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Boyd indices for quasi-normed function spaces with some bounds

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Abstract
We calculate the Boyd indices for quasi-normed rearrangement invariant function spaces with some bounds. An application to Lorentz type spaces is also given.

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1 Introduction

Let L_{loc} be the space of all locally integrable functions f on $\mathbf{R}^n$ and M^+ be the cone of all locally integrable functions $g \geq 0$ on $(0,1)$ with the Lebesgue measure.

Let f^* be the decreasing rearrangement of f given by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}, \quad t > 0,$$

and μ_f be the distribution function of f defined by

$$\mu_f(\lambda) = \bigl|\{x \in \mathbf{R}^n : |f(x)| > \lambda\}\bigr|_n,$$

$|\cdot|_n$ denoting Lebesgue n-measure.

Also,

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,ds.$$

We use the notations $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

We consider rearrangement invariant quasi-normed spaces $E \hookrightarrow L^1(\Omega)$ such that $\|f\|_E = \rho_E(f^*) < \infty$, where ρ_E is a quasi-norm rearrangement invariant defined on M^+.

For simplicity, we assume that Ω is a bounded Lebesgue measurable subset of $\mathbf{R}^n$ with Lebesgue measure equal to 1 and origin lies in Ω.

There is an equivalent quasi-norm $\rho_p \approx \rho_E$ that satisfies the triangle inequality $\rho_p^p(g_1 + g_2) \leq \rho_p^p(g_1) + \rho_p^p(g_2)$ for some $p \in (0,1]$ that depends only on the space E (see [1]). We say that the quasi-norm ρ_E satisfies Minkowski's inequality if for the equivalent quasi-norm ρ_p,

$$\rho_p^p\Big(\sum g_j\Big) \lesssim \sum \rho_p^p(g_j), \quad g_j \in M^+.$$

Usually we apply this inequality for functions $g \in M^+$ with some kind of monotonicity.

Recall the definition of the lower and upper Boyd indices α_E and β_E. Let $g_u(t) = g(t/u)$ if $t < \min(1,u)$ and $g_u(t) = 0$ if $\min(1,u) < t < 1$, where $g \in M^+$, and let

$$h_E(u) = \sup\left\{\frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in M^+\right\}, \quad u > 0$$

be the dilation function generated by ρ_E. Suppose that it is finite. Then

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \quad \text{and} \quad \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}.$$

The function h_E is sub-multiplicative, increasing, $h_E(1) = 1$, $h_E(u)h_E(1/u) \geq 1$ and hence $0 \leq \alpha_E \leq \beta_E$. We suppose that $0 < \alpha_E = \beta_E \leq 1$.

If $\beta_E < 1$, we have by using Minkowski's inequality that $\rho_E(f^*) \approx \rho_E(f^{**})$. In particular, $\|f\|_E \approx \rho_E(f^{**})$ if $\beta_E < 1$. For example, consider the gamma spaces $E = \Gamma^q(w)$, $0 < q \leq \infty$, w-positive weight, that is, a positive function from M^+, with a quasi-norm $\|f\|_{\Gamma^q(w)} := \rho_E(f^*)$, $\rho_E(g) := \rho_{w,q}(\int_0^1 g(tu)\,du)$, where

$$\rho_{w,q}(g) := \left(\int_0^1 \big[g(t)w(t)\big]^q \,dt/t\right)^{1/q}, \quad g \in M^+ \tag{1.1}$$

and

$$\left(\int_0^1 w^q(t)\,dt/t\right)^{1/q} < \infty.$$

Then $L^\infty(\Omega) \hookrightarrow \Gamma^q(w) \hookrightarrow L^1(\Omega)$. If $w(t) = t^{1/p}$, $1 < p < \infty$, we write as usual $L^{p,q}$ instead of $\Gamma^q(t^{1/p})$. Consider also the classical Lorentz spaces $\Lambda^q(w)$, $0 < q \leq \infty$; $f \in \Lambda^q(w)$ if $\|f\|_{\Lambda^q_w} := \rho_{w,q}(f^*) < \infty$, $w(2t) \approx w(t)$. We suppose that $L^\infty(\Omega) \hookrightarrow \Lambda^q(w) \hookrightarrow L^1(\Omega)$.

The Boyd indices are useful in various problems concerning continuity of operators acting in rearrangement invariant spaces [2] or in optimal couples of rearrangement invariant spaces [3–5], and in the problems of optimal embeddings [6–8]. The main goal of this paper is to provide formulas for the Boyd indices with some bounds of rearrangement invariant quasi-normed spaces and to apply these results to the case of Lorentz type spaces.

