Yang et al. *Journal of Inequalities and Applications* (2016) 2016:221
DOI 10.1186/s13660-016-1160-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Monotonicity and inequalities involving the incomplete gamma function

Zhen-Hang Yang[1,2], Wen Zhang[3] and Yu-Ming Chu[1*]

*Correspondence:
chuyuming2005@126.com
[1] School of Mathematics and Computation Sciences, Hunan City University, Yiyang, 413000, China
Full list of author information is available at the end of the article

Abstract

In the article, we deal with the monotonicity of the function $x \to [(x^p + a)^{1/p} - x]/I_p(x)$ on the interval $(0, \infty)$ for $p > 1$ and $a > 0$, and present the necessary and sufficient condition such that the double inequality $[(x^p + a)^{1/p} - x]/a < I_p(x) < [(x^p + b)^{1/p} - x]/b$ for all $x > 0$ and $p > 1$, where $I_p(x) = e^{x^p} \int_x^\infty e^{-t^p}\,dt$ is the incomplete gamma function.

MSC: 33B20; 26D07; 26D15

Keywords: incomplete gamma function; gamma function; psi function

1 Introduction

Let $a > 0$ and $x > 0$. Then the classical gamma function $\Gamma(x)$, incomplete gamma function $\Gamma(a, x)$ and psi function $\psi(x)$ are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt, \qquad \Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t}\,dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. It is well known that the identities

$$\int_x^\infty e^{-t^p}\,dt = \frac{1}{p}\Gamma\left(\frac{1}{p}, x^p\right), \qquad \int_0^x e^{-t^p}\,dt = \frac{1}{p}\Gamma\left(\frac{1}{p}\right) - \frac{1}{p}\Gamma\left(\frac{1}{p}, x^p\right) \tag{1.1}$$

hold for all $x, p > 0$.

Recently the bounds for the integral $\int_x^\infty e^{-t^p}\,dt$ or $\int_0^x e^{-t^p}\,dt$ have attracted the attention of many researchers. In particular, many remarkable inequalities for bounding both integrals can be found in the literature [1–12]. Let

$$I_p(x) = e^{x^p} \int_x^\infty e^{-t^p}\,dt. \tag{1.2}$$

Then $I_2(x)$ is actually the Mills ratio and it has been investigated by many researchers [13–19], and the functions $I_3(x)$ and $I_4(x)$ can be used to research the heat transfer problem [20] and electrical discharge in gases [21], respectively.

Komatu [22] and Pollak [23] proved that the double inequality

$$\frac{1}{\sqrt{x^2 + 2} + x} < I_2(x) < \frac{1}{\sqrt{x^2 + 4/\pi} + x}$$

holds for all $x > 0$.

Springer

In [24], Gautschi proved that the double inequality

$$\frac{1}{2}\left[\left(x^p+2\right)^{1/p}-x\right] < I_p(x) < \frac{1}{a_0}\left[\left(x^p+a_0\right)^{1/p}-x\right] \tag{1.3}$$

holds for all $x > 0$ and $p > 1$, where

$$a_0 = \Gamma^{p/(1-p)}\left(1+\frac{1}{p}\right). \tag{1.4}$$

An application of inequality (1.3) was given in [25]. Alzer [26] proved that the double inequality

$$\Gamma\left(1+\frac{1}{p}\right)\left[1-\left(1-e^{-\alpha x^p}\right)^{1/p}\right] < I_p(x) < \Gamma\left(1+\frac{1}{p}\right)\left[1-\left(1-e^{-\beta x^p}\right)^{1/p}\right]$$

holds for all $x > 0$ and $p > 0$ with $p \neq 1$ if and only if $\alpha \geq \max\{1, \Gamma^{-p}(1+1/p)\}$ and $\beta \leq \min\{1, \Gamma^{-p}(1+1/p)\}$.

