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Some identities on the higher-order-twisted q-Euler numbers and polynomials with weight α

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Abstract

In this article, we introduce some properties of higher-order-twisted q-Euler numbers and polynomials with weight α, and we observe some properties of higher-order-twisted q-Euler numbers and polynomials with weight α for several cases. In particular, by using the the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give a new concept of twisted q-Euler numbers and polynomials with weight α.

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1. Introduction

Let p be a fixed odd prime. Throughout this article $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$, will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$ (see [1-14]). When one speaks of q-extension, q can be regarded as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$; it is always clear from context. If $q \in \mathbb{C}$, we assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|1 - q|_p < 1$ (see [1-14]).

In this article, we use the notation of q-number as follows (see [1-14]):

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \to 1} [x]_q = x$ for any x with $|x|_p \le 1$ in the p-adic case.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows (see [6,7]):

$$\begin{aligned} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) &= \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N - 1} f(x)(-q)^x, \\ &= \lim_{N \to \infty} \frac{[2]_q}{1 + q^{p^N}} \sum_{x=0}^{p^N - 1} f(x)(-q)^x. \end{aligned} \tag{1}$$

From (1), we note that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0),$$

where $f_1(x) = f(x+1)$.

It is well known that the ordinary Euler polynomials are defined by

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$.

In the special case, $x = 0$, $E_n(0) = E_n$ are called the nth Euler numbers (see [1-14]).

By (2), we get the following recurrence relation as follows:

$$E_0 = 1, \text{ and } (E+1)^n + E = \begin{cases} 2, \text{ if } n = 0, \\ 0, \text{ if } n > 0. \end{cases} \tag{2}$$

Recently, (h, q)-Euler numbers are defined by

$$E_{0,q}^{(h)} = \frac{2}{1+q^h}, \text{ and } q^h(qE_q^{(h)}+1)^n + E_q^{(h)} = \begin{cases} 2, \text{ if } n = 0, \\ 0, \text{ if } n > 0, \end{cases}$$

with the usual convention about replacing $\left(E_q^{(h)}\right)^n$ by $E_{n,q}^{(h)}$ (see [1-16]).

Note that $\lim_{q\to 1} E_{n,q}^{(h)} = E_n$.

Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{w | w^{p^N} = 1\}$ is the cyclic group of order p^N. For $w \in T_p$, we denote by $\varphi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto w^x$.

For $\alpha \in \mathbb{N}$ and $w \in T_p$, the twisted q-Euler numbers with weight α are also defined by

$$\tilde{E}_{0,q,w}^{(\alpha)} = \frac{[2]_q}{wq+1}, \text{ and } wq\left(q^\alpha \tilde{E}_{q,w}^{(\alpha)} + 1\right)^n + \tilde{E}_{n,q,w}^{(\alpha)} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

with the usual convention about replacing $\left(\tilde{E}_{q,w}^{(\alpha)}\right)^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}$ (see [2,5]).

The main purpose of this article is to present a systemic study of some families of higher-order-twisted q-Euler numbers and polynomials with weight α. In Section 2, we investigate higher-order-twisted q-Euler numbers and polynomials with weight α and establish interesting properties. In Sections 3, 4, and 5, we observe some properties for special cases.

2. Higher-order-twisted q-Euler numbers and polynomials with weight α

For $h \in \mathbb{Z}$, $\alpha, k \in \mathbb{N}$, $w \in T_p$ and $n \in \mathbb{Z}_+$, let us consider the expansion of higher-order-twisted q-Euler polynomials with weight α as follows:

$$\tilde{E}_{n,q,w}^{(\alpha)}(h,k|x) = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}} w^{\sum_{i=1}^k x_i} \left[\sum_{i=1}^k x_i + x\right]_{q^\alpha}^n q^{x_1(h-1)+\cdots+x_k(h-k)} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k). \tag{3}$$

From (1) and (3), we note that

$$\tilde{E}_{n,q,w}^{(\alpha)}(h,k|x) = \frac{[2]_q^k}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{q^{\alpha l x}}{(1+wq^{\alpha l+h})\cdots(1+wq^{\alpha l+h-k+1})}. \tag{4}$$

In the special case, $x = 0$ $\tilde{E}^{(\alpha)}_{n,q,w}(h,k|0) = \tilde{E}^{(\alpha)}_{n,q,w}(h,k)$ are called the higher-order-twisted q-Euler numbers with weight α.