2 Boyd indices for quasi-normed function spaces

Let ρ_E be a monotone quasi-norm on M^+ and let E be the corresponding rearrangement invariant quasi-normed space consisting of all $f \in L^1(\Omega)$ such that $\|f\|_E = \rho_E(f^*) < \infty$.

Theorem 2.1 *Let*

$$g_u(t) = \begin{cases} g(t/u) & \textit{if } 0 < t < \min(1,u), \\ 0 & \textit{if } \min(1,u) \leq t < 1, \end{cases}$$

where $g \in M^+$, and let

$$h_E(u) = \sup\left\{\frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in M^+\right\}, \quad u > 0,$$

be the dilation function generated by ρ_E. Suppose that it is finite. Then the Boyd indices are well defined

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \quad \textit{and} \quad \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}$$

and they satisfy

$$\alpha_E = \lim_{t\to 0} \frac{\log h_E(t)}{\log t}, \tag{2.1}$$

$$\beta_E = \lim_{t\to\infty} \frac{\log h_E(t)}{\log t}. \tag{2.2}$$

In particular, $0 \le \alpha_E \le \beta_E \le \frac{\log h_E(2)}{\log 2}$.

Proof We have

$$g_{uv} = (g_u)_v \quad \text{if } u < v. \tag{2.3}$$

Indeed, since $\min(1, uv) \le \min(1, v)$ for $u < v$, we find $(g_u)_v(t) = g_u(t/(uv))$ if $0 < t < \min(1, uv)$ and $(g_u)_v(t) = 0$ if $\min(1, uv) \le t < 1$. Thus (2.3) is proved. This implies that the function h_E is sub-multiplicative.

Further, the function $\omega(x) = \log h_E(e^x)$ is sub-additive increasing on $(-\infty, \infty)$ and $\omega(0) = 0$. Hence, by [2], Lemma 5.8, (2.2) is satisfied and evidently $\beta_E \le \frac{\log h_E(2)}{\log 2}$.

Since $h_E(1) = 1$ and h_E is sub-multiplicative, therefore

$$h_E(u_1u_2) \le h_E(u_1)h_E(u_2).$$

Replacing u_2 by $\frac{1}{u_1}$, we get

$$h_E(1) \le h_E(u_1)h_E\left(\frac{1}{u_1}\right),$$

which implies that

$$1 \le h_E(u_1)h_E\left(\frac{1}{u_1}\right); \quad \text{because } h_E(1) = 1,$$

it follows that $1 \le h_E(u)h_E(1/u)$.

We have

$$\alpha_E \le \beta_E.$$

Indeed

$$\log\bigl(h_E(u)\bigr) \geq \log\left(\frac{1}{h_E(\frac{1}{u})}\right),$$

if $u > 1$, then

$$\frac{\log(h_E(u))}{\log u} \geq \frac{\log(\frac{1}{h_E(\frac{1}{u})})}{\log u} = \frac{\log(h_E(\frac{1}{u}))}{\log\frac{1}{u}},$$

which implies that

$$\lim_{u\to\infty}\frac{\log(h_E(u))}{\log u} \geq \lim_{u\to\infty}\frac{\log(h_E(\frac{1}{u}))}{\log\frac{1}{u}}.$$

Since β_E is finite, therefore α_E is also finite. Since $h_E(1) = 1$ and we know that h_E is increasing function, so

$$h_E(u) \leq 1 \quad \text{for } 0 < u < 1,$$

which implies that

$$\log\bigl(h_E(u)\bigr) \leq 0,$$

which implies that

$$\frac{\log(h_E(u))}{\log u} \geq 0,$$

which implies that

$$\alpha_E = \sup_{0<u<1}\frac{\log(h_E(u))}{\log u} \geq 0,$$

and hence

$$0 \leq \alpha_E \leq \beta_E. \qquad \square$$

Let ρ_H be a monotone quasi-norm on M^+ and let H be the corresponding quasi-normed space, consisting of all locally integrable functions on $(0,1)$ with a finite quasi-norm $\|g\|_H = \rho_H(|g|)$.