Motivated by inequality (1.3), in the article we deal with the monotonicity of the function

$$R(x) = \frac{(x^p+a)^{1/p}-x}{e^{x^p}\int_x^\infty e^{-t^p}\,dt} = \frac{(x^p+a)^{1/p}-x}{I_p(x)} \tag{1.5}$$

and prove that the double inequality

$$\frac{1}{a}\left[\left(x^p+a\right)^{1/p}-x\right] < I_p(x) < \frac{1}{b}\left[\left(x^p+b\right)^{1/p}-x\right] \tag{1.6}$$

holds for all $x > 0$ and $p > 1$ if and only if $a \geq 2$ and $b \leq a_0 = \Gamma^{p/(1-p)}(1+1/p)$.

2 Lemmas

In order to prove our main results, we need to introduce an auxiliary function at first.

Let $-\infty \leq a < b \leq \infty$, f and g be differentiable on (a,b), and $g' \neq 0$ on (a,b). Then the function $H_{f,g}$ [27, 28] is defined by

$$H_{f,g}(x) = \frac{f'(x)}{g'(x)}g(x) - f(x). \tag{2.1}$$

Lemma 2.1 (See [28], Theorem 9) *Let $\infty \leq a < b \leq \infty$, f and g be differentiable on (a,b) with $f(b^-) = g(b^-) = 0$ and $g'(x) < 0$ on (a,b), $H_{f,g}$ be defined by* (2.1), *and there exists $\lambda \in (a,b)$ such that $f'(x)/g'(x)$ is strictly increasing on (a,λ) and strictly decreasing on (λ,b). Then the following statements are true*:

(1) *if $H_{f,g}(a^+) \geq 0$, then $f(x)/g(x)$ is strictly decreasing on (a,b)*;

(2) *if $H_{f,g}(a^+) < 0$, then there exists $x_0 \in (a,b)$ such that $f(x)/g(x)$ is strictly increasing on (a,x_0) and strictly decreasing on (x_0,b).*

Lemma 2.2 (See [29], Theorem 1.25) *Let $-\infty < a < b < \infty$, $f,g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing* (*decreasing*) *on* (a,b), *then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.3 *The inequality*

$$\Gamma^{1/(1-x)}(1+x) > \frac{1}{2} \tag{2.2}$$

holds for all $x \in (0,1)$.

Proof We clearly see that inequality (2.2) is equivalent to

$$\log\Gamma(1+x) + (1-x)\log 2 > 0 \tag{2.3}$$

for $x \in (0,1)$.

Let

$$h(x) = \log\Gamma(1+x) + (1-x)\log 2. \tag{2.4}$$

Then simple computations lead to

$$h(1) = 0, \tag{2.5}$$

$$h'(x) = \psi(x+1) - \log 2 < \psi(2) - \log 2 = 1 - \gamma - \log 2 < 0 \tag{2.6}$$

for $x \in (0,1)$, where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant.

Therefore, inequality (2.3) follows easily from (2.4)-(2.6). □

Lemma 2.4 *The function* $\Gamma^{1/x}(1+x)$ *is strictly increasing on* $(0,\infty)$, *and the double inequality*

$$x < \Gamma^{1/x}(1+x) < 1 \tag{2.7}$$

holds for all $x \in (0,1)$.

Proof Let

$$\varphi_1(x) = \log\Gamma(1+x), \qquad \varphi_2(x) = x, \qquad \varphi(x) = \frac{\varphi_1(x)}{\varphi_2(x)} = \frac{\log\Gamma(1+x)}{x}, \tag{2.8}$$

$$\phi(x) = \log\Gamma(1+x) - x\log x. \tag{2.9}$$

Then simple computations lead to

$$\varphi_1(0) = \varphi_2(0) = 0, \tag{2.10}$$

$$\phi(0^+) = \phi(1) = 0, \tag{2.11}$$

$$\left[\frac{\varphi_1'(x)}{\varphi_2'(x)}\right]' = \psi'(x+1) > 0 \tag{2.12}$$

for $x \in (0,\infty)$, and

$$\phi''(x) = \psi'(1+x) - \frac{1}{x} < 0 \tag{2.13}$$

for $x \in (0,1)$.