By (3), we get

$$\tilde{E}^{(\alpha)}_{n,q,w}(h,k) = (q^{\alpha}-1)\tilde{E}^{(\alpha)}_{n+1,q,w}(h-\alpha,k) + \tilde{E}^{(\alpha)}_{n,q,w}(h-\alpha,k). \tag{5}$$

From (5) and mathematical induction, we get the following theorem.

Theorem 1. For $\alpha, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we have

$$\begin{aligned}&\sum_{i=0}^{n-1}(-1)^{i-1}(q^{\alpha}-1)^{n-1-i}\tilde{E}^{(\alpha)}_{n-i,q,w}(h,k)\\ &= (q^{\alpha}-1)^{n-1}\tilde{E}^{(\alpha)}_{n,q,w}(h-\alpha,k) + (-1)^{n}E^{(\alpha)}_{1,q,w}(h-\alpha,k).\end{aligned}$$

For complex number $q \in \mathbb{C}_p$, $m \in \mathbb{Z}_+$, we get the following;

$$\begin{aligned} q^{\alpha(x_1+\cdots+x_{k+1})m} &= \left(1-(1-q^{\alpha(x_1+\cdots+x_{k+1})})\right)^m \\ &= \sum_{l=0}^{m}\binom{m}{l}(-1)^l\left(1-q^{\alpha(x_1+\cdots+x_{k+1})}\right)^l \\ &= \sum_{l=0}^{m}\binom{m}{l}(-1)^l(1-q^{\alpha})^l\frac{(1-q^{\alpha(x_1+\cdots+x_{k+1})})^l}{(1-q^{\alpha})^l} \\ &= \sum_{l=0}^{m}\binom{m}{l}(-1)^l(1-q^{\alpha})^l[x_1+x_2+\cdots+x_{k+1}]^l_{q^{\alpha}}. \end{aligned}$$

From (3), (4), and above property, we have

$$\begin{aligned} &\tilde{E}^{(\alpha)}_{0,q,w}(m\alpha,k+1)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{j=1}^{k+1}x_j}q^{\sum_{j=1}^{k+1}(m\alpha-j)x_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_{k+1})\\ &= \sum_{l=0}^{m}\binom{m}{l}(q^{\alpha}-1)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{j=1}^{k+1}x_j}\left[\sum_{j=1}^{k+1}x_j\right]^l_{q^{\alpha}} q^{-\sum_{j=1}^{k+1}jx_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_{k+1})\\ &= \sum_{l=0}^{m}\binom{m}{l}(q^{\alpha}-1)^l\tilde{E}^{(\alpha)}_{l,q,w}(0,k+1)\\ &= \frac{[2]_q^{k+1}}{(1+wq^{\alpha m})(1+wq^{\alpha m-1})\ldots(1+wq^{\alpha m-k})}. \end{aligned} \tag{6}$$

From (3), we can derive the following equation.

$$\begin{aligned} &\sum_{j=0}^{i}\binom{i}{j}(q^{\alpha}-1)^j\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{s=1}^{k}x_s}\left[\sum_{s=1}^{k}x_s\right]^{n-i+j}_{q^{\alpha}} q^{\sum_{s=1}^{k}(h-\alpha-s)x_s}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{s=1}^{k}x_s}\left[\sum_{s=1}^{k}x_s\right]^{n-i}_{q^{\alpha}} q^{\sum_{s=1}^{k}(h-s)x_s}q^{\alpha\left(\sum_{s=1}^{k}x_s\right)(i-1)}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\ &= \sum_{j=0}^{i-1}(q^{\alpha}-1)^j\binom{i-1}{j}\tilde{E}^{(\alpha)}_{n-i+j,q,w}(h,k). \end{aligned} \tag{7}$$

By (3), (4), (5), and (6), we see that

$$\sum_{j=0}^{i}(q^{\alpha}-1)^{j}\binom{i}{j}\tilde{E}_{n-1+j,q,w}^{(\alpha)}(h-\alpha,k)=\sum_{j=0}^{i-1}(q^{\alpha}-1)^{j}\binom{i-1}{j}\tilde{E}_{n-i+j,q,w}^{(\alpha)}(h,k).$$

Therefore, we obtain the following theorem.