Theorem 2.2 *Let*

$$(\Psi_u g)(t) = \begin{cases} g(ut), & \textit{if } 0 < t < \min(1,\frac{1}{u}), \\ g(1), & \textit{if } \min(1,\frac{1}{u}) \leq t < 1, \end{cases}$$

where $g \in M^+$, *and let*

$$h_H(u) = \sup\left\{\frac{\rho_H(\Psi_u g)}{\rho_H(g)} : g \in G_a\right\}, \quad u > 0,$$

be the dilation function generated by ρ_H. Suppose that it is finite, where

$$G_a := \{g \in M^+ : t^{-a/n} g(t) \text{ is decreasing}\}, \quad a > 0.$$

Then the Boyd indices are well defined

$$\alpha_H := \sup_{0<t<1} \frac{\log h_H(t)}{\log t} \quad \text{and} \quad \beta_H := \inf_{1<t<\infty} \frac{\log h_H(t)}{\log t}$$

and they satisfy

$$\alpha_H = \lim_{t\to 0} \frac{\log h_H(t)}{\log t}, \tag{2.4}$$

$$\beta_H = \lim_{t\to\infty} \frac{\log h_H(t)}{\log t}. \tag{2.5}$$

In particular, $\frac{\log h_H(1/2)}{\log 1/2} \leq \alpha_H \leq \beta_H \leq a/n$.

Proof We have

$$\Psi_{uv} g = \Psi_u(\Psi_v g) \quad \text{if } u < v. \tag{2.6}$$

Indeed, since $\min(1, 1/(uv)) \leq \min(1, 1/u)$ for $u < v$, we find $\Psi_u(\Psi_v g)(t) = g(t/(uv))$ if $0 < t < \min(1, 1/(uv))$ and $\Psi_u(\Psi_v g)(t) = g(1)$ if $\min(1, 1/(uv)) \leq t < 1$. Thus (2.6) is proved. This implies that the function h_H is sub-multiplicative. Since the function $u^{-a/n} h_H(u)$ is decreasing, it follows that the function $u^{a/n} h_H(1/u)$ is increasing and sub-multiplicative. Hence we can apply the results from Theorem 2.1. This establishes Theorem 2.2. □

Example 2.3 If $E = \Lambda^q(t^a w)$, $0 \leq a \leq 1$, $0 < q \leq \infty$, where w is slowly varying, then $\alpha_E = \beta_E = a$.

Proof We give a proof for $0 < q < \infty$, the case $q = \infty$ is analogous. We have, for $g \in M^+$,

$$\rho_E(g_u^*) = \left(\int_0^1 [g_u^*(t) t^a w(t)]^q \, dt/t\right)^{1/q} = \left(\int_0^{\min(1,u)} [g^*(t/u) t^a w(t)]^q \, dt/t\right)^{1/q}$$

and by a change of variables,

$$\rho_E(g_u^*) \leq \left(\int_0^1 [g^*(t)(tu)^a w(tu)]^q \, dt/t\right)^{1/q}. \tag{2.7}$$

From the definition of a slowly varying function it follows that for every $\varepsilon > 0$, $t^{-\varepsilon} w(t) \approx d(t)$, where d is a decreasing function. Then, for $u > 1$, we have $d(tu) \leq d(t)$, thus

$$(tu)^{-\varepsilon} w(tu) \lesssim d(tu) \lesssim t^{-\varepsilon} w(t),$$

which implies that

$$w(tu) \lesssim u^{\varepsilon} w(t), \quad u > 1. \tag{2.8}$$

Inserting this estimate in (2.7), we arrive at

$$\rho_E(g_u^*) \lesssim u^{a+\varepsilon}\rho_E(g^*), \quad u > 1,$$

which yields $h_E(u) \lesssim u^{a+\varepsilon}$, $u > 1$. Then it follows that $\beta_E \le a + \varepsilon$. Analogously, $\alpha_E \ge a - \varepsilon$. Since $\varepsilon > 0$ is arbitrary and $\alpha_E \le \beta_E$, we obtain $\alpha_E = \beta_E = a$. □

Example 2.4 If $H = L_*^q(w(t)t^{-\alpha})$, $0 \le \alpha < a/n$, $0 < q \le \infty$, where w is slowly varying, then $\alpha_H = \beta_H = \alpha$.

Proof We give a proof for $0 < q < \infty$, the case $q = \infty$ is analogous. We have, for $g \in G_a$,

$$\begin{aligned}\rho_H(\Psi_u g) &= \left(\int_0^1 \left[\Psi_u g(t)t^{-\alpha}w(t)\right]^q\,dt/t\right)^{1/q} \\ &= \left(\int_0^{\min(1,1/u)} \left[g(tu)t^{-\alpha}w(t)\right]^q\,dt/t\right)^{1/q} + I(u),\end{aligned}$$

where $I(u) = (\int_{\min(1,1/u)}^1 [t^{-\alpha}w(t)]^q\,dt/t)^{1/q}g(1)$. Note that $I(u) = 0$ for $0 < u < 1$. Since for every $\varepsilon > 0$ we have $w(t) \lesssim t^{\varepsilon}$, it follows that $I(u) \lesssim u^{\alpha+\varepsilon}g(1)$, $u > 1$. Also, $g(1)\rho_H(t^{a/n}) \le \rho_H(g)$ and $\rho_H(t^{a/n}) < \infty$ due to $\alpha < a/n$.