It follows from (2.8), (2.10), (2.12), and Lemma 2.2 that $\varphi(x)$ and $e^{\varphi(x)} = \Gamma^{1/x}(1+x)$ is strictly increasing on $(0,\infty)$.

Inequality (2.13) leads to the conclusion that the function $\phi(x)$ is strictly concave on the interval $(0,1)$ and the inequality

$$\phi(x) > \phi(0)(1-x) + \phi(1)x \tag{2.14}$$

holds for all $x \in (0,1)$.

Therefore, $\phi(x) > 0$ and the first inequality of (2.7) holds for all $x \in (0,1)$ follows from (2.9), (2.11), and (2.14). While the second inequality of (2.7) can be derived from the monotonicity of the function $\Gamma^{1/x}(1+x)$ on the interval $(0,1)$. □

Lemma 2.5 *Let $p > 1$ and $x > 0$. Then the function $a \to [(x^p + a)^{1/p} - x]/a$ is strictly decreasing on* $(0,\infty)$.

Proof Let

$$\omega_1(a) = (x^p + a)^{1/p} - x, \qquad \omega_2(a) = a, \qquad \omega(a) = \frac{\omega_1(a)}{\omega_2(a)} = \frac{(x^p + a)^{1/p} - x}{a}. \tag{2.15}$$

Then we clearly see that

$$\omega_1(0) = \omega_2(0) = 0, \tag{2.16}$$

$$\left[\frac{\omega_1'(a)}{\omega_2'(a)}\right]' = \frac{1-p}{p^2(x^p + a)^{(2p-1)/p}} < 0 \tag{2.17}$$

for all $p > 1$, $x > 0$ and $a > 0$.

Therefore, Lemma 2.5 follows easily from Lemma 2.2 and (2.15)-(2.17). □

Lemma 2.6 *Let $p > 1$, $a > 0$ and $x > 0$, $H_{f,g}(x)$ be defined by* (2.1), *and $f_1(x)$ and $g_1(x)$ be defined by*

$$f_1(x) = \left[(x^p + a)^{1/p} - x\right]e^{-x^p}, \qquad g_1(x) = \int_x^\infty e^{-t^p}\,dt, \tag{2.18}$$

respectively. Then $H_{f_1,g_1}(0^+) = \Gamma(1+1/p) - a^{1/p}$.

Proof Let

$$u = u(x) = \left(\frac{x^p + a}{x^p}\right)^{1/p} \in (1,\infty). \tag{2.19}$$

Then from (2.18) and (2.19) one has

$$f_1(0) = a^{1/p}, \qquad g_1(0) = \frac{1}{p}\Gamma\left(\frac{1}{p}\right) = \Gamma\left(1+\frac{1}{p}\right), \tag{2.20}$$

$$\begin{aligned}\frac{f_1'(x)}{g_1'(x)} &= -\left(\frac{x^p+a}{x^p}\right)^{1/p-1} + px^p\left[\left(\frac{x^p+a}{x^p}\right)^{1/p} - 1\right] + 1 \\ &= 1 + \frac{(pa-1)u + u^{1-p} - pa}{u^p - 1}.\end{aligned} \tag{2.21}$$

It follows from (2.1), (2.20), and (2.21) that

$$\begin{aligned}H_{f_1,g_1}(0^+) &= \lim_{x\to 0^+}\frac{f_1'(x)}{g_1'(x)}\lim_{x\to 0^+} g_1(x) - \lim_{x\to 0^+} f_1(x) \\ &= \Gamma\left(1+\frac{1}{p}\right)\left[1 + \lim_{u\to\infty}\frac{(pa-1)u + u^{1-p} - pa}{u^p - 1}\right] - a^{1/p} \\ &= \Gamma\left(1+\frac{1}{p}\right) - a^{1/p}.\end{aligned}$$

□

3 Main results

Theorem 3.1 *Let $p > 1$, $a > 0$, $x > 0$ and $R(x)$ be defined by* (1.5). *Then the following statements are true*:

(1) *if $a \geq 2$, then $R(x)$ is strictly increasing on* $(0, \infty)$;
(2) *if $a \leq \Gamma^p(1 + 1/p)$, then $R(x)$ is strictly decreasing on* $(0, \infty)$;
(3) *if $\Gamma^p(1 + 1/p) < a < 2$, then there exists $x_0 \in (0, \infty)$ such that $R(x)$ is strictly increasing on* $(0, x_0)$ *and strictly decreasing on* (x_0, ∞).

