Multiloop Noncommutative Open String Theory and their QFT limit.

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Abstract: The multiloop amplitudes for open bosonic string in presence of a constant B-field are derived from first principles. The basic ingredients of the construction are the commutation relations for the string modes and the Reggeon vertex describing the interaction among three generic string states. The modifications due to the presence of the B–field affect non–trivially only the zero modes. This makes it possible to write in a simple and elegant way the general expression for multiloop string amplitudes in presence of a constant B-field. The field theory limit of these string amplitudes is also considered. We show that it reproduces exactly the Feynman diagrams of noncommutative field theories. Issues of UV/IR are briefly discussed.

1 Multiloop NCOS and Reggeon Formalism

The first basic ingredient for the construction of string amplitudes in the operator formalism are the commutation relations for the string modes. As usual, these commutation relations can be derived from the the tree level world–sheet action. We consider an open string ending on a D-brane in presence of a constant B-field. The open string mode expansion is

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha'(p_0^\mu \tau - p_0^\nu F_\nu{}^\mu \sigma) + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{e^{-in\tau}}{n}(ia_n^k \cos n\sigma - a_n^\nu F_\nu{}^\mu \sin n\sigma) , \qquad (1)$$

where $F = B - dA$ is the modified Born-Infeld field strength. Canonical quantization yields the following commutation relations [1]

$$[a_n^\mu, x_0^\nu] = [a_n^\mu, p_0^\nu] = [p_0^\mu, p_0^\nu] = 0, \qquad (2)$$

$$[a_m^\mu, a_n^\nu] = mM^{-1\mu\nu}\delta_{m+n}, \quad [x_0^\mu, p_0^\nu] = iM^{-1\mu\nu}, \quad [x_0^\mu, x_0^\nu] = i\,\Theta^{\mu\nu}, \qquad (3)$$

where $M_{\mu\nu} = g_{\mu\nu} - (Fg^{-1}F)_{\mu\nu}$ is the open string metric and

$$(M^{-1})^{\mu\nu} = \Big(\frac{1}{g+F}g\frac{1}{g-F}\Big)^{\mu\nu}, \quad \Theta^{\mu\nu} = 2\pi\alpha'\Big(\frac{1}{g+F}F\frac{1}{g-F}\Big)^{\mu\nu} \qquad (4)$$

are the symmetric and antisymmetric part of the matrix $(\frac{1}{g+F})^{\mu\nu} = (M^{-1})^{\mu\nu} - \frac{\Theta^{\mu\nu}}{2\pi\alpha'}$. Although these commutation relations were derived at the tree level, they are valid at all loops [2, 3] and therefore one can use them directly to construct the higher loop string amplitudes. Due to the limitation of vertex operator formalism to go beyond 1-loop, it is necessary to introduce the Reggeon vertex formalism for string amplitudes.

The basic object in the Reggeon formalism is the 3-Reggeon vertex which describes, at the tree level, the interaction among three generic string states [4]

$$V_{3;0}^\theta(\zeta) = \int \frac{dp}{\sqrt{\det M}} \; \langle p, 0; q = 3 | : \mathrm{e}^{\left\{\oint_0 dz(-X^v(\zeta+z)\partial_z X(z) - c^v(\zeta+z)b(z) + b^v(\zeta+z)c(z))\right\}} : . \tag{5}$$

Here the bra indicates the vacuum of the emitted string with momentum p and the label $q = 3$ specifies the ghost number. X^v is the virtual propagating string. X and X^v both have an expansion of the form (1) with commutations relations given by (2), (3), while the virtual and the external strings simply commute among themselves. Notice that that (5) is almost identical to the standard 3-Reggeon vertex in the trivial background $B = 0$. The only modification with the respect to the usual case is the appearence of a factor of $\det M$ in the measure of momentum integrals which is due to the fact that the open string metric is flat, but non trivial[1]. More generally one has to modify the usual ($B = 0$) normalizations every time the volume of the space seen by the open strings appears: for instance, the open string vacuum has to be normalized as $\langle 0|0 \rangle = \sqrt{\det M} \, V$ and, thus, the generic momentum state satisfies $\langle p|p' \rangle = \sqrt{\det M} \, \delta^d(p - p')$. However, even if (5) is formally unchanged, it contains a non-trivial dependence on B through the mode expansion (1) and the new commutation relations (2), (3). It should also be stressed that the zero-mode x_0 (or y_0 if the interaction is at $\sigma = \pi$) appears only in the expansion of the virtual string and this is the only source of the non-trivial dependence on Θ. We will employ the usual physical states having ghost number 1; thus, the (b, c) system is not affected by the background field F and one recovers the well-known results for the ghost contributions. Because of this, in what follows, ghosts will no longer be mentioned, and we will focus only on the F-dependent modifications coming from the orbital part.

