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# A new class of analytic functions defined by means of a generalization of the Srivastava-Attiya operator

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Full list of author information is available at the end of the article**Abstract**

In this paper, we introduce a new class of analytic functions defined by a new convolution operator  $J_{(\lambda, \rho), (\mu, q), b}^{s, a, \lambda}$  which generalizes the well-known Srivastava-Attiya operator investigated by Srivastava and Attiya (*Integral Transforms Spec. Funct.* 18:207-216, 2007). We derive coefficient inequalities, distortion theorems, extreme points and the Fekete-Szegő problem for this new function class.

**MSC:** Primary 30C45; 11M35; secondary 30C10**Keywords:** analytic functions; starlike functions; generalized Hurwitz-Lerch zeta function; Srivastava-Attiya operator; Hadamard product

## 1 Introduction

Let  $\mathcal{A}$  denote the class of functions  $f(z)$  normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}.$$

A function  $f(z)$  in the class  $\mathcal{A}$  is said to be in the class  $\mathcal{S}^*(\alpha)$  of starlike functions of order  $\alpha$  in  $\mathbb{U}$  if it satisfies the following inequality:

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1). \quad (1.2)$$

The largely investigated Srivastava-Attiya operator is defined as [1] (see also [2–4]):

$$J_{s, a}(f)(z) = z + \sum_{k=2}^{\infty} \left( \frac{1+a}{k+a} \right)^s a_k z^k, \quad (1.3)$$

where  $z \in \mathbb{U}$ ,  $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ,  $s \in \mathbb{C}$  and  $f \in \mathcal{A}$ .

In fact, the linear operator  $J_{s,a}(f)$  can be written as

$$J_{s,a}(f)(z) := G_{s,a}(z) * f(z) \tag{1.4}$$

in terms of the Hadamard product (or convolution), where  $G_{s,a}(z)$  is given by

$$G_{s,a}(z) := (1+a)^s [\Phi(z, s, a) - a^{-s}] \quad (z \in \mathbb{U}). \tag{1.5}$$

The function  $\Phi(z, s, a)$  involved in the right-hand side of (1.5) is the well-known Hurwitz-Lerch zeta function defined by (see, for example, [5, p.121 *et seq.*]; see also [6] and [7, p.194 *et seq.*])

$$\begin{aligned} \Phi(z, s, a) &:= \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \\ (a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1). \end{aligned} \tag{1.6}$$

Recently, a new family of  $\lambda$ -generalized Hurwitz-Lerch zeta functions was investigated by Srivastava [8] (see also [9–13]). Srivastava considered the following function:

$$\begin{aligned} &\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z, s, a; b, \lambda) \\ &= \frac{1}{\lambda \Gamma(s)} \cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^q (\mu_j)_{n\sigma_j}} H_{0,2}^{2,0} \left[ (a+n)b^{\frac{1}{\lambda}} \mid \overline{(s, 1), \left(0, \frac{1}{\lambda}\right)} \right] \frac{z^n}{n!} \\ &(\min\{\Re(a), \Re(s)\} > 0; \Re(b) > 0; \lambda > 0), \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} &\left( \lambda_j \in \mathbb{C} \ (j = 1, \dots, p) \text{ and } \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ (j = 1, \dots, q); \rho_j > 0 \ (j = 1, \dots, p); \right. \\ &\left. \sigma_j > 0 \ (j = 1, \dots, q); 1 + \sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j \geq 0 \right) \end{aligned}$$

and the equality in the convergence condition holds true for suitably bounded values of  $|z|$  given by

$$|z| < \nabla := \left( \prod_{j=1}^p \rho_j^{-\rho_j} \right) \cdot \left( \prod_{j=1}^q \sigma_j^{\sigma_j} \right).$$

Here, and for the remainder of this paper,  $(\lambda)_\kappa$  denotes the Pochhammer symbol defined, in terms of the gamma function, by

$$(\lambda)_\kappa := \frac{\Gamma(\lambda + \kappa)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda + 1) \cdots (\lambda + \kappa - 1) & (\kappa = n \in \mathbb{N}; \lambda \in \mathbb{C}), \\ 1 & (\kappa = 0; \lambda \in \mathbb{C} \setminus \{0\}), \end{cases} \tag{1.8}$$

it being understood *conventionally* that  $(0)_0 := 1$  and assumed *tacitly* that the  $\Gamma$ -quotient exists (see, for details, [14, p.21 *et seq.*]).

**Definition 1** The  $H$ -function involved in the right-hand side of (1.7) is the well-known Fox’s  $H$ -function [15, Definition 1.1] (see also [14, 16]) defined by

$$\begin{aligned}
 H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n} \left[ z \mid \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right] \\
 &= \frac{1}{2\pi i} \int_{\mathcal{L}} \Xi(s) z^{-s} ds \quad (z \in \mathbb{C} \setminus \{0\}; |\arg(z)| < \pi),
 \end{aligned}
 \tag{1.9}$$

where

$$\Xi(s) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \cdot \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\prod_{j=n+1}^p \Gamma(a_j + A_j s) \cdot \prod_{j=m+1}^q \Gamma(1 - b_j - B_j s)},$$