Proof Let $f_1(x)$, $g_1(x)$, $u = u(x) \in (1, \infty)$ be defined by (2.18) and (2.19), and $h(u)$ and $h_1(u)$ be defined by

$$h(u) = (p-1)(ap-1)u^{2p} - ap^2u^{2p-1} + (2p + ap - 2)u^p + 1 - p, \tag{3.1}$$

$$h_1(u) = 2(p-1)(ap-1)u^p - ap(2p-1)u^{p-1} + 2p + ap - 2. \tag{3.2}$$

Then from (1.2), (1.5), (2.18), (2.21), (3.1), (3.2), and Lemma 2.4 we have

$$R(x) = \frac{f_1(x)}{g_1(x)}, \tag{3.3}$$

$$h(1) = h_1(1) = 0, \tag{3.4}$$

$$\left[\frac{f_1'(x)}{g_1'(x)}\right]' = \frac{\frac{d}{du}[1 + \frac{(pa-1)u+u^{1-p}-pa}{u^p-1}]}{\frac{dx}{du}} = \frac{(u^p-1)^{1/p-1}}{a^{1/p}u^{2p-1}}h(u), \tag{3.5}$$

$$h'(u) = pu^{p-1}h_1(u), \tag{3.6}$$

$$h_1'(u) = p(p-1)u^{p-2}\left[2(ap-1)(u-1) + (a-2)\right], \tag{3.7}$$

$$\frac{1}{p} < \Gamma^p\left(1+\frac{1}{p}\right) < 2 \tag{3.8}$$

for $p > 1$.

We divide the proof into four cases.

Case 1: $a \geq 2$. Then from (3.4)-(3.7) we clearly see that the function $f_1'(x)/g_1'(x)$ is strictly increasing on $(0,\infty)$. Therefore, $R(x)$ is strictly increasing on $(0,\infty)$ follows from Lemma 2.2 and (3.3) together with the monotonicity of the function $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$ and $f_1(\infty) = g_1(\infty) = 0$.

Case 2: $a \leq 1/p$. Then from (3.4)-(3.8) we clearly see that the function $f_1'(x)/g_1'(x)$ is strictly decreasing on $(0,\infty)$. Therefore, $R(x)$ is strictly decreasing on $(0,\infty)$ follows from Lemma 2.2 and (3.3) together with the monotonicity of the function $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$ and $f_1(\infty) = g_1(\infty) = 0$.

Case 3: $1/p < a \leq \Gamma^p(1 + 1/p)$. Then (3.1), (3.2), and Lemma 2.6 lead to

$$\lim_{u\to\infty} h(u) = \infty, \qquad \lim_{u\to\infty} h_1(u) = \infty, \tag{3.9}$$

$$H_{f_1,g_1}(0^+) \geq 0. \tag{3.10}$$

Note that (3.7) can be rewritten as

$$h_1'(u) = 2p(ap-1)(p-1)u^{p-2}(u-u_0) \tag{3.11}$$

with $u_0 = 1 + (2-a)/[2(ap-1)] \in (1,\infty)$.

From (3.11) we clearly see that $h_1(u)$ is strictly decreasing on $(1,u_0)$ and strictly increasing on (u_0,∞). Then from (3.4), (3.6), and (3.9) we know that there exists $\lambda \in (1,\infty)$ such that $h(u) < 0$ for $u \in (1,\lambda)$ and $h(u) > 0$ for $u \in (\lambda,\infty)$.