The tree level N-Reggeon vertex is obtained by simply multiplying N 3-Reggeon vertices in different positions ζ, but with the common propagating string X^v. Finally one takes the vacuum expectation value in the Hilbert space of the propagating string in order to obtain a symmetric object in the N external states. The new Θ-dependent part comes when one collects together the zero mode factors. In particular, if the external legs of all the original 3-Reggeon vertices are emitted from the border $\sigma = 0$, one obtains the new phase factor

$$e^{ip^1 x_0} \cdots e^{ip^N x_0} = e^{-\frac{i}{2}\sum_{r<s}^N p^r \Theta p^s}, \tag{6}$$

where momentum conservation has been used. As a result, we obtain the N-Reggeon vertex with all legs emitted from the $\sigma = 0$ border

$$V_{N;0}^\Theta = \sqrt{\det M} \; V_{N;0}^0 \exp(-\frac{i}{2}\sum_{i<j}^N p^i \Theta p^j), \tag{7}$$

where p_μ^i is the momentum *operator* of the i-th leg, in the direction flowing towards the boundary. Here $V_{N;0}^0$ indicates the N-Reggeon vertex derived for the usual "commutative" case $F = 0$

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[1]We thank O. Andreev for pointing out to us this fact. In Ref. [2] this factor was absent because the noncommutative Reggeon-Vertices and amplitudes were written in terms of rescaled string coordinates, as we do in the following Section 3.1.

in [5]. This vertex is a bra in the direct product of the N distinct Fock spaces for the external strings; like the 3-Reggeon vertex, it gives the scattering amplitude when the external legs are saturated with physical states. Note that the momentum dependent phase factor is exactly the same modification as the one introduced by the Moyal $*$-product in the tree level vertex of a noncommutative field theory [6].

The h-loop N-Reggeon vertex in the presence of a constant F-field can be constructed by sewing together pairs of legs in a tree level $(N + 2h)$-Reggeon vertex. The sewing is achieved by using the BRST invariant operator $P(x)$, which is a function of L_0 and $L_{\pm 1}$ and of the ghosts. Since L_n does not involve the zero modes x_0, we conclude that the string propagator $P(x)$ and the sewing procedure are not modified by the presence of F-field. Therefore the only new feature for $\Theta \neq 0$ is in the zero modes part of the Reggeon vertex. And we obtain [2]

$$V_{N;h}^\Theta = \sqrt{\det M} \; \widetilde{V}_{N;h}^0 \prod_{I=1}^h \int \frac{dp^I}{\sqrt{\det M}} \exp\left(\frac{1}{2} \sum_{I,J=1}^h p_\mu^I A_{IJ}^{\mu\nu} p_\nu^J + \sum_{I=1}^h B_I^\mu p_\mu^I + C \right), \qquad (8)$$

where $\widetilde{V}_{N;h}^0$ [8, 5] contains the ghost contribution and only the nonzero mode piece of the orbital part of N-Reggeon vertex in absence of background. Here

$$A_{IJ}^{\mu\nu} = A_{IJ}^{0\mu\nu} - i\Theta^{\mu\nu}\mathcal{J}_{IJ}, \quad B_I^\mu = B_I^{0\mu} - i\Theta^{\mu\nu}P_{I\nu}, \quad C = C^0 - \frac{i}{2}\sum_{i<j}^N p^i \Theta p^j, \qquad (9)$$

where p_I are the loop momenta with loop indices $I = 1, \cdots h$; p_i are the external momenta with legs labelled by $i = 1, \cdots, N$; $\mathcal{J}_{IJ}$ is the intersection matrix for the internal loops; P_I is the sum of the external momenta leaving the Ith loop, and μ, ν stands for the spacetime indices. A^0, B^0, C^0 are independent of Θ and are given by [8],

$$A_{IJ}^{0\mu\nu} = 2\alpha'(2\pi i\tau_{IJ})(M^{-1})^{\mu\nu}, \qquad B_I^{0\mu} = \frac{1}{2\pi}\sum_{i=1}^N \oint_0 dz \partial X^{(i)\mu}(z) \int_{z_0}^{V_i(z)} \omega_I, \qquad (10)$$

$$\begin{aligned}
C^0 = &-\frac{1}{2}\sum_{i=1}^N \oint_0 dz \partial X^{(i)}(z) p_0^{(i)} \ln V_i'(z) \\
&+\frac{1}{2\alpha'}\sum_{i<j}^N \oint_0 dz \oint_0 dy \partial X^{(i)}(z) \ln[V_i(z) - V_j(y)] \partial X^{(j)}(y) \\
&+\frac{1}{4\alpha'}\sum_{i,j=1}^N \oint_0 dz \oint_0 dy \partial X^{(i)}(z) \ln\left(\frac{E(V_i(z), V_j(y))}{V_i(z) - V_j(y)} \right) \partial X^{(j)}(y),
\end{aligned}$$

where, due to (2) and (3), all indices are contracted by the open string metric M^{-1}. Here $V_i(z)$ is chosen to satisfy $V_i^{-1}(z) = 0$ for $z = z_i$. ω_I is the normalized Abelian differential and τ_{IJ} is the period matrix and $E(z, w)$ is the prime form. Their explicit expressions in term of the Schottky parameters can be found in [8].