an empty product is interpreted as 1,  $m, n, p$  and  $q$  are integers such that  $1 \leq m \leq q, 0 \leq n \leq p, A_j > 0 (j = 1, \dots, p), B_j > 0 (j = 1, \dots, q), a_j \in \mathbb{C} (j = 1, \dots, p), b_j \in \mathbb{C} (j = 1, \dots, q)$  and  $\mathcal{L}$  is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\{\Gamma(b_j + B_j s)\}_{j=1}^m$$

from the poles of the gamma functions

$$\{\Gamma(1 - a_j + A_j s)\}_{j=1}^n.$$

It is worthy to mention that using the fact that [8, p.1496, Remark 7]

$$\lim_{b \rightarrow 0} \left\{ H_{0,2}^{2,0} \left[ (a+n)b^{\frac{1}{\lambda}} \mid \overline{(s, 1), \left(0, \frac{1}{\lambda}\right)} \right] \right\} = \lambda \Gamma(s) \quad (\lambda > 0),
 \tag{1.10}$$

equation (1.7) reduces to

$$\begin{aligned}
 \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z, s, a; 0, \lambda) &:= \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z, s, a) \\
 &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^q (\mu_j)_{n\sigma_j}} \frac{z^n}{n!}.
 \end{aligned}
 \tag{1.11}$$

**Definition 2** The function  $\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z, s, a)$  involved in (1.11) is the multiparameter extension and generalization of the Hurwitz-Lerch zeta function  $\Phi(z, s, a)$  introduced by Srivastava *et al.* [13, p.503, Eq. (6.2)] defined by

$$\begin{aligned}
 \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z, s, a) &:= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^q (\mu_j)_{n\sigma_j}} \frac{z^n}{n!} \\
 &\left( p, q \in \mathbb{N}_0; \lambda_j \in \mathbb{C} (j = 1, \dots, p); a, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- (j = 1, \dots, q); \right. \\
 &\left. \rho_j, \sigma_k \in \mathbb{R}^+ (j = 1, \dots, p; k = 1, \dots, q); \right)
 \end{aligned}$$

$$\begin{aligned}
 \Delta &> -1 \text{ when } s, z \in \mathbb{C}; \\
 \Delta &= -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \nabla^*; \\
 \Delta &= -1 \text{ and } \Re(\Xi) > \frac{1}{2} \text{ when } |z| = \nabla^*
 \end{aligned}
 \tag{1.12}$$

with

$$\nabla^* := \left( \prod_{j=1}^p \rho_j^{-\rho_j} \right) \cdot \left( \prod_{j=1}^q \sigma_j^{\sigma_j} \right),
 \tag{1.13}$$

$$\Delta := \sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j \quad \text{and} \quad \Xi := s + \sum_{j=1}^q \mu_j - \sum_{j=1}^p \lambda_j + \frac{p-q}{2}.
 \tag{1.14}$$

We propose to consider the following linear operator

$$J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}(f) : \mathcal{A} \rightarrow \mathcal{A},$$

defined by

$$J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}(f)(z) = G_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}(z) * f(z),
 \tag{1.15}$$

where  $*$  denotes the Hadamard product (or convolution) of analytic functions, and the function  $G_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}(z)$  is given by

$$\begin{aligned}
 &G_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}(z) \\
 &:= \frac{\lambda \prod_{j=1}^q (\mu_j) \Gamma(s)(a+1)^s}{\prod_{j=1}^p (\lambda_j)} \cdot \Lambda(a+1, b, s, \lambda)^{-1} \\
 &\quad \cdot \left[ \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(1, \dots, 1, 1, \dots, 1)}(z, s, a; b, \lambda) - \frac{a^{-s}}{\lambda \Gamma(s)} \Lambda(a, b, s, \lambda) \right] \\
 &= z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \left( \frac{a+1}{a+k} \right)^s \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \frac{z^k}{k!}
 \end{aligned}
 \tag{1.16}$$

with

$$\Lambda(a, b, s, \lambda) := H_{0,2}^{2,0} \left[ ab^{\frac{1}{\lambda}} \mid (s, 1), \left( 0, \frac{1}{\lambda} \right) \right].$$

Combining (1.15) and (1.16), we obtain

$$\begin{aligned}
 &J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}(f)(z) \\
 &= z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \left( \frac{a+1}{a+k} \right)^s \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) a^k \frac{z^k}{k!} \\
 &\quad \left( \lambda_j \in \mathbb{C} \ (j = 1, \dots, p) \text{ and } \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ (j = 1, \dots, q); p \leq q + 1; z \in \mathbb{U} \right),
 \end{aligned}
 \tag{1.17}$$

with

$$\min\{\Re(a), \Re(s)\} > 0; \quad \lambda > 0 \quad \text{if } \Re(b) > 0$$

and

$$s \in \mathbb{C}; \quad a \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad \text{if } b = 0.$$

**Remark 1** It follows from (1.15) and (1.17) that the operator  $J_{(\lambda_p),(\mu_q),0}^{s,a,\lambda}(f)$  (special case of (1.17) when  $b = 0$ ) can be defined for  $a \in \mathbb{C} \setminus \mathbb{Z}^-$  by the following limit relationship:

$$J_{(\lambda_p),(\mu_q),0}^{s,0,\lambda}(f)(z) := \lim_{a \rightarrow 0} \{J_{(\lambda_p),(\mu_q),0}^{s,a,\lambda}(f)(z)\}. \tag{1.18}$$

