{2}} \|y\|^{\frac{q}{2}} + \left(\|x\|^q + \|y\|^q\right)\right)$$

for all $x, y \in \mathbb{K}^*$. Then there exists a unique reciprocal mapping $R : \mathbb{K}^* \to \mathbb{M}$ satisfying (1.22) and

$$\|f(x) - R(x)\| \leq \begin{cases} \frac{3c_3}{|2|^q} \|x\|^q, & \text{for } s < -1 \\ \frac{6c_3}{|k|^{q+1}} \|x\|^q, & \text{for } s > -1 \end{cases}$$

for every $x \in \mathbb{K}^*$.

Proof The proof follows immediately by taking $\varphi(x, y) = (\|x\|^{\frac{q}{2}} \|y\|^{\frac{q}{2}} + (\|x\|^q + \|y\|^q))$ in Theorem 4.1 with $q < -1$ and in Theorem 4.2 with $q > -1$. □

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