Note that all the dependence in Θ is localized in the zero modes loop momentum integration. Carrying out the loop momentum integration, one obtains finally

$$V_{N;h}^\Theta = \left[\sqrt{\det M}\right]^{1-h} \widetilde{V}_{N;h}^0 \frac{1}{\sqrt{\det \frac{-A}{2}}} \exp(-\frac{1}{2}B^T A^{-1} B + C), \qquad (11)$$

where the determinant is taken over the space of Lorentz and loop indices (μI). The effects of Θ are summarized elegantly in (9).

3

<u>Modifications of Θ to the string amplitude</u>

As an illustration of how $\Theta \neq 0$ modify the string amplitude, it is instructive to recall the explicit form of the higher loop string amplitude for $\Theta = 0$. For example, the string amplitude for M-gauge bosons can be obtained by saturating the $V_{M;h}^{\Theta=0}$ with M external gluon states. Using the explicit formula (10), one obtains

$$A_M^{(h)}(p_1, \ldots, p_M) = C_h \mathcal{N}_0^M \int [dm]_h \frac{\prod_{i=1}^M d\rho_i/\rho_i}{dV_{abc}} \prod_{i<j} \exp\left(\sum_{i<j} 2\alpha' p_i^\mu p_j^\nu (G_{\mu\nu}(\rho_i, \rho_j))\right)$$

$$\times \left[\exp\sum_{i\neq j} \left(\sqrt{2\alpha'}\,\epsilon_i^\mu \rho_i \partial_{\rho_i} G_{\mu\nu}(\rho_i, \rho_j) p_j^\nu + \frac{1}{2}\epsilon_i^\mu \rho_i\rho_j \partial_{\rho_i}\partial_{\rho_j} G_{\mu\nu}(\rho_i, \rho_j)\epsilon_j^\nu\right)\right], \tag{12}$$

where only terms linear in each polarization should be kept. In the general case of having other external states, one need to change correspondingly the form of the second line in (12), however the first line is universal. Here in (12) $dV_{abc} = d\psi_a d\psi_b d\psi_c (\psi_a - \psi_b)^{-1}(\psi_a - \psi_c)^{-1}(\psi_b - \psi_c)^{-1}$ is the projective invariant volume element, where ψ_a, ψ_b, ψ_c are any three of the M punctures ρ_i or of the $2h$ fixed points ξ_μ, η_μ of the generators of the Schottky group. The normalization constants $\mathcal{N}_0$ and C_h are given by [7] and one can check that their functional form is unmodified by the presence of F

$$\mathcal{N}_0 = \sqrt{2}g_o(2\alpha')^{\frac{d-2}{4}}, \qquad C_h = \frac{1}{(2\pi)^{dh}}g_o^{2h-2}\frac{1}{(2\alpha')^{d/2}}, \tag{13}$$

where g_o is the open string coupling. To relate the open string coupling g_o with the closed string coupling g_s, one can study the factorization of the annulus diagrams. In Ref. [9] the relation between the two couplings is explicitly written in terms of the volume V seen by the open strings (see Eq. (3.10)). As we have already said, in the presence of a background F-field V has to be replaced by $\sqrt{\det M}\,V$. Thus from the result of [9] one can derive the F-dependence of g_o

$$g_o \propto (\det M)^{1/8} g_s \sim \left[\det(1 + F)\right]^{1/4} g_s, \tag{14}$$

which gives an implicit dependence on F to all of the above open string quantities. This relation was derived in [17] by looking at the Born-Infeld actions. Here we give a string derivation of this relation within the operator formalism. The measure factor $[dm]_h$ for $\Theta = 0$ is given by

$$[dm]_h = \prod_{\mu=1}^h \left[\frac{dk_\mu d\xi_\mu d\eta_\mu}{k_\mu^2(\xi_\mu - \eta_\mu)^2}(1 - k_\mu)^2\right] \left[\det\left(-2\pi i\alpha'\tau_{IJ}\right)\right]^{-d/2} \prod_\alpha' \left[\prod_{n=1}^\infty (1 - k_\alpha^n)^{-d} \prod_{n=2}^\infty (1 - k_\alpha^n)^2\right]. \tag{15}$$

where k_μ are the multipliers and ξ_μ, η_μ are the fixed points of the Schottky generators of the surface [8]; the primed product over α denotes a product over conjugacy classes of elements of the Schottky group, where only elements that cannot be written as powers of other elements must be included. Three of $2h + M$ parameters ξ_μ, η_μ and ρ_i can be fixed using an overall projective invariance of the amplitude. The fixing of this invariance introduces the projective invariant volume element dV_{abc} in (12). Including the "multipliers" k_μ, one is left with $3h-3+M$ variables, the correct number of independent moduli for a Riemann surface of genus h with M punctures.