We can see that the operator  $J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}$  generalizes several recently investigated operators such as:

- (i) If  $p = 2, q = 1$  and  $b = 0$ , then  $J_{(\lambda_1,\lambda_2),(\mu_1),0}^{s,a,\lambda} = J_{\lambda_1,\lambda_2;\mu_1}^{s,a}$ , where  $J_{\lambda_1,\lambda_2;\mu_1}^{s,a}$  is the linear operator introduced by Prajapat and Bulboacă [17, p.571, Eq. (1.8)].
- (ii)  $J_{(\gamma-1,1),(\nu),0}^{s,a,\lambda} = I_{a,\nu,\gamma}^s$ , where  $I_{a,\nu,\gamma}^s$  is the generalized operator recently studied by Noor and Bukhari [18, p.2, Eq. (1.3)].
- (iii)  $J_{(\gamma-1,1),(\nu),0}^{0,0,\lambda} = I_{\nu,\gamma}^s$ , where  $I_{\nu,\gamma}^s$  is the Choi-Saigo-Srivastava operator [19].
- (iv)  $J_{(\gamma,1),(\gamma),0}^{s,a,\lambda} = J_{s,a}$ , where  $J_{s,a}$  is the Srivastava-Attiya operator [1].
- (v)  $J_{(\gamma,1),(\gamma),0}^{-r,a,\lambda} = I(r, a)$  ( $a \geq 0, r \in \mathbb{Z}$ ), where the operator  $I(r, a)$  is the one introduced by Cho and Srivastava [20].
- (vi)  $J_{(\beta,1),(\alpha+\beta),0}^{0,a,\lambda} = \mathcal{Q}_\beta^\alpha$  ( $\alpha \geq 0, \beta > -1$ ), where the operator  $\mathcal{Q}_\beta^\alpha$  was studied by Jung *et al.* [21].
- (vii)  $J_{(\gamma,1),(\gamma),0}^{1,a,\lambda} = J_a$  ( $a \geq -1$ ), where  $J_a$  denotes the Bernardi operator [22].
- (viii)  $J_{(\gamma,1),(\nu),0}^{0,0,\lambda} = \mathcal{L}(\gamma, \nu)$ , where  $\mathcal{L}(\gamma, \nu)$  is the well-known Carlson-Shaffer operator [23].
- (ix)  $J_{(2,1),(2-\gamma),0}^{0,0,\lambda} = \Omega_z^\gamma$  ( $0 \leq \gamma < 1$ ), where  $\Omega_z^\gamma$  is the fractional integral operator investigated by Owa and Srivastava [24].
- (x)  $J_{(\lambda_1-1,\dots,\lambda_p-1),(\mu_1-1,\dots,\mu_q-1),0}^{0,a,\lambda} = H_1(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q)$  ( $p \leq q + 1$ ), where the operator  $H_1(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q)$  is the Dziok-Srivastava operator [25, 26] which contains as special cases the Hohlov operator [27] and the Ruscheweyh operator [28].

We say that a function  $f \in \mathcal{A}$  is in the class  $\mathcal{S}_{(\lambda_p),(\mu_q),b}^{s,a,\lambda,*}(\alpha)$  if  $J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}(f)$  is in the class  $\mathcal{S}^*(\alpha)$ , that is, if

$$\Re\left(\frac{z(J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}(f))'}{J_{(\lambda_p),(\mu_q),b}^{s,a,\lambda}(f)}\right) > \alpha$$

$$(\lambda_j \in \mathbb{C} (j = 1, \dots, p) \text{ and } \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- (j = 1, \dots, q);$$

$$z \in \mathbb{U}; 0 \leq \alpha < 1; p \leq q + 1), \tag{1.19}$$

with

$$\min\{\Re(a), \Re(s)\} > 0; \quad \lambda > 0 \quad \text{if } \Re(b) > 0$$

and

$$s \in \mathbb{C}; \quad a \in \mathbb{C} \setminus \mathbb{Z}^- \quad \text{if } b = 0.$$

In this paper, we systematically investigate the class  $\mathcal{S}_{(\lambda,p),(\mu,q),b}^{s,a,\lambda,*}(\alpha)$  of analytic functions defined above by means of the new generalized Srivastava-Attiya convolution operator  $J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}$ . Especially, we derive coefficient inequalities, distortion theorems, extreme points and the Fekete-Szegő problem for this new function class.

### 2 Coefficient inequalities

**Theorem 1** *Let  $\alpha \in [0, 1)$ . If  $f(z) \in \mathcal{A}$  satisfies the following equality*

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)}{k!} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \right| \left| \left( \frac{a+1}{a+k} \right)^s \right| \left| \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| |a_k| \leq 1 - \alpha, \tag{2.1}$$

then

$$f \in \mathcal{S}_{(\lambda,p),(\mu,q),b}^{s,a,\lambda,*}(\alpha).$$

*Proof* Suppose that inequality (2.1) holds for  $\alpha \in [0, 1)$ . Let us define the function  $F(z)$  by

$$F(z) := \frac{z J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f)'(z)}{J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f)(z)} - \alpha \quad (f(z) \in \mathcal{A}). \tag{2.2}$$

It is sufficient to prove that

$$\left| \frac{F(z) - 1}{F(z) + 1} \right| < 1 \quad (z \in \mathbb{U}) \tag{2.3}$$

to prove that  $f(z) \in \mathcal{S}_{(\lambda,p),(\mu,q),b}^{s,a,\lambda,*}(\alpha)$ .