Theorem 2. For α, $k \in \mathbb{N}$ and n, $i \in \mathbb{Z}_{+}$, we have

$$\sum_{j=0}^{i}\binom{i}{j}(q^{\alpha}-1)^{j}\tilde{E}_{n-i+j,q,w}^{(\alpha)}(h-\alpha,k)=\sum_{j=0}^{i-1}(q^{\alpha}-1)^{j}\binom{i-1}{j}\tilde{E}_{n-i+j,q,w}^{(\alpha)}(h,k).$$

By simple calculation, we easily see that

$$\sum_{j=0}^{m}\binom{m}{j}(q^{\alpha}-1)^{j}\tilde{E}_{j,q,w}^{(\alpha)}(0,k)=\frac{[2]_q^k}{(1+wq^{\alpha m})(1+wq^{\alpha m-1})\cdots(1+wq^{\alpha m-k+1})}.$$

3. Polynomials $\tilde{E}_{n,q,w}^{(\alpha)}(0,k|x)$

We now consider the polynomials $\tilde{E}_{n,q,w}^{(\alpha)}(0,k|x)$ (in q^x) by

$$\begin{aligned}&\tilde{E}_{n,q,w}^{(\alpha)}(0,k|x)\\&=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}w^{x_1+\cdots+x_k}\left[x+\sum_{i=1}^{k}x_i\right]_{q^{\alpha}}^{n}q^{-\sum_{j=1}^{k}jx_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k).\end{aligned}\tag{8}$$

By (8) and (4), we get

$$(q^{\alpha}-1)^{n}\tilde{E}_{n,q,w}^{(\alpha)}(0,k|x)=[2]_q^k\sum_{l=0}^{n}\binom{n}{l}q^{\alpha lx}(-1)^{n-1}\frac{1}{(1+wq^{\alpha l})\cdots(1+wq^{\alpha l-k+1})}.\tag{9}$$

From (8) and (9), we can derive the following equation.

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}w^{x_1+\cdots+x_k}q^{\sum_{j=1}^{k}(\alpha n-j)x_j+\alpha nx}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\&=\sum_{j=0}^{n}\binom{n}{j}[\alpha]_q^j(q-1)^{j}\tilde{E}_{j,q,w}^{(\alpha)}(0,k|x),\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}w^{x_1+\cdots+x_k}q^{\sum_{j=1}^{k}(\alpha n-j)x_j+\alpha nx}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\&=\frac{[2]_q^k q^{\alpha nx}}{(1+wq^{\alpha n})\cdots(1+wq^{\alpha n-k+1})}.\end{aligned}\tag{10}$$

Therefore, by (9) and (10), we obtain the following theorem.

Theorem 3. For $\alpha \in \mathbb{N}$ and n, $k \in \mathbb{Z}_{+}$, we have

$$\tilde{E}_{n,q,w}^{(\alpha)}(0,k|x)=\frac{[2]_q^k}{[\alpha]_q^n(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}q^{\alpha lx}\frac{1}{(-wq^{\alpha l-k+1}:q)_k},$$

and

$$\sum_{l=0}^{n}\binom{n}{l}[\alpha]_q^l(q-1)^l\tilde{E}_{l,q,w}^{(\alpha)}(0,k|x)=\frac{q^{\alpha nx}[2]_q^k}{(-wq^{\alpha n-k+1}:q)_k},$$

where $(a:q)_0 = 1$ and $(a:q)_k = (1-a)(1-aq) \ldots (1-aq^{k-1})$.

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k}\left[x+\sum_{j=1}^{k}x_j\right]_{q^\alpha}^n q^{-\sum_{j=1}^{k}jx_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
&=\frac{[d]_{q^\alpha}^n}{[d]_{-q}^k}\sum_{a_1,\cdots,a_k=0}^{d-1}w^{a_1+\cdots+a_k}q^{-\sum_{j=2}^{k}(j-1)a_j}(-1)^{\sum_{j=1}^{k}a_j}\times \qquad (11)\\
&\quad\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}w^{d(x_1+\cdots+x_k)}\left[\frac{x+\sum_{j=1}^{k}a_j}{d}+\sum_{j=1}^{k}x_j\right]_{q^{\alpha d}}^n q^{-d\sum_{j=1}^{k}jx_j}d\mu_{-q^d}(x_1)\cdots d\mu_{-q^d}(x_k)
\end{aligned}$$

Thus, by (11), we obtain the following theorem.