When Θ is turned on, the shift in (9) for A, B, C modify the exponent of (11). This part of the modification may be summarized in terms of a modified Green function depending on

Θ. In addition, the shift in A gives rise to a Θ dependent measure factor $\sqrt{\det(-A/2)}$ in (11). Notice that in the particular case of the h-loop bubble backbone with no intersections $\mathcal{J} = 0$, the det M factors cancel out and the F-dependence is completely determined by that of g_0 and by the universal factor $\sqrt{\det M}$ present in (8),

$$V_{0;h}^{\Theta} = [\det(1 + F)]^{\frac{1+h}{2}} \times V_{0;h}^{0}. \tag{16}$$

For $h = 1$, this agrees with the computation of [10], and, in the general case, we recover the result of [11].

2 One-loop Amplitudes and Green Function

Having the general form (11) of the multiloop Reggeon vertex, it is straightforward to write down explicitly the 1-loop amplitude and extract the 1-loop Green function.

First, there is no modification to A^0 and the modification to C^0 is simply the usual field theory Filk phase. The 1-loop measure is given by

$$[dm]_1 \frac{\prod_{i=1}^{M} d\rho_i / \rho_i}{dV_{abc}} = \prod_{i=2}^{M} \frac{d\rho_i}{\rho_i} \frac{dk}{k^2} \left(-\frac{\ln k}{2}\right)^{-d/2} \prod_{n=1}^{\infty} (1 - k^n)^{2-d} \tag{17}$$

where k is the multiplier of the annulus in the Schottky representation. We have exploited the projective invariance to fix $\rho_1 = 1, \quad \eta = 0, \quad \xi = \infty$ in (17). Since the measure is not modified, the 1-loop amplitude can be written in the familiar form (12) with all the Θ-dependence taken care of by the Green function. Thus we concentrate on the effects from the shift of B^0. The one-loop Abelian differential is $\omega = dz/z$. With the convenient choice of $V_i(\rho) = \rho + \rho_i$, we have

$$B_I^{0\mu} = \sum_{r \in I_1 \cup I_2} \mathcal{B}^{r\,\mu}(\rho) \ln|\rho + \rho_r| \quad \text{with} \quad \mathcal{B}^{r\,\mu}(\rho) = \frac{1}{2\pi i} \oint_0 d\rho \partial X^{(r)\mu}(\rho), \tag{18}$$

where $\mathcal{B}^{r\mu}$ is an operator in which the integration is carried out on any function that multiply it on the right. One obtains finally

$$\begin{aligned} \frac{-1}{2A} B^2 &= 2\alpha' \cdot \frac{1}{8\alpha'^2 \ln k} \Big(\sum_{\substack{r<s \\ r,s \in I_1 \cup I_2}} \mathcal{B}^r(\rho) \mathcal{B}^s(\rho') \ln^2 \left|\frac{\rho + \rho_r}{\rho' + \rho_s}\right| \\ &+ \frac{i}{\alpha'} \sum_{\substack{r \in I_1 \\ s \in I_2}} \mathcal{B}^r(\rho) \Theta \mathcal{B}^s(\rho') \ln|(\rho + \rho_r)(\rho' + \rho_s)| + \frac{1}{4\alpha'^2} \sum_{\substack{r \in I_1 \\ s \in I_2}} \mathcal{B}^r(\rho) \Theta^2 \mathcal{B}^s(\rho') \Big), \end{aligned} \tag{19}$$

where I_1, I_2 denote the two boundaries of the annulus. The second line contains all the Θ-dependence of the one-loop N-Reggeon vertex $V_{N;1}^{\Theta}$.

Written in terms of the Schottky representation of the annulus, the planar and nonplanar open string Green functions are

$$G_P^{\mu\nu}(\rho, \rho') = I_0^P (M^{-1})^{\mu\nu} - \frac{i\Theta^{\mu\nu}}{4\alpha'} \epsilon(\rho - \rho'), \tag{20}$$

$$G_{NP}^{\mu\nu}(\rho, \rho') = I_0^{NP} (M^{-1})^{\mu\nu} + \frac{(\Theta^2)^{\mu\nu}}{8\alpha'^2} \frac{1}{\ln k} \pm \frac{i\Theta^{\mu\nu}}{2\alpha'} \frac{\ln|\rho\rho'|}{\ln k}, \tag{21}$$

where the Θ-independent piece is given by

$$I_0^P(\rho, \rho') = \frac{\ln^2 \rho/\rho'}{2\ln k} + \ln\left|\sqrt{\frac{\rho}{\rho'}} - \sqrt{\frac{\rho'}{\rho}}\right| + \ln\prod_{n=1}^{\infty}\left|\frac{(1 - k^n\rho/\rho')(1 - k^n\rho'/\rho)}{(1 - k^n)^2}\right| \tag{22}$$

for the planar case, while for non–planar contractions one has

$$I_0^{NP}(\rho, \rho') = \frac{\ln^2 |\rho/\rho'|}{2\ln k} + \ln(\sqrt{\frac{|\rho|}{|\rho'|}} + \sqrt{\frac{|\rho'|}{|\rho|}}) + \ln\prod_{n=1}^{\infty}\left|\frac{(1 + k^n|\rho/\rho'|)(1 + k^n|\rho'/\rho|)}{(1 - k^n)^2}\right|. \tag{23}$$

In the last term of the nonplanar Green function in (21), positive sign is taken when $\rho > 0, \rho' < 0$ and negative sign is taken for the opposite case $\rho' > 0, \rho < 0$.