In fact, we have that

$$\begin{aligned} |\mathcal{F}(z)| &:= \left| \frac{F(z) - 1}{F(z) + 1} \right| = \left| \frac{\frac{z J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f)'(z)}{J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f)(z)} - \alpha - 1}{\frac{z J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f)'(z)}{J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f)(z)} - \alpha + 1} \right| \\ &= \left| \frac{\alpha z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \left( \frac{a+1}{a+k} \right)^s \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \frac{(\alpha+1-k)}{k!} a_k z^k}{(2-\alpha)z - \sum_{k=2}^{\infty} \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \left( \frac{a+1}{a+k} \right)^s \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \frac{(\alpha-1-k)}{k!} a_k z^k} \right|, \end{aligned}$$

and thus

$$\begin{aligned} |\mathcal{F}(z)| &\leq \frac{\alpha |z| + \sum_{k=2}^{\infty} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \right| \left| \left( \frac{a+1}{a+k} \right)^s \right| \left| \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| \frac{(\alpha+1-k)}{k!} |a_k| \cdot |z|^k}{(2-\alpha)|z| - \sum_{k=2}^{\infty} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \right| \left| \left( \frac{a+1}{a+k} \right)^s \right| \left| \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| \frac{(\alpha-1-k)}{k!} |a_k| \cdot |z|^k} \\ &< \frac{\alpha + \sum_{k=2}^{\infty} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \right| \left| \left( \frac{a+1}{a+k} \right)^s \right| \left| \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| \frac{(k-\alpha-1)}{k!} |a_k|}{(2-\alpha) - \sum_{k=2}^{\infty} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \right| \left| \left( \frac{a+1}{a+k} \right)^s \right| \left| \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| \frac{(k-\alpha+1)}{k!} |a_k|} \leq 1, \end{aligned}$$

provided that (2.1) is satisfied. □

The next theorem aims to provide coefficient inequalities for functions  $f(z)$  belonging to the class  $\mathcal{S}_{(\lambda,p),(\mu,q),b}^{s,a,\lambda,*}(\alpha)$ .

**Theorem 2** *Let  $\alpha \in [0, 1)$ . If  $f(z) \in \mathcal{S}_{(\lambda,p),(\mu,q),b}^{s,a,\lambda,*}(\alpha)$ , then*

$$|a_k| \leq k! \left( \frac{2(1-\alpha)}{k-1} \right) \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \quad (k \in \mathbb{N} \setminus \{1\}). \tag{2.4}$$

The result is sharp.

*Proof* Let

$$p(z) := \frac{z(J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f))'(z)}{J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f)(z)} - \alpha = 1 + c_1z + c_2z^2 + \dots$$

Then  $p(z)$  is analytic and

$$p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \quad (z \in \mathbb{U}).$$

We note easily that

$$z(J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f))'(z) = [(1-\alpha)p(z) + \alpha]J_{(\lambda,p),(\mu,q),b}^{s,a,\lambda}(f)(z).$$

With the help of (1.17), we find

$$\begin{aligned} & \frac{(k-1)}{k!} \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \left( \frac{a+1}{a+k} \right)^s \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) a_k \\ &= (1-\alpha) \cdot \left[ c_{k-1} + \sum_{m=2}^{k-1} \frac{\prod_{j=1}^p (\lambda_j + 1)_{m-1}}{\prod_{j=1}^q (\mu_j + 1)_{m-1}} \left( \frac{a+1}{a+m} \right)^s \left( \frac{\Lambda(a+m, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \frac{a_m c_{k-m}}{m!} \right] \end{aligned} \tag{2.5}$$

for  $k \in \mathbb{N} \setminus \{1\}$ .

By making use of the Carathéodory lemma [29, p.41], we have

$$\begin{aligned} & \frac{(k-1)}{k!} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \right| \left| \left( \frac{a+1}{a+k} \right)^s \right| \left| \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| \cdot |a_k| \\ & \leq 2(1-\alpha) \\ & \cdot \left[ 1 + \sum_{m=2}^{k-1} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{m-1}}{\prod_{j=1}^q (\mu_j + 1)_{m-1}} \right| \left| \left( \frac{a+1}{a+m} \right)^s \right| \left| \left( \frac{\Lambda(a+m, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| \frac{|a_m|}{m!} \right]. \end{aligned} \tag{2.6}$$

We have to prove that inequality (2.4) holds true for  $k \in \mathbb{N} \setminus \{1\}$ . We will proceed by the principle of mathematical induction. If  $k = 2$  in (2.6), we obtain

$$|a_2| \leq 4(1-\alpha) \left| \frac{\prod_{j=1}^q (\mu_j + 1)}{\prod_{j=1}^p (\lambda_j + 1)} \right| \left| \left( \frac{a+2}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+2, b, s, \lambda)} \right) \right|. \tag{2.7}$$