On the other hand, by a change of variables,

$$\rho_H(\Psi_u g) \lesssim \left(\int_0^1 \left[g(t)(t/u)^{-\alpha}w(t/u)\right]^q\,dt/t\right)^{1/q} + u^{\alpha+\varepsilon}\rho_H(g).$$

As in the proof of the previous example, we have

$$w(t/u) \lesssim u^{\varepsilon}w(t), \quad u > 1,$$

therefore

$$\rho_H(\Psi_u g) \lesssim u^{\alpha+\varepsilon}\rho_H(g), \quad u > 1, g \in G_a.$$

Hence $h_H(u) \lesssim u^{\alpha+\varepsilon}$, $u > 1$. Then it follows that $\beta_H \le \alpha + \varepsilon$. Analogously, $\alpha_H \ge \alpha - \varepsilon$. Since $\varepsilon > 0$ is arbitrary and $\alpha_H \le \beta_H$, we obtain $\alpha_H = \beta_H = \alpha$. □

3 Basic inequalities

Here we prove a few inequalities, which are of independent interest.

Theorem 3.1 *If* $\alpha < \alpha_H$, *then*

$$\rho_H\left(t^{\alpha}\int_0^t s^{-\alpha}g(s)\frac{ds}{s}\right) \lesssim \rho_H(g), \quad g \in G_a$$

and if $\beta_H < \beta$, *then*

$$\rho_H\left(t^{\beta}\int_t^1 s^{-\beta}g(s)\frac{ds}{s}\right) \lesssim \rho_H(g), \quad g \in G_a.$$

Proof We are going to use Minkowski's inequality for the equivalent p-norm of ρ_H. To this end, first we replace the integrals by sums using monotonicity properties of $g \in G_a$.

Thus

$$
\begin{aligned}
t^{\alpha} \int_0^t s^{-\alpha} g(s) \frac{ds}{s} &= \int_0^1 v^{-\alpha} g(tv) \frac{dv}{v} \\
&= \sum_{l=-\infty}^{0} \int_{2^{l-1}}^{2^l} v^{-\alpha} g(tv) \frac{dv}{v} \\
&\lesssim \sum_{l=-\infty}^{0} 2^{-l\alpha} g\bigl(t2^l\bigr).
\end{aligned}
$$

Applying Minkowski's inequality, we get

$$
\begin{aligned}
\rho_H^p \biggl(t^{\alpha} \int_0^t s^{-\alpha} g(s) \frac{ds}{s} \biggr) &\lesssim \sum_{l=-\infty}^{0} 2^{-lp\alpha} \rho_H^p \bigl(g\bigl(t2^l\bigr)\bigr) \\
&\lesssim \rho_H^p(g) \sum_{l=-\infty}^{0} 2^{-p\alpha l} h_H^p\bigl(2^l\bigr) \\
&\lesssim \rho_H^p(g) \sum_{l=-\infty}^{0} 2^{-p\alpha l} 2^{lp(\alpha_H - \varepsilon)} \\
&\lesssim \rho_H^p(g) \sum_{l=-\infty}^{0} 2^{lp(\alpha_H - \varepsilon - \alpha)}.
\end{aligned}
$$

The above series is convergent if we choose $\varepsilon > 0$ such that $\varepsilon < \alpha_H - \alpha$, so we have

$$
\rho_H \biggl(t^{\alpha} \int_0^t s^{-\alpha} g(s) \frac{ds}{s} \biggr) \lesssim \rho_H(g).
$$

On the other hand, for $g \in G_a$,

$$
\begin{aligned}
t^{\beta} \int_t^1 s^{-\beta} g(s) \frac{ds}{s} &= \int_1^{\infty} \chi_{(0,1)}(tv) v^{-\beta} g(tv) \frac{dv}{v} \\
&= \sum_{l=0}^{\infty} \int_{2^l}^{2^{l+1}} \chi_{(0,1)}(tv) v^{-\beta} g(tv) \frac{dv}{v} \\
&\lesssim \sum_{l=0}^{\infty} 2^{-l\beta} g\bigl(t2^l\bigr) \chi_{(0,1)}\bigl(t2^l\bigr).
\end{aligned}
$$