2.1 Remarks on the Green function

In the above, the open string amplitudes and the open string Green function in the presence of B-field were obtained directly using the basic commutation relations (2), (3) and the Reggeon formalism. A different approach of using the *closed* string Green function as input was adopted at the one loop level [12, 13, 14, 15, 16] to calculate the open string amplitude. The idea is to obtain the *open* string Green function by letting the arguments of the closed string Green function to approach the boundary and then use it as input (in, *e.g.* (12)) to calculate the open string amplitude. However as discussed in [14, 2], there are ambiguities associated with this approach.

1. Approaching the boundary

We first recall the procedure of [17] in obtaining the boundary correlator, in particular its antisymmetric part, from the bulk. The boundary correlator obtained there is

$$\mathcal{G}^{\mu\nu}(\tau, \tau') := \langle X^\mu(\tau)X^\nu(\tau')\rangle = -\alpha'(M^{-1})^{\mu\nu}\ln(\tau - \tau')^2 + \frac{i}{2}\Theta^{\mu\nu}\epsilon(\tau - \tau'), \tag{24}$$

where M^{-1} and Θ are given in (4). The antisymmetric piece was obtained from the term

$$\Theta^{\mu\nu}\ln\frac{z - \bar{z}'}{\bar{z} - z'} \tag{25}$$

in the bulk correlator by letting z, z' to *approach* the boundary in an appropriate manner. Notice that restricting the arguments z, z' directly to the real axis yields $\ln 1 = 2n\pi i$ instead of the moduli dependent piece in (24). Note also that there is no uniform shift one can perform to the bulk correlator so that (24) is obtained when restricted to the boundary. A limiting procedure must be adopted. We also remark that radial ordering (on $|z|$) is employed for operators in the bulk, while a time ordering on τ is employed on the boundary. The two orderings agree on the positive τ axis, but is opposite to each other on the negative axis.

Thus there is a certain ambiguity in the antisymmetric part of (24) relating to the choice of the branch cut of the *ln*. This ambiguity can be fixed easily at the tree level and the form (24) is the correct one.

2. Choices of closed string Green functions

As noted in [14], there is an additional source of ambiguity at one loop: there is a freedom in the definition of the closed string Green function.

At the one loop level, it was first noted by [10] that it is not possible to strictly impose on the Green function the same boundary condition as imposed on the string coordinates $(\sigma = 0, \pi)$

$$\partial_\sigma X^\mu + \partial_\tau X^\nu F_\nu{}^\mu = 0, \quad \mu, \nu = 0, 1, \cdots, p, \tag{26}$$

and hence there is a certain degree of freedom in the choice of what constraint is satisfied by G. If one insists on

$$\Delta G_{\mu\nu}(z, z') = -2\pi\alpha'\eta_{\mu\nu}\delta(z, z') \tag{27}$$

then since $\oint \partial_\perp G ds = -2\pi\alpha'$ that is fixed by the Gauss theorem, one obtains the following set of compatible boundary conditions

$$\frac{\partial}{\partial r} G_{\mu\nu}(z, z') - \frac{i}{r}\frac{\partial}{\partial\theta} F_\mu{}^\lambda G_{\lambda\nu}(z, z') = \begin{cases} -\frac{\alpha'}{a}\beta & \text{at } r = a, \\ -\frac{\alpha'}{a}(1-\beta) & \text{at } r = b, \end{cases} \tag{28}$$

where β is arbitrary. The choice $\beta = 0$ was first derived by [10] using the method of image and was used in [16] in the computation of the tachyon amplitudes, while the choice $\beta = 1/2$ was adopted in [12]. One can also introduce a background charge so that a Green function

$$\Delta G_{\mu\nu}(z, z') = -2\pi\alpha'\eta_{\mu\nu}\delta(z, z') + \frac{2\pi\alpha'}{A}, \tag{29}$$

where $A = \pi(b^2 - a^2)$ is the area of the annulus, satisfying the boundary condition

$$\frac{\partial}{\partial r} G_{\mu\nu}(z, z') - \frac{i}{r}\frac{\partial}{\partial\theta} F_\mu{}^\lambda G_{\lambda\nu}(z, z') = 0, \quad r = a, b \tag{30}$$

can be constructed. This closed string Green function was used in [13, 14, 15] to derive the open string Green function and in computing the 2 point function for noncommutative photons. The origin of this freedom in defining the Green function was analyzed from the point of view of the boundary state approach in [14], where the closed string Green functions was derived from the closed string worldsheet with boundary states inserted so as to create an open string worldsheet. There it was noted that the freedom in the definition of the closed string Green function is related to a freedom in interpreting the tachyon amplitudes in terms of contributions from the closed string Green function and contributions from the self-contraction $\mathcal{C}$. A shifted