Theorem 4. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have

$$\begin{aligned}
&\tilde{E}_{n,q,w}^{(\alpha)}(0,k|x)\\
&=\frac{[d]_{q^\alpha}^n}{[d]_{-q}^k}\sum_{a_1,\cdots,a_k=0}^{d-1}(-w)^{a_1+\cdots+a_k}q^{-\sum_{j=2}^{k}(j-1)a_j}\tilde{E}_{n,q^d,w^d}^{(\alpha)}\left(0,k|\frac{x+a_1+\cdots+a_k}{d}\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\tilde{E}_{n,q,w}^{(\alpha)}(0,k|dx)\\
&=\frac{[d]_{q^\alpha}^n}{[d]_{-q}^k}\sum_{a_1,\cdots,a_k=0}^{d-1}(-w)^{a_1+\cdots+a_k}q^{-\sum_{j=2}^{k}(j-1)a_j}\tilde{E}_{n,q^d,w^d}^{(\alpha)}\left(0,k|x+\frac{a_1+\cdots+a_k}{d}\right).
\end{aligned}$$

By (8), we get

$$\begin{aligned}
\tilde{E}_{n,q,w}^{(\alpha)}(0,k|x&=\sum_{l=0}^{n}\binom{n}{l}[x]_{q^\alpha}^{n-l}q^{\alpha lx}\tilde{E}_{l,q,w}^{(\alpha)}(0,k)\\
&=\left([x]_{q^\alpha}+q^{\alpha x}\tilde{E}_{q,w}^{(\alpha)}(0,k)\right)^n,
\end{aligned}$$

where $\tilde{E}_{n,q,w}^{(\alpha)}(0,k|0)=\tilde{E}_{n,q,w}^{(\alpha)}(0,k)$.

Thus, we note that

$$\begin{aligned}
\tilde{E}_{n,q,w}^{(\alpha)}(0,k|x+y&=\sum_{l=0}^{n}\binom{n}{l}[y]_{q^\alpha}^{n-l}q^{\alpha ly}\tilde{E}_{l,q,w}^{(\alpha)}(0,k|x)\\
&=\left([y]_{q^\alpha}+q^{\alpha y}\tilde{E}_{q,w}^{(\alpha)}(0,k|x)\right)^n.
\end{aligned}$$

4. Polynomials $\tilde{E}_{n,q,w}^{(\alpha)}(h,1|x)$

Let us define polynomials $\tilde{E}_{n,q,w}^{(\alpha)}(h,1|x)$ as follows:

$$\tilde{E}_{n,q,w}^{(\alpha)}(h,1|x)=\int_{\mathbb{Z}_p}w^{x_1}[x+x_1]_{q^\alpha}^n q^{x_1(h-1)}d\mu_{-q}(x_1). \qquad (12)$$

From (12), we have

$$\tilde{E}_{n,q,w}^{(\alpha)}(h,1|x)=\frac{[2]_q}{(1-q^{\alpha})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{1}{(1+wq^{\alpha l+h})}.$$

By the calculation of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we see that

$$\begin{aligned}&q^{\alpha x}\int_{\mathbb{Z}_p} w^{x_1}[x+x_1]_{q^{\alpha}}^{n} q^{x_1(h-1)}d\mu_{-q}(x_1)\\&=(q^{\alpha}-1)\int_{\mathbb{Z}_p} w^{x_1}[x+x_1]_{q^{\alpha}}^{n+1} q^{x_1(h-\alpha-1)}d\mu_{-q}(x_1)+\int_{\mathbb{Z}_p} w^{x_1}[x+x_1]_{q^{\alpha}}^{n} q^{x_1(h-\alpha-1)}d\mu_{-q}(x_1).\end{aligned}\tag{13}$$

Thus, by (13), we obtain the following theorem.

Theorem 5. For $\alpha\in\mathbb{N}$ and $h\in\mathbb{Z}$, we have

$$q^{\alpha x}\tilde{E}_{n,q,w}^{(\alpha)}(h,1|x)=(q^{\alpha}-1)\tilde{E}_{n+1,q,w}^{(\alpha)}(h-\alpha,1|x)+\tilde{E}_{n,q,w}^{(\alpha)}(h-\alpha,1|x).$$