Again applying Minkowski's inequality, we get

$$
\begin{aligned}
\rho_H^p \biggl(t^{\beta} \int_t^1 s^{-\beta} g(s) \frac{ds}{s} \biggr) &\lesssim \sum_{l=0}^{\infty} 2^{-l\beta p} \rho_H^p \bigl(g\bigl(t2^l\bigr) \chi_{(0,1)}\bigl(t2^l\bigr)\bigr) \\
&\lesssim \rho_H^p(g) \sum_{l=0}^{\infty} 2^{-l\beta p} h_H^p\bigl(2^l\bigr)
\end{aligned}
$$

$$\lesssim \rho_H^p(g) \sum_{l=0}^{\infty} 2^{-l\beta p} 2^{pl(\beta_H+\varepsilon)}$$

$$\lesssim \rho_H^p(g) \sum_{l=0}^{\infty} 2^{lp(\beta_H+\varepsilon-\beta)}.$$

The above series is finite if we choose a suitable $\varepsilon > 0$ such that $\varepsilon < \beta - \beta_H$. The proof is finished. □

Theorem 3.2 *If* $\beta_E < a$, *then*

$$\rho_E\left(t^{-a} \int_0^t s^a g(s) \frac{ds}{s}\right) \lesssim \rho_E(g), \quad g \in D_0,$$

where $D_0 := \{g \in M^+ : g(t) \text{ is decreasing and } g(t) = 0 \text{ for } t \geq 1\}$.

Proof We are going to use Minkowski's inequality for the equivalent p-norm of ρ_E. To this end, first we replace the integral by sums using monotonicity properties of $g \in D_0$.

Thus

$$\begin{aligned} t^{-a} \int_0^t s^a g(s) \frac{ds}{s} &= \int_0^1 v^a g(tv) \frac{dv}{v} \\ &= \sum_{l=-\infty}^{0} \int_{2^l}^{2^{l+1}} v^a g(tv) \frac{dv}{v} \\ &\lesssim \sum_{l=-\infty}^{0} 2^{al} g\bigl(t2^l\bigr). \end{aligned}$$

Applying Minkowski's inequality, we get

$$\begin{aligned} \rho_E^p\left(t^{-a} \int_0^t s^a g(s) \frac{ds}{s}\right) &\lesssim \sum_{l=-\infty}^{0} 2^{pal} \rho_E^p\bigl(g\bigl(t2^l\bigr)\bigr) \\ &\lesssim \rho_E^p(g) \sum_{l=-\infty}^{0} 2^{pal}\, h_E^p\bigl(2^1\bigr) \\ &\lesssim \rho_E^p(g) \sum_{l=-\infty}^{0} 2^{pal}\, 2^{-1p(\beta_E+\varepsilon)} \\ &\lesssim \rho_E^p(g) \sum_{l=-\infty}^{0} 2^{lp(a-\beta_E-\varepsilon)}. \end{aligned}$$

The above series is finite if we choose $\varepsilon > 0$ such that $\varepsilon < a - \beta_E$, and this concludes the proof. □

Theorem 3.3 *If* $\alpha_E > 0$, *then*

$$\rho_E\left(\int_t^1 g(u) \frac{du}{u}\right) \lesssim \rho_E(g), \quad g \in D_0.$$

Proof We are going to use Minkowski's inequality for the equivalent p-norm of ρ_E. To this end, first we replace the integral by sums using monotonicity properties of $g \in D_0$.

Thus

$$\begin{aligned} \int_t^1 g(u)\frac{du}{u} &\lesssim \int_1^\infty \chi_{(0,1)}(tv)g(tv)\frac{dv}{v} \\ &= \sum_{l=0}^\infty \int_{2^l}^{2^{l+1}} \chi_{(0,1)}(tv)g(tv)\frac{dv}{v} \\ &\lesssim \sum_{l=0}^\infty \chi_{(0,1)}\bigl(t2^l\bigr)g\bigl(t2^l\bigr). \end{aligned}$$

Applying Minkowski's inequality, we get

$$\begin{aligned} \rho_E^p\biggl(\int_t^1 g(u)\frac{du}{u}\biggr) &\lesssim \sum_{l=0}^\infty \rho_E^p\bigl(\chi_{(0,1)}\bigl(t2^l\bigr)g\bigl(t2^l\bigr)\bigr) \\ &\lesssim \rho_E^p(g)\sum_{l=0}^\infty h_E^p\bigl(2^{-l}\bigr) \\ &\lesssim \rho_E^p(g)\sum_{l=0}^\infty 2^{-l(\alpha_E-\varepsilon)}. \end{aligned}$$

Choosing $\varepsilon > 0$ such that $\alpha_E > \varepsilon$, we conclude the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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