$$G'_{\mu\nu}(z, z') = G_{\mu\nu}(z, z') + \mathcal{M}_{\mu\nu}(z, z') \tag{31}$$

with a $\mathcal{M}$ satisfying certain conditions [14] gives the same closed string tachyon amplitude. The reason that G and G' can gives the same tachyon amplitude is because of momentum conservation. For higher closed string states, the Green function may also be contracted with other available quantum numbers like polarization which doesn't have a conservation law. So the shift is indeed possible only for tachyon amplitudes. For the amplitude of higher massive closed string states, it is (20) and (21) which are to be used. As was shown in [14], they can be obtained from an appropriate limiting procedure of the bulk Green function that solves (29) and (30). Other form of Green function gives incorrect amplitudes.

This kind of ambiguity persists at higher loops. While it can be fixed easily at the tree level, it is more subtle at the loop level. When one go to higher loop, there are many more boundaries and the above mentioned ambiguities with the Green function will be much harder to resolve [18].

At tree and the 1-loop level, all the effects of Θ can be summarized in terms of a modified Green function. One may therefore also use other approaches to obtain the Green function and use it as input in, for example (12), to calculate the string amplitude. However this is no longer the case for two and higher loops. As we mentioned above, there are new modifications to the measure that is not presented at the tree and 1-loop level. We stress that this Θ-dependent modification to the measure cannot be obtained from the Green function approach. The advantage of the Reggeon operator formalism [2] is that one obtains the string amplitude (11) in one step and there is no need to isolate a Green function from it, which is where the ambiguities lie.

3 Field Theory Limit

The relations between string and field theory amplitudes have been thoroughly studied since the early days of dual models [19]. It turned out that string amplitudes contain very precise information on various perturbative quantities of different field theories. In fact, even using the simple bosonic string as a starting point, it is possible to recover, with a suitable definition of the low energy limit, the results of the usual Feynman diagrams for scalar [20, 21], Yang–Mills [22] or gravity [23] field theories (see also the references in these papers).

In this section we briefly review the basic steps allowing to derive field theories amplitudes from string expressions like the one in (12). In fact, the same procedure normally used to recover the Feynman diagrams of commutative theories can be also applied to the results derived in the previous sections. Of course the background F-field of the string calculations is now related to the appearance of the non commutative parameters Θ in the field results. However, the algorithm one uses to perform the low energy limit is essentially not affected by the presence of the F-field [12, 13, 14, 15, 16]: all important modifications related to the new feature of noncommutativity are already encoded in the string result and in particular in Eq. (9).

It is usually said that field theory results are recovered from string amplitudes simply by taking the limit $\alpha' \to 0$. Of course, this prescription has to be suitably interpreted to yield sensible answers. First, α' is a dimensionful parameter and what the above limit really means is that the typical energy scale of the external states is very small in comparison with the string scale ($\alpha' p_i \cdot p_j \to 0$). Moreover, it is not possible to blindly perform this limit on the amplitudes. In fact, string results are written as integrals over the moduli space of the Riemann surface representing the world–sheet. Also, the integrand contains various divergences which make the limiting procedure delicate. However, the general idea underlying the derivation of field amplitudes from string ones is to obtain also the answer for field diagrams in an integral form. In fact, the world–sheet moduli are strictly related to the usual Schwinger parameters introduced in diagrammatic perturbative computations. Moreover, only the corners of moduli space where the integrand diverges contribute to the field theory limit. Thus, a single string amplitude decomposes in a sum of different contributions coming from different corners of the full integration region. It turns out that each of these terms encodes the result of all Feynman diagrams of a given topology (for instance, in the Yang–Mills case the string approach automatically sums the contribution of ghost and gluon propagation).

Summarizing the low energy limit on string amplitudes is basically done in three steps:

- First it is necessary to express in terms of string quantities all the parameters appearing in the lagrangian of the field theory one wants to reproduce. The dictionary between the two sets of parameters can be easily derived by looking at the simplest diagram in

both theories; for instance, by matching the 3-point amplitudes, one can usually find the relation between the string and the field theory coupling constants.

- Then one has to focus on the different relevant corners of the region of integration. In each case, the string moduli are transformed into Schwinger parameters with a relation of the type $t = -\alpha' \ln f(z)$, for some function f which may depend on the details of the worldsheet parameterization.