It is easy to show that

$$\begin{aligned}\tilde{E}_{n,q,w}^{(\alpha)}(h,1|x)&=\int_{\mathbb{Z}_p} w^{x_1}[x+x_1]_{q^{\alpha}}^{n} q^{x_1(h-1)}d\mu_{-q}(x_1)\\&=\sum_{l=0}^{n}\binom{n}{l}[x]_{q^{\alpha}}^{n-1}q^{\alpha l x}\int_{\mathbb{Z}_p} w^{x_1}[x_1]_{q^{\alpha}}^{l} q^{x_1(h-1)}d\mu_{-q}(x_1)\\&=\sum_{l=0}^{n}\binom{n}{l}[x]_{q^{\alpha}}^{n-1}q^{\alpha l x}\tilde{E}_{l,q,w}^{(\alpha)}(h,1)\\&=\left(q^{\alpha x}\tilde{E}_{q,w}^{(\alpha)}(h,1)+[x]_{q^{\alpha}}\right)^n,\quad \text{for } n\geq 1,\end{aligned}\tag{14}$$

with the usual convention about replacing $(\tilde{E}_{q,w}^{(\alpha)}(h,1))^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}(h,1)$.

From $qI_{-q}(f_1)+I_{-q}(f)=[2]_q f(0)$, we have

$$\begin{aligned}&wq^h\int_{\mathbb{Z}_p} w^{x_1}[x+x_1+1]_{q^{\alpha}}^{n} q^{x_1(h-1)}d\mu_{-q}(x_1)+\int_{\mathbb{Z}_p} w^{x_1}[x+x_1]_{q^{\alpha}}^{n} q^{x_1(h-1)}d\mu_{-q}(x_1)\\&=[2]_q[x]_{q^{\alpha}}^{n}.\end{aligned}\tag{15}$$

By (13) and (15), we get

$$wq^h\tilde{E}_{n,q,w}^{(\alpha)}(h,1|x+1)+\tilde{E}_{n,q,w}^{(\alpha)}(h,1|x)=[2]_q[x]_{q^{\alpha}}^{n}.\tag{16}$$

For $x=0$ in (16), we have

$$wq^h\tilde{E}_{n,q,w}^{(\alpha)}(h,1|1)+\tilde{E}_{n,q,w}^{(\alpha)}(h,1)=\begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n>0.\end{cases}\tag{17}$$

Therefore, by (14) and (17), we obtain the following theorem.

Theorem 6. For $h\in\mathbb{Z}$ and $n\in\mathbb{Z}_+$, we have

$$wq^h(q^{\alpha}\tilde{E}_{q,w}^{(\alpha)}(h,1)+1)^n\tilde{E}_{n,q,w}^{(\alpha)}(h,1)=\begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n>0,\end{cases}$$

with the usual convention about replacing $(\tilde{E}_{q,w}^{(\alpha)}(h,1))^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}(h,1)$.

From the fermionic p-adic q-integral on $\mathbb{Z}_p$, we easily get

$$\tilde{E}^{(\alpha)}_{0,q,w}(h,1)=\int_{\mathbb{Z}_p} w^{x_1}q^{x_1(h-1)}d\mu_{-q}(x_1)=\frac{[2]_q}{[2]_{wq^h}}.$$

By (12), we see that

$$\begin{aligned}\tilde{E}^{(\alpha)}_{n,q^{-1},w^{-1}}(h,1|1-x)&=\int_{\mathbb{Z}_p} w^{-1}[1-x+x_1]^n_{q^{-\alpha}}q^{-x_1(h-1)}d\mu_{-q^{-1}}(x_1)\\&=(-1)^n wq^{\alpha n+h-1}\frac{[2]_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{1}{1+wq^{\alpha l+h}}\\&=(-1)^n wq^{\alpha n+h-1}\tilde{E}^{(\alpha)}_{n,q,w}(h,1|x)\end{aligned}\tag{18}$$

Therefore, by (18), we obtain the following theorem.

Theorem 7. For $\alpha\in\mathbb{N}$, $h\in\mathbb{Z}$ and $n\in\mathbb{Z}_+$, we have

$$\tilde{E}^{(\alpha)}_{n,q^{-1},w^{-1}}(h,1|1-x)=(-1)^n wq^{\alpha n+h-1}\tilde{E}^{(\alpha)}_{n,q,w}(h,1|x).$$

In particular, for $x = 1$, we get

$$\begin{aligned}&\tilde{E}^{(\alpha)}_{n,q,w}(h,1)\\&=(-1)^n wq^{\alpha n+h-1}\tilde{E}^{(\alpha)}_{n,q,w}(h,1|1)\\&=(-1)^{n+1}q^{\alpha n-1}\tilde{E}^{(\alpha)}_{n,q,w}(h,1)\text{ if }n\geq 1.\end{aligned}$$

Let $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$. Then we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} w^{x_1}q^{x_1(h-1)}[x+x_1]^n_{q^\alpha}d\mu_{-q}(x_1)\\&=\frac{[d]^n_{q^\alpha}}{[d]_{-q}}\sum_{a=0}^{d-1}w^a q^{ha}(-1)^a\int_{\mathbb{Z}_p} w^{dx_1}\left[\frac{x+a}{d}+x_1\right]^n_{q^{\alpha d}}q^{x_1(h-1)d}d\mu_{-q^d}(x_1).\end{aligned}\tag{19}$$

Therefore, by (19), we obtain the following theorem.