Now suppose that (2.4) is satisfied for  $k \leq n$ . Then, from (2.4) and (2.6), we have that

$$\begin{aligned}
 & \frac{n}{(n+1)!} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_n}{\prod_{j=1}^q (\mu_j + 1)_n} \right| \left| \left( \frac{a+1}{a+n+1} \right)^s \right| \left| \left( \frac{\Lambda(a+n+1, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| \cdot |a_{n+1}| \\
 & \leq 2(1-\alpha) \left[ 1 + \sum_{m=2}^n \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{m-1}}{\prod_{j=1}^q (\mu_j + 1)_{m-1}} \right| \left| \left( \frac{a+1}{a+m} \right)^s \right| \left| \left( \frac{\Lambda(a+m, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| \frac{|a_m|}{m!} \right] \\
 & \leq 2(1-\alpha) \left[ 1 + \sum_{m=2}^n \frac{2(1-\alpha)}{m-1} \prod_{j=2}^{m-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \right] \\
 & \leq 2(1-\alpha) \prod_{j=2}^n \left( 1 + \frac{2(1-\alpha)}{j-1} \right), \tag{2.8}
 \end{aligned}$$

whence

$$\begin{aligned}
 |a_k| & \leq k! \left( \frac{2(1-\alpha)}{k-1} \right) \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \\
 & \quad \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \quad (k \in \mathbb{N} \setminus \{1\}).
 \end{aligned}$$

The result is sharp for the function  $f(z)$  given by

$$\begin{aligned}
 f(z) & = z + \frac{2(1-\alpha)}{k-1} \left( \frac{a+k}{a+1} \right)^s \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \\
 & \quad \cdot \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) z^k \quad (k \in \mathbb{N} \setminus \{1\}). \tag{2.9}
 \end{aligned}$$

□

### 3 Distortion inequalities for the function class $\mathcal{S}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(\alpha)$

In this section, we establish distortion inequalities for functions belonging to the class  $\mathcal{S}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(\alpha)$ . These inequalities are given in the following theorem.

**Theorem 3** *Let  $f(z) \in \mathcal{S}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(\alpha)$  and  $0 \leq \alpha < 1$ . Then*

$$\begin{aligned}
 & r - 2(1-\alpha)r^2 \sum_{k=2}^{\infty} \frac{k!}{k-1} \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \\
 & \quad \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \\
 & \leq |f(z)| \\
 & \leq r + 2(1-\alpha)r^2 \sum_{k=2}^{\infty} \frac{k!}{k-1} \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \\
 & \quad \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1) \tag{3.1}
 \end{aligned}$$



and

$$\begin{aligned}
 & 1 - 2(1 - \alpha)r \sum_{k=2}^{\infty} \frac{k \cdot k!}{k-1} \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \\
 & \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \\
 & \leq |f'(z)| \\
 & \leq 1 + 2(1 - \alpha)r \sum_{k=2}^{\infty} \frac{k \cdot k!}{k-1} \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \\
 & \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1).
 \end{aligned} \tag{3.2}$$

*Proof* Let  $f(z) \in \mathcal{A}$  be given by (1.1). Then, making use of Theorem 2, we find

$$\begin{aligned}
 |f(z)| & \leq |z| + \sum_{k=2}^{\infty} |a_k| \cdot |z^k| \\
 & \leq r + 2(1 - \alpha)r^2 \sum_{k=2}^{\infty} \frac{k!}{k-1} \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \\
 & \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1)
 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
 |f(z)| & \geq |z| - \sum_{k=2}^{\infty} |a_k| \cdot |z^k| \\
 & \geq r - 2(1 - \alpha)r^2 \sum_{k=2}^{\infty} \frac{k!}{k-1} \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \\
 & \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1).
 \end{aligned} \tag{3.4}$$

From (1.1), we also have that

$$\begin{aligned}
 |f'(z)| & \leq 1 + \sum_{k=2}^{\infty} k \cdot |a_k| \cdot |z^{k-1}| \\
 & \leq 1 + 2(1 - \alpha)r \sum_{k=2}^{\infty} \frac{k \cdot k!}{k-1} \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \\
 & \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1)
 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
 |f'(z)| &\geq 1 - \sum_{k=2}^{\infty} k \cdot |a_k| \cdot |z|^k \\
 &\geq 1 - 2(1-\alpha)r \sum_{k=2}^{\infty} \frac{k \cdot k!}{k-1} \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| \\
 &\quad \cdot \left| \frac{\prod_{j=1}^q (\mu_j + 1)_{k-1}}{\prod_{j=1}^p (\lambda_j + 1)_{k-1}} \right| \prod_{j=2}^{k-1} \left( 1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1).
 \end{aligned}
 \tag{3.6}$$

We thus obtain the results (3.1) and (3.2) asserted by Theorem 3. □

#### 4 Extreme points

This section is devoted to presenting the extreme points of the function class  $\mathcal{S}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(a)$ . Let  $\tilde{\mathcal{S}}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(a)$  be the subclass of  $\mathcal{S}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(a)$  that consists in all functions  $f(z) \in \mathcal{A}$ , which satisfy inequality (2.1). Then the extreme points of  $\tilde{\mathcal{S}}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(a)$  are given by the following theorem.