- Finally one can send $\alpha' \to 0$, but has to keep all field theory parameters finite (even if they are dimensionful, like the Schwinger proper parameters)

3.1 Taking the field theory limits

As it turns out, there are two ways one can derive the field theory limit. In [17], the Seiberg-Witten limit

$$\alpha' \sim \epsilon, \quad g \sim \epsilon^2, \quad F \sim \epsilon \tag{32}$$

with $\epsilon \to 0$ is considered so that

$$M_{\mu\nu} = -(Bg^{-1}B)_{\mu\nu}, \quad \Theta^{\mu\nu} = -2\pi\alpha' \left(\frac{1}{F}\right)^{\mu\nu} \tag{33}$$

are fixed in this limit. They show that in this limit the tree level, amplitudes which can be computed from (7), yields the result of a noncommutative field theory in a metric $M_{\mu\nu}$ and with the noncommutative parameter $\Theta^{\mu\nu}$ given by (33). For example,

$$S = \int \sqrt{\det M} \left[(\partial\Phi)^2 + V_\Theta(\Phi)\right]. \tag{34}$$

Since M and Θ are finite in this limit, all our multiloop formula also have a well defined limit. On the other hand, if one prefers (see for example [14, 2]), one can also rescale the string coordinates by a factor $\hat{X}^\mu = X^\nu(g - F)_\nu{}^\mu$ and use $\hat{X}$ to construct the corresponding string amplitude. For example, at the tree level, one gets

$$\hat{\mathcal{G}}^{\mu\nu}(\tau, \tau') := \langle \hat{X}^\mu(\tau)\hat{X}^\nu(\tau')\rangle = -\alpha' g^{\mu\nu} \ln(\tau - \tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau - \tau'), \tag{35}$$

where

$$\theta := (g + F)\Theta(g - F) = 2\pi\alpha' F \tag{36}$$

and the metric g is the closed string metric. For higher loops, one simply has to replace everywhere (e.g. in (9), (10), (20), (21)) $(M^{-1})^{\mu\nu}$, $\Theta^{\mu\nu}$ by $g^{\mu\nu}, \theta^{\mu\nu}$ in the string theoretic expressions. Thus one can also take the following *noncommutative field theory limit* [14, 2]

$$\alpha' \sim \epsilon, \quad g \text{ fixed}, \quad F \sim 1/\epsilon. \tag{37}$$

so that

$$g_{\mu\nu}, \quad \theta^{\mu\nu} = 2\pi\alpha' F^{\mu\nu} \tag{38}$$

are fixed in the limit. In this limit, the noncommutative field theory

$$S = \int \sqrt{\det g} \left[(\partial\Phi)^2 + V_\theta(\Phi)\right], \tag{39}$$

in a background metric g and with noncommutative parameter θ is resulted. We stressed that the two limits are complitely equivalent. No matter which limit one takes, there is always a pair of parameters (metric and the noncommutative parameter) in the string amplitude, and the field theory limit is always taken in such a way that they remain finite. In the end, we obtain a field theory with a background metric and a noncommutative parameter.

3.2 Examples

In what follows, we aim at producing a field theory limit with a flat Minkowskian metric $\eta_{\mu\nu}$, which is more often considered in the literatures of noncommutative quantum field theory. This can be most easily achieved from a string theory having a metric $g_{\mu\nu} = \eta_{\mu\nu}$ and using the rescaled $\hat{X}$ to construct the string amplitudes. The field theory noncommutativity parameter θ is then related to the string moduli F by equation (36).

In order to give some concrete examples, we will now consider two Feynman diagrams of the noncommutative scalar theory with cubic interaction in six dimensions. We will focus first on the non–planar contribution to the 2–point function at one loop and then on the non–planar 2-loop vacuum bubble. Many other examples are thoroughly described in [12, 14, 2] also for scalar theories with quartic interactions and for Yang–Mills theory. Eq. (12) can be used also as a starting point for studying scalar (i.e. tachyon) amplitudes simply by setting all the polarizations ϵ_i to zero. If we fix $\rho_1 = 1$ on the first border and the second puncture on the other border $\rho_2 \in [-1, -k]$, we have to use in the master formula the non–planar Green function (20), obtaining

$$A_{2,1}^{\text{NP}}(p_1, -p_1) = \frac{\alpha'^{2-d/2}}{(4\pi)^{d/2}} \frac{g_3^2}{4} \int_0^1 \frac{dk}{k} \, \mathrm{e}^{\alpha' m^2 \ln k} (-\ln k)^{-d/2} \int_{-1}^{-k} \frac{d\rho_2}{\rho_2} \, \mathrm{e}^{\left[-2\alpha' p_{1\mu} p_{1\nu} G_{\text{NP}}^{\mu\nu}(1, \rho_2)\right]} , \qquad (40)$$

where we have already translated the string coupling constant into field theory one by using the relation $g_3 = 2^{5/2} g_{\text{op}} (2\alpha')^{\frac{d-6}{4}}$, found from the matching of the 3-point functions. As usual, the logarithmic divergences in the integrand are related to the dimensionful Schwinger parameters via a factor of α'. In particular, at 1-loop one always associates $\ln k$ to the total length of the loop by taking, $\ln k = -T/\alpha'$ where T has to be kept finite as $\alpha' \to 0$. In this limit k goes to zero exponentially which means shrinking the annulus to a one-loop Feynman graph. After having replaced string quantities with field theory ones, the α' dependence of (40) simplifies and the whole amplitude is just proportional to a single power of α'. This means that, in order to have a finite answer for our field theory limit, it is necessary to introduce one more Schwinger parameter. Since we want to reproduce the irreducible diagram, we perform the $\alpha' \to 0$ limit by keeping fixed also $t_2 \ln \rho_2 = -t_2/\alpha'$ and get

$$A_{2,1}^{\text{NP}}(p_1, -p_1) = \frac{g_3^2}{4} \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{dT}{T^{d/2}} \mathrm{e}^{-m^2 T} \int_0^T dt_2 \exp\left[-p_1^2 t_2 \left(1 - \frac{t_2}{T}\right) + p_1^\mu p_1^\nu \frac{\theta_{\mu\nu}^2}{4T}\right] . \qquad (41)$$