From (2.19) we clearly see that the function $x \to u(x)$ is strictly decreasing from $(0,\infty)$ onto $(1,\infty)$. Then (3.5) and $h(u) < 0$ for $u \in (1,\lambda)$ and $h(u) > 0$ for $u \in (\lambda,\infty)$ lead to the conclusion that $f_1'(x)/g_1'(x)$ is strictly increasing on $(0,\mu)$ and strictly decreasing on (μ,∞), where $\mu = [a/(\lambda^p - 1)]^{1/p}$.

Therefore, $R(x)$ is strictly decreasing on $(0,\infty)$ follows from (3.3), (3.10), Lemma 2.1(1), and the piecewise monotonicity of the function $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$ together with the fact that $g_1'(x) = -e^{-x^p} < 0$ and $f_1(\infty) = g_1(\infty) = 0$.

Case 4: $\Gamma^p(1 + 1/p) < a < 2$. Then we clearly see that (3.9) and (3.11) again hold. Making use of the same method as in Case 3 we know that there exists $\eta > 0$ such that $f_1'(x)/g_1'(x)$ is strictly increasing on $(0,\eta)$ and strictly decreasing on (η,∞).

It follows from Lemma 2.6 that

$$H_{f_1,g_1}(0^+) < 0. \tag{3.12}$$

Therefore, there exists $x_0 \in (0,\infty)$ such that $R(x)$ is strictly increasing on $(0,x_0)$ and strictly decreasing on (x_0,∞) follows from (3.3), (3.12), Lemma 2.1(2), and the piecewise monotonicity of the function $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$ together with the fact that $g_1'(x) = -e^{-x^p} < 0$ and $f_1(\infty) = g_1(\infty) = 0$. □

Let $p > 1$, $x > 0$, $a > 0$, $R(x)$, $f_1(x)$, $g_1(x)$ and $u = u(x)$ be defined by (1.5), (2.18), and (2.19), respectively. Then we clearly see that

$$f_1(\infty) = g_1(\infty) = 0. \tag{3.13}$$

It follows from (2.20), (2.21), (3.3), and (3.13) that

$$R(0^+) = \frac{a^{1/p}}{\Gamma(1+\frac{1}{p})}, \tag{3.14}$$

$$\begin{aligned} R(\infty) &= \lim_{x\to\infty}\frac{f_1(x)}{g_1(x)} = \lim_{x\to\infty}\frac{f_1'(x)}{g_1'(x)} \\ &= 1 + \lim_{u\to 1^+}\frac{(pa-1)u + u^{1-p} - pa}{u^p - 1} = a. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15) together with Theorem 3.1 we get Corollary 3.2 immediately.

Corollary 3.2 *Let $p > 1$, $a, x > 0$, $I_p(x)$ and $R(x)$ be defined by* (1.2) *and* (1.5), *and x_0 be the unique solution of the equation $R'(x) = 0$ on the interval $(0,\infty)$ for $\Gamma^p(1+1/p) < a < 2$. Then the following statements are true*:

(1) *if $a \geq 2$, then the double inequality*

$$\frac{1}{a}\left[\left(x^p + a\right)^{1/p} - x\right] < I_p(x) < a^{-1/p}\Gamma\left(1+\frac{1}{p}\right)\left[\left(x^p + a\right)^{1/p} - x\right]$$

holds for all $p > 1$ and $x > 0$;

(2) *if $0 < a \leq \Gamma^p(1+1/p)$, then the double inequality*

$$a^{-1/p}\Gamma\left(1+\frac{1}{p}\right)\left[\left(x^p + a\right)^{1/p} - x\right] < I_p(x) < \frac{1}{a}\left[\left(x^p + a\right)^{1/p} - x\right]$$

holds for all $p > 1$ and $x > 0$;

(3) *if $\Gamma^p(1+1/p) < a < 2$, then the two-sided inequality*

$$\frac{1}{R(x_0)}\left[\left(x^p + a\right)^{1/p} - x\right] \leq I_p(x) < \max\left\{\frac{1}{a}, \frac{\Gamma(1+\frac{1}{p})}{a^{1/p}}\right\}\left[\left(x^p + a\right)^{1/p} - x\right]$$

is valid for all $p > 1$ and $x > 0$.