**Theorem 4** *Let*

$$f_1(z) := z \tag{4.1}$$

and

$$\begin{aligned}
 f_k(z) &:= z + \frac{k!(1-\alpha)}{(k-\alpha)} \left| \frac{\prod_{j=1}^p (\mu_j + 1)_{k-1}}{\prod_{j=1}^q (\lambda_j + 1)_{k-1}} \right| \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right) \right| z^k \\
 &\quad (k \in \mathbb{N} \setminus \{1\}).
 \end{aligned}
 \tag{4.2}$$

Then

$$f(z) \in \tilde{\mathcal{S}}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(a) \quad (0 \leq \alpha < 1)$$

if and only if it can be expressed in the following form:

$$f(z) = \sum_{k=1}^{\infty} \gamma_k f_k(z) \quad \left( \gamma_k > 0; \sum_{k=1}^{\infty} \gamma_k = 1 \right). \tag{4.3}$$

*Proof* Suppose that

$$\begin{aligned}
 f(z) &= \sum_{k=1}^{\infty} \gamma_k f_k(z) \\
 &= z + \sum_{k=2}^{\infty} \gamma_k \frac{k!(1-\alpha)}{(k-\alpha)} \left| \frac{\prod_{j=1}^p (\mu_j + 1)_{k-1}}{\prod_{j=1}^q (\lambda_j + 1)_{k-1}} \right| \\
 &\quad \cdot \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right| z^k.
 \end{aligned}
 \tag{4.4}$$

Then

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{(k-\alpha)}{k!} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \right| \left| \left( \frac{a+1}{a+k} \right)^s \right| \left| \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| \\ & \cdot \gamma_k \frac{k!(1-\alpha)}{(k-\alpha)} \left| \frac{\prod_{j=1}^p (\mu_j + 1)_{k-1}}{\prod_{j=1}^q (\lambda_j + 1)_{k-1}} \right| \left| \left( \frac{a+k}{a+1} \right)^s \right| \left| \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+k, b, s, \lambda)} \right| \\ & = (1-\alpha) \sum_{k=2}^{\infty} \gamma_k = (1-\alpha)(1-\gamma_1) \\ & \leq 1-\alpha. \end{aligned} \tag{4.5}$$

Thus, by the definition of the function class  $\tilde{\mathcal{S}}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(\alpha)$ , we have

$$f \in \tilde{\mathcal{S}}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(\alpha) \quad (0 \leq \alpha < 1).$$

Conversely, if

$$f \in \tilde{\mathcal{S}}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(\alpha) \quad (0 \leq \alpha < 1),$$

then, by using (2.1), we may set

$$\begin{aligned} \gamma_k &= \frac{(k-\alpha)}{(1-\alpha)k!} \left| \frac{\prod_{j=1}^p (\lambda_j + 1)_{k-1}}{\prod_{j=1}^q (\mu_j + 1)_{k-1}} \right| \left| \left( \frac{a+1}{a+k} \right)^s \right| \left| \left( \frac{\Lambda(a+k, b, s, \lambda)}{\Lambda(a+1, b, s, \lambda)} \right) \right| |a_k| \\ & \quad (k \in \mathbb{N} \setminus \{1\}), \end{aligned} \tag{4.6}$$

which implies that

$$f(z) = \sum_{k=1}^{\infty} \gamma_k f_k.$$

The proof of Theorem 4 is thus completed. □

### 5 The Fekete-Szegő problem

In this section, we shall obtain the Fekete-Szegő inequality for functions in the class  $\mathcal{S}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(\alpha)$  when

$$s > 0, \quad a > 0 \quad \text{and} \quad 0 \leq \alpha < 1.$$

We need to recall an important lemma due to Ma and Minda [30] in order to prove our result involving Fekete-Szegő inequality.

**Lemma 1** *If*

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots$$

is an analytic function in  $\mathbb{U}$  such that

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}),$$

then

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2 & (\nu \leq 0), \\ 2 & (0 \leq \nu \leq 1), \\ 4\nu - 2 & (\nu \geq 1). \end{cases}$$

When  $\nu < 0$  or  $\nu > 1$ , the equality holds true if and only if

$$p(z) = \frac{1+z}{1-z} \tag{5.1}$$

or one of its rotations. If  $0 < \nu < 1$ , then the equality holds true if and only if

$$p(z) = \frac{1+z^2}{1-z^2} \tag{5.2}$$

or one of its rotations. If  $\nu = 0$ , then the equality holds true if and only if

$$p(z) = \left(\frac{1+\omega}{2}\right)\left(\frac{1+z}{1-z}\right) + \left(\frac{1-\omega}{2}\right)\left(\frac{1-z}{1+z}\right) \quad (0 \leq \omega \leq 1) \tag{5.3}$$

or one of its rotations. If  $\nu = 1$ , then the equality holds true if and only if  $p(z)$  is the reciprocal of one of the functions such that the equality holds true in the case  $\nu = 0$ .