Thus we recover the standard Schwinger proper time integral, but with the additional factor typical of the noncommutative diagrams. As noted in [24], for $\tilde{p}^2 \neq 0$ this serves as an effective UV cutoff and is at the origin of the UV/IR mixing.

In the above example the non-planarity of the diagram is related to the insertion of the external legs. From the string point of view this follows from the fact that the two insertions are done on the two different borders of the annulus. In field theory this means that the

two 3-point vertices in the graph have different cyclical orientation (and so the Filk phase has opposite sign). At one loop, this kind of non–planarity is the only possible one. As we saw in the previous string calculation, at multiloop level a new feature appears, namely the internal legs can intersect and bring a non–trivial modification of the string measure (11). A first simple check of this string result is to show that, at low energies, it can reproduce the result of noncommutative Feynman diagrams.

The simplest field theory displaying this feature is the irreducible vacuum bubble that is obtained by sewing together two 3–point vertices with different relative orientations. As in the above 1–loop diagram, the two Filk factors combine and give an additional phase factor $e^{ir\theta q}$ with the respect to the usual computation. However, now we have to integrate over both momenta contracted with θ. Carrying out the r integral, we obtain

$$A_{0,2}^{\mathrm{NP}} = \int dq \int \frac{dt_i}{l^{d/2}} e^{-m^2 \sum_i t_i} \, \exp\left(-\frac{1}{l}[q^\mu \Delta_{\theta\mu\nu} q^\nu]\right), \tag{42}$$

where t_i are the Schwinger parameters related to the three propagators, $l := t_2 + t_3$. Finally Δ_θ is the following diagonal matrix

$$(\Delta_\theta)_{\mu\nu} := \delta_{\mu\nu}(t_1 t_2 + t_1 t_3 + t_2 t_3) - \frac{\theta_{\mu\nu}^2}{4}, \tag{43}$$

which matches exactly the leading contribution in the multiplier of the expression for $A_{IJ}^{\mu\nu}$ in (9), once the string parameters have been translated in Schwinger proper times [2].

4   Some Remarks on UV/IR

In this paper, we studied open string amplitudes in the presence of a constant B-field. We derived from first principles a closed expression (11) for the N-Reggeon vertex encoding the presence of the non-trivial background. From this Reggeon vertex one can generate all string loop diagrams by the usual techniques. Then we considered the field theory limits ((33) and (37)) which, at disk level, give rise to the star-product and to noncommutative Lagrangians. In the same limits, the multiloop open string amplitudes derived from the new Reggeon vertex reduces to the loop diagrams of a noncommutative field theory. Moreover, as in the commutative case, the matching of the string and the field theory calculations is evident at the level of integrands and is diagram by diagram. These observations are further evidence that non commutative field theories can be embedded in string theory in a consistent way.

However within this point of view, there is still one important point which deserves a better understanding, that is the UV/IR mixing [24]. In fact, if string theory provides a microscopical definition for noncommutative field theories, then in principle one should be able to apply Wilson approach and obtain a low energy action describing the physics in the IR. However, in the noncommutative case, one typically finds that (for $\tilde{p}^2$) there are massless poles which do not correspond to any degree of freedom present in the effective action.

There have been various different proposals to explain this puzzle from a string point of view. A possibility is that other string states, besides those of open strings, are not compltely decoupled either because in the non planar diagrams open strings can never be really point-like [16] or because closed strings play a relevant role [24], see also [15]. However, even if this is true, it has not been possible so far to identify clearly the nature of the states entering in the effective action. Another possibility is that the UV/IR is just an artifact of the perturbative

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expansion. For instance, the pure scalar field theory we considered can be embedded easily in the bosonic open string theory, which is tachyonic (here we expanded around the usual flat vacuum). However, it is by now clear that open string tachyon can condense giving rise to stable vacua and it is possible that a field theory limit around these new vacua is not pathological. On the other hand UV/IR mixing can also be presented in supersymmetric gauge theories which can be embedded in perfectly consistent string models. Perturbative analysis of supersymmetric gauge theory with focus on aspects of UV/IR can be found in [25]. From the knowledge accumulated on the commutative gauge theory, it is perhaps not surprising that nonperturbative contribution may modifies substantially the physics in the IR and may provide a cure to the UV/IR mixing. Recently the case of noncommutative pure $N = 2$ Yang-Mills theory has been analyzed beyond the perturbative level [26], and particularly in [27]. It seems that the contribution of instantons makes the IR behvior less pathological.

Acknowledgments

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