Theorem 3.3 *Let $p > 1$, $a, b, x > 0$, $I_p(x)$ and a_0 be defined by* (1.2) *and* (1.4), *respectively. Then the bilateral inequality*

$$\frac{1}{a}\left[\left(x^p + a\right)^{1/p} - x\right] < I_p(x) < \frac{1}{b}\left[\left(x^p + b\right)^{1/p} - x\right] \tag{3.16}$$

holds for all $p > 1$ and $x > 0$ if and only if $a \geq 2$ and $b \leq a_0$.

Proof If $a \geq 2$ and $b \leq a_0$, then inequality (3.16) is valid for all $p > 1$ and $x > 0$ follows easily from (1.3) and Lemma 2.5.

If the inequality $I_p(x) < [(x^p + b)^{1/p} - x]/b$ takes place for $p > 1$ and $x > 0$, then (3.14) leads to

$$\lim_{x\to 0^+}\frac{(x^p + b)^{1/p} - x}{I_p(x)} = \frac{b^{1/p}}{\Gamma(1+\frac{1}{p})} \geq b,$$

which implies $b \leq a_0$.

Next, we use the proof by contradiction to prove that $a \geq 2$ if the inequality $I_p(x) > [(x^p + b)^{1/p} - x]/a$ holds for all $x > 0$ and $p > 1$.

From Lemmas 2.3 and 2.4 we clearly see that

$$\Gamma^p\left(1 + \frac{1}{p}\right) < a_0 < 2. \tag{3.17}$$

We divide the proof into two cases.

Case 1: $a \leq a_0$. Then it follows from the sufficiency of Theorem 3.3 which was proved previously that $I_p(x) < [(x^p + b)^{1/p} - x]/a$ for all $p > 1$ and $x > 0$.

Case 2: $a_0 < a < 2$. Let $R(x)$ be defined by (1.5), then Theorem 3.1(3), (3.15), and (3.17) lead to the conclusion that there exists $x_0 \in (0, \infty)$ such that $R(x)$ is strictly decreasing on (x_0, ∞) and

$$\frac{(x^p + a)^{1/p} - x}{I_p(x)} = R(x) > R(\infty) = a$$

or

$$I_p(x) < \frac{1}{a}\left[\left(x^p + a\right)^{1/p} - x\right]$$

for all $p > 1$ and $x \in (x_0, \infty)$. □

Let $p > 1$, $a > 0$, $x > 0$, $q = 1/p \in (0, 1)$, and $u = x^p > 0$. Then from (1.1) and (1.2) one has

$$I_p(x) = qe^u\Gamma(q, u), \qquad \left(x^p + a\right)^{1/p} - x = (u + a)^q - u^q,$$

and Corollary 3.2 and Theorem 3.3 can be rewritten as follows.

Corollary 3.4 *Let $q \in (0, 1)$, $a > 0$, and $u > 0$. Then the following statements are true*:

(1) *if $a \geq 2$, then the double inequality*

$$\frac{(u + a)^q - u^q}{qa} < e^u\Gamma(q, u) < \frac{\Gamma(1 + q)[(u + a)^q - u^q]}{qa^q} \tag{3.18}$$

holds for all $q \in (0, 1)$ and $u > 0$, and inequality (3.18) *is reversed if* $0 < a \leq \Gamma^{1/q}(1 + q)$;

(2) *if $\Gamma^{1/q}(1 + q) < a < 2$, then the two-sided inequality*

$$\frac{(u + a)^q - u^q}{q\theta(q, u_0, a)} \leq e^u\Gamma(q, u) < \max\left\{\frac{1}{a}, \frac{\Gamma(1 + q)}{a^q}\right\}\frac{(u + a)^q - u^q}{q}$$

holds for all $q \in (0, 1)$ and $u > 0$, where $\theta(q, u_0, a) = [(u_0 + a)^q - u_0^q]/[qe^{u_0}\Gamma(q, u_0)]$ and u_0 is the unique solution of the equation

$$\frac{d\left[\frac{(u+a)^q - u^q}{qe^u\Gamma(q,u)}\right]}{du} = 0$$

on the interval $(0, \infty)$ for $\Gamma^{1/q}(1 + q) < a < 2$.