**Theorem 5** *Let*

$$s > 0, \quad a > 0, \quad 0 \leq \alpha < 1$$

and

$$\lambda_j > -1 \quad (j = 1, \dots, p), \quad \mu_j > -1 \quad (j = 1, \dots, q).$$

If  $f \in \mathcal{S}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}(a)$ , then

$$|a_3 - \tau a_2^2| \leq \begin{cases} 3(1-\alpha) \frac{\prod_{j=1}^q (\mu_j+1)_2}{\prod_{j=1}^p (\lambda_j+1)_2} \cdot \left(\frac{a+3}{a+1}\right)^s \left(\frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+3, b, s, \lambda)}\right) (-4\nu + 2) & (\tau \leq \sigma_1), \\ 6(1-\alpha) \frac{\prod_{j=1}^q (\mu_j+1)_2}{\prod_{j=1}^p (\lambda_j+1)_2} \cdot \left(\frac{a+3}{a+1}\right)^s \left(\frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+3, b, s, \lambda)}\right) & (\sigma_1 \leq \tau \leq \sigma_2), \\ 3(1-\alpha) \frac{\prod_{j=1}^q (\mu_j+1)_2}{\prod_{j=1}^p (\lambda_j+1)_2} \cdot \left(\frac{a+3}{a+1}\right)^s \left(\frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+3, b, s, \lambda)}\right) (4\nu - 2) & (\tau \geq \sigma_2), \end{cases}$$

where

$$\nu := (1 - \alpha) \left( \frac{4\tau}{3} \prod_{j=1}^p \left( \frac{\lambda_j + 2}{\lambda_j + 1} \right) \prod_{j=1}^q \left( \frac{\mu_j + 1}{\mu_j + 2} \right) \left( \frac{a + 2}{a + 3} \right)^s \left( \frac{a + 2}{a + 1} \right)^s \cdot \left( \frac{\Lambda(a + 3, b, s, \lambda)}{\Lambda(a + 2, b, s, \lambda)} \right) \left( \frac{\Lambda(a + 1, b, s, \lambda)}{\Lambda(a + 2, b, s, \lambda)} - 1 \right), \tag{5.4}$$

$$\sigma_1 = \frac{3}{4} \prod_{j=1}^p \left( \frac{\lambda_j + 1}{\lambda_j + 2} \right) \prod_{j=1}^q \left( \frac{\mu_j + 2}{\mu_j + 1} \right) \left( \frac{a + 1}{a + 2} \right)^s \cdot \left( \frac{\Lambda(a + 2, b, s, \lambda)}{\Lambda(a + 3, b, s, \lambda)} \right) \left( \frac{\Lambda(a + 2, b, s, \lambda)}{\Lambda(a + 1, b, s, \lambda)} \right) \tag{5.5}$$

and

$$\sigma_2 = \frac{3(2 - \alpha)}{4(1 - \alpha)} \prod_{j=1}^p \left( \frac{\lambda_j + 1}{\lambda_j + 2} \right) \prod_{j=1}^q \left( \frac{\mu_j + 2}{\mu_j + 1} \right) \left( \frac{a + 1}{a + 2} \right)^s \cdot \left( \frac{\Lambda(a + 2, b, s, \lambda)}{\Lambda(a + 3, b, s, \lambda)} \right) \left( \frac{\Lambda(a + 2, b, s, \lambda)}{\Lambda(a + 1, b, s, \lambda)} \right). \tag{5.6}$$

The result is sharp.

*Proof* For  $f \in \mathcal{S}_{(\lambda_p), (\mu_q), b}^{s, a, \lambda, *}( \alpha )$ , let

$$p(z) = \frac{z^{s, a, \lambda} J_{(\lambda_p), (\mu_q), b}^{s, a, \lambda}(f)(z) - \alpha}{1 - \alpha} = 1 + c_1 z + c_2 z^2 + \dots \tag{5.7}$$

Then, with the help of (2.5), we have

$$a_2 = 2(1 - \alpha) c_1 \frac{\prod_{j=1}^q (\mu_j + 1)}{\prod_{j=1}^p (\lambda_j + 1)} \left( \frac{a + 2}{a + 1} \right)^s \left( \frac{\Lambda(a + 1, b, s, \lambda)}{\Lambda(a + 2, b, s, \lambda)} \right) \tag{5.8}$$

and

$$a_3 = 3(1 - \alpha) \frac{\prod_{j=1}^q (\mu_j + 1)_2}{\prod_{j=1}^p (\lambda_j + 1)_2} \left( \frac{a + 3}{a + 1} \right)^s \left( \frac{\Lambda(a + 1, b, s, \lambda)}{\Lambda(a + 3, b, s, \lambda)} \right) [c_2 + (1 - \alpha)c_1^2]. \tag{5.9}$$

Therefore, we find

$$\begin{aligned} a_3 - \tau a_2^2 &= 3(1 - \alpha) \frac{\prod_{j=1}^q (\mu_j + 1)_2}{\prod_{j=1}^p (\lambda_j + 1)_2} \left( \frac{a + 3}{a + 1} \right)^s \left( \frac{\Lambda(a + 1, b, s, \lambda)}{\Lambda(a + 3, b, s, \lambda)} \right) [c_2 + (1 - \alpha)c_1^2] \\ &\quad - 4\tau(1 - \alpha)^2 c_1^2 \frac{\prod_{j=1}^q (\mu_j + 1)^2}{\prod_{j=1}^p (\lambda_j + 1)^2} \left( \frac{a + 2}{a + 1} \right)^{2s} \left( \frac{\Lambda(a + 1, b, s, \lambda)}{\Lambda(a + 2, b, s, \lambda)} \right)^2 \\ &= 3(1 - \alpha) \frac{\prod_{j=1}^q (\mu_j + 1)_2}{\prod_{j=1}^p (\lambda_j + 1)_2} \left( \frac{a + 3}{a + 1} \right)^s \left( \frac{\Lambda(a + 1, b, s, \lambda)}{\Lambda(a + 3, b, s, \lambda)} \right) \left[ c_2 - c_1^2(1 - \alpha) \right] \end{aligned}$$

$$\begin{aligned} & \cdot \left( \frac{4\tau}{3} \prod_{j=1}^p \left( \frac{\lambda_j + 2}{\lambda_j + 1} \right) \prod_{j=1}^q \left( \frac{\mu_j + 1}{\mu_j + 2} \right) \left( \frac{a+2}{a+3} \right)^s \left( \frac{a+2}{a+1} \right)^s \right. \\ & \left. \cdot \left( \frac{\Lambda(a+3, b, s, \lambda)}{\Lambda(a+2, b, s, \lambda)} \right) \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+2, b, s, \lambda)} - 1 \right) \right]. \end{aligned} \tag{5.10}$$

We thus write

$$a_3 - \tau a_2^2 = 3(1 - \alpha) \frac{\prod_{j=1}^q (\mu_j + 1)_2}{\prod_{j=1}^p (\lambda_j + 1)_2} \left( \frac{a+3}{a+1} \right)^s \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+3, b, s, \lambda)} \right) (c_2 - \nu c_1^2), \tag{5.11}$$

where

$$\begin{aligned} \nu := & (1 - \alpha) \left( \frac{4\tau}{3} \prod_{j=1}^p \left( \frac{\lambda_j + 2}{\lambda_j + 1} \right) \prod_{j=1}^q \left( \frac{\mu_j + 1}{\mu_j + 2} \right) \left( \frac{a+2}{a+3} \right)^s \left( \frac{a+2}{a+1} \right)^s \right. \\ & \left. \cdot \left( \frac{\Lambda(a+3, b, s, \lambda)}{\Lambda(a+2, b, s, \lambda)} \right) \left( \frac{\Lambda(a+1, b, s, \lambda)}{\Lambda(a+2, b, s, \lambda)} - 1 \right) \right). \end{aligned} \tag{5.12}$$

The result asserted by Theorem 5 follows by applying Lemma 1.

Moreover, if  $\tau < \sigma_1$  or  $\tau > \sigma_2$ , then the equality holds true if and only if

$$J_{(\lambda_p), (\tau_q), b}^{s, a, \lambda}(f)(z) = \frac{z}{(1 - e^{i\theta} z)^{2(1-\alpha)}} \quad (\theta \in \mathbb{R}). \tag{5.13}$$

For  $\sigma_1 < \tau < \sigma_2$ , the equality holds true if and only if

$$J_{(\lambda_p), (\tau_q), b}^{s, a, \lambda}(f)(z) = \frac{z}{(1 - e^{i\theta} z)^{1-\alpha}} \quad (\theta \in \mathbb{R}). \tag{5.14}$$

If  $\tau = \sigma_1$ , then the equality holds true if and only if

$$\begin{aligned} J_{(\lambda_p), (\tau_q), b}^{s, a, \lambda}(f)(z) &= \left( \frac{z}{(1 - e^{i\theta} z)^{2(1-\alpha)}} \right)^{[(1+\omega)/2]} \left( \frac{z}{(1 + e^{i\theta} z)^{2(1-\alpha)}} \right)^{[(1-\omega)/2]} \\ &= \frac{z}{[(1 - e^{i\theta} z)^{1+\omega} (1 + e^{i\theta} z)^{1-\omega}]^{1-\alpha}} \quad (\theta \in \mathbb{R}; 0 \leq \omega \leq 1). \end{aligned} \tag{5.15}$$

Finally, when  $\tau = \sigma_2$ , the equality holds true if and only if  $J_{(\lambda_p), (\tau_q), b}^{s, a, \lambda}(f)(z)$  satisfies the following condition:

$$\frac{z(J_{(\lambda_p), (\tau_q), b}^{s, a, \lambda}(f))'(z)}{J_{(\lambda_p), (\tau_q), b}^{s, a, \lambda}(f)(z)} = (1 - \alpha)p(z) + \alpha, \tag{5.16}$$

where

$$\frac{1}{p(z)} = \left( \frac{1 + \omega}{2} \right) \left( \frac{1 + z}{1 - z} \right) + \left( \frac{1 - \omega}{2} \right) \left( \frac{1 - z}{1 + z} \right) \quad (0 < \omega < 1). \tag{5.17}$$

□

We conclude this paper by mentioning that, by suitably specializing the parameters involved, our main results (Theorem 1 to Theorem 5) would yield a number of (known or

new) results for much simpler function classes, which were investigated in several earlier works by employing many special cases of the new generalized Srivastava-Attiya operator.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors completed the paper together. All authors read and approved the final manuscript.

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