Corollary 3.5 *Let $a, b, u > 0$, $q \in (0,1)$ and a_0 be defined by* (1.4). *Then the double inequality*

$$\frac{(u+a)^q - u^q}{qa} < e^u \Gamma(q,u) < \frac{(u+b)^q - u^q}{qb}$$

holds for all $q \in (0,1)$ and $u > 0$ if and only if $a \geq 2$ and $b \leq a_0$.

Let $q \to 0^+$ and $Ei(u) = \lim_{q\to 0^+} \Gamma(q,u)$. Then Corollaries 3.4 and 3.5 lead to Remarks 3.6 and 3.7.

Remark 3.6 Let $a > 0$ and $u > 0$, then the following statements are true:

(1) if $a \geq 2$, then the double inequality

$$\frac{\log(1+\frac{a}{u})}{a} < e^u Ei(u) < \log\left(1+\frac{a}{u}\right) \tag{3.19}$$

holds for all $u > 0$, and inequality (3.19) is reversed if $0 < a < e^{-\gamma}$;

(2) if $e^{-\gamma} < a < 2$, then we have the sided inequality

$$\frac{e^{u_0} Ei(u_0)}{\log(1+\frac{a}{u_0})} \log\left(1+\frac{a}{u}\right) \leq e^u Ei(u) < \max\left\{\frac{1}{a}, 1\right\} \log\left(1+\frac{a}{u}\right) \tag{3.20}$$

for all $u > 0$, where u_0 is the unique solution of the equation

$$\frac{d}{du}\frac{\log(1+\frac{a}{u})}{e^u Ei(u)} = 0 \tag{3.21}$$

on the interval $(0,\infty)$ for $e^{-\gamma} < a < 2$.

Remark 3.7 Let $a, b > 0$ and a_0 be defined by (1.4). Then the double inequality

$$\frac{\log(1+\frac{a}{u})}{a} < e^u Ei(u) < \frac{\log(1+\frac{b}{u})}{b}$$

holds for all $u > 0$ if and only if $a \geq 2$ and $b \leq a_0$.

In particular, if $a = 1$, then numerical computations show that $u_0 = 0.23855\ldots$ is the unique solution of the equation

$$\frac{d}{du}\frac{\log(1+\frac{1}{u})}{e^u Ei(u)} = 0$$

and $e^{u_0} Ei(u_0)/\log(1 + 1/u_0) = 0.83311\ldots > 8{,}331/10{,}000$. Therefore, Remark 3.7 leads to Remark 3.8.

Remark 3.8 The double inequality

$$\frac{8{,}331}{10{,}000}\log\left(1+\frac{1}{u}\right) < e^u Ei(u) < \log\left(1+\frac{1}{u}\right)$$

is valid for all $u > 0$.

Remark 3.9 Unfortunately, in the article we cannot deal with the monotonicity for the function $R(x)$ defined by (1.5) and present the bounds for the function $I_p(x)$ given by (1.2) in the case of $p \in (0,1)$; we leave it as an open problem to the reader.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Computation Sciences, Hunan City University, Yiyang, 413000, China. [2]Customer Service Center, State Grid Zhejiang Electric Power Research Institute, Hangzhou, 310009, China. [3]Albert Einstein College of Medicine, Yeshiva University, New York, NY 10033, USA.

Acknowledgements
The research was supported by the Natural Science Foundation of China under Grants 61374086, 11371125, and 11401191.

Received: 14 June 2016 Accepted: 2 September 2016 Published online: 13 September 2016

References
1. Neuman, E: Inequalities and bounds for the incomplete gamma function. Results Math. **63**(3-4), 1209-1214 (2013)
2. Alzer, H, Baricz, Á: Functional inequalities for the incomplete gamma function. J. Math. Anal. Appl. **385**(1), 167-178 (2013)
3. Borwein, JM, Chan, OY: Uniform bounds for the complementary incomplete gamma function. Math. Inequal. Appl. **12**(1), 115-121 (2009)
4. Ismall, MEH, Laforgia, A: Functional inequalities for incomplete gamma and related functions. Math. Inequal. Appl. **9**(2), 299-302 (2006)
5. Laforgia, A, Natalini, P: Supplements to known monotonicity results and inequalities for the gamma and incomplete gamma function. J. Inequal. Appl. **2006**, Article ID 48727 (2006)
6. Paris, RB: Error bounds for the uniform asymptotic expansion of the incomplete gamma function. J. Comput. Appl. Math. **147**(1), 215-231 (2002)
7. Qi, F: Monotonicity results and inequalities for the gamma and incomplete gamma functions. Math. Inequal. Appl. **5**(1), 61-67 (2002)
8. Elbert, Á, Laforgia, A: An inequality for the product of two integrals related to the incomplete gamma function. J. Inequal. Appl. **5**(1), 39-51 (2000)
9. Natalini, P, Palumbo, B: Inequalities for the incomplete gamma function. Math. Inequal. Appl. **3**(1), 69-77 (2000)
10. Qi, F, Mei, J-Q: Some inequalities of the incomplete gamma and related functions. Z. Anal. Anwend. **18**(3), 793-799 (1999)
11. Qi, F, Guo, S-L: Inequalities for the incomplete gamma and related functions. Math. Inequal. Appl. **2**(1), 47-53 (1999)
12. Gupta, SS, Waknis, MN: A system of inequalities for the incomplete gamma function and the normal integral. Ann. Math. Stat. **36**, 139-149 (1965)
13. Gordon, RD: Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. Ann. Math. Stat. **12**, 364-366 (1941)
14. Birnbaum, ZW: An inequality for Mill's ratio. Ann. Math. Stat. **13**, 245-246 (1942)
15. Sampford, MR: Some inequalities on Mill's ratio and related functions. Ann. Math. Stat. **24**, 130-132 (1953)
16. Shenton, LR: Inequalities for the normal integral including a new continued fraction. Biometrika **41**, 177-189 (1954)
17. Boyd, AV: Inequalities for Mills' ratio. Rep. Stat. Appl. Res. UJSE **6**, 44-46 (1959)
18. Gasull, A, Utzet, F: Approximating Mills ratio. J. Math. Anal. Appl. **420**(2), 1832-1853 (2014)
19. Yang, Z-H, Chu, Y-M: On approximating Mills ratio. J. Inequal. Appl. **2015**, Article ID 273 (2015)
20. Yamagata, K: A contribution to the theory of non-isothermal laminar flow of fluids inside a straight tube of circular cross section. Mem. Fac. Eng., Kyushu Imp. Univ. **8**, 365-449 (1940)
21. Schumann, WO: Elektrische Durchbruchfeldstäke von Gasen. Springer, Berlin (1923)
22. Komatu, Y: Elementary inequalities for Mills' ratio. Rep. Stat. Appl. Res. UJSE **4**, 69-70 (1955)
23. Pollak, HO: A remark on 'Elementary inequalities for Mills' ratio' by Yûsaku Komatu. Rep. Stat. Appl. Res. UJSE **4**, 110 (1956)
24. Gautschi, W: Some elementary inequalities relating to the gamma and incomplete gamma function. J. Math. Phys. **38**, 77-81 (1959/60)
25. Montgomery, GF: On the transmission error function for meteor-burst communication. Proc. IRE **46**, 1423-1424 (1958)
26. Alzer, H: On some inequalities for the incomplete gamma function. Math. Comput. **218**, 771-778 (1997)
27. Yang, Z-H, Chu, Y-M, Wang, M-K: Monotonicity criterion for the quotient of power series with applications. J. Math. Anal. Appl. **428**(1), 587-604 (2015)
28. Yang, Z-H: A new way to prove L'Hospital monotone rules with applications. arXiv:1409.6408 [math.CA]
29. Anderson, GD, Vamanamurthy, MK, Vuorinen, M: Conformal Invariants, Inequalities, and Quasiconformal Maps. Wiley, New York (1997)