Theorem 8 (Multiplication formula). For $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$, we have

$$\tilde{E}^{(\alpha)}_{n,q,w}(h,1|x)=\frac{[d]^n_{q^\alpha}}{[d]_{-q}}\sum_{a=0}^{d-1}w^a q^{ha}(-1)^q\tilde{E}^{(\alpha)}_{n,q^d,w^d}\left(h,1|\frac{x+a}{d}\right).$$

5. Polynomials $\tilde{E}^{(\alpha)}_{n,q,w}(h,k|x)$ and $k = h$

In (3), we know that

$$\begin{aligned}&\tilde{E}^{(\alpha)}_{n,q,w}(h,k|x)\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k}[x_1+\cdots+x_k+x]^n_{q^\alpha}q^{(h-1)x_1+\cdots+(h-k)x_k}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k).\end{aligned}$$

Thus, we get

$$(q^\alpha-1)^n\tilde{E}^{(\alpha)}_{n,q,w}(h,k|x)=[2]^k_q\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}\frac{q^{\alpha l x}}{(1+wq^{\alpha l+h})\cdots(1+wq^{\alpha l+h-k+1})},$$

and

$$\begin{aligned} & wq^h \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x+1+\sum_{i=1}^k x_i]_{q^\alpha}^n q^{\sum_{i=1}^k (h-i)x_i} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ & = - \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x+\sum_{i=1}^k x_i]_{q^\alpha}^n q^{\sum_{i=1}^k (h-i)x_i} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ & \quad + [2]_q \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_2+\cdots+x_k} [x+\sum_{i=2}^k x_i]_{q^\alpha}^n q^{\sum_{i=2}^k (h-i)x_i} d\mu_{-q}(x_2)\cdots d\mu_{-q}(x_k). \end{aligned} \tag{20}$$

Therefore, by (3) and (20), we obtain the following theorem.

Theorem 9. For $h \in \mathbb{Z}$, $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we have

$$wq^h \tilde{E}_{n,q,w}^{(\alpha)}(h,k|x+1) + \tilde{E}_{n,q,w}^{(\alpha)}(h,k|x) = [2]_q \tilde{E}_{n,q,w}^{(\alpha)}(h-1,k-1|x).$$

Note that

$$\begin{aligned} & q^{\alpha x} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+w_k} [x+\sum_{i=1}^k x_i]_{q^\alpha}^n q^{\sum_{i=1}^k (h-i)x_i} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ & = (q^\alpha - 1) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x+\sum_{i=1}^k x_i]_{q^\alpha}^{n+1} q^{\sum_{i=1}^k (h-\alpha-i)x_i} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ & \quad + \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x+\sum_{i=1}^k x_i]_{q^\alpha}^n q^{\sum_{i=1}^k (h-\alpha-i)x_i} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ & = (q^\alpha - 1)\tilde{E}_{n+1,q,w}^{(\alpha)}(h-\alpha,k|x) + \tilde{E}_{n,q,w}^{(\alpha)}(h-\alpha,k|x). \end{aligned} \tag{21}$$

Therefore, by (21), we obtain the following theorem.

Theorem 10. For $n \in \mathbb{Z}_+$, we have

$$q^{\alpha x}\tilde{E}_{n,q,w}^{(\alpha)}(h,k|x) = (q^\alpha - 1)\tilde{E}_{n+1,q,w}^{(\alpha)}(h-\alpha,k|x) + \tilde{E}_{n,q,w}^{(\alpha)}(h-\alpha,k|x).$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x+\sum_{j=1}^k x_j]_{q^\alpha}^n q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ & = \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1,\cdots,a_k=0}^{d-1} w^{a_1+\cdots+a_k} q^{h\sum_{j=1}^k aj - \sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \times \\ & \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{d(x_1+\cdots+x_k)} \left[\frac{x+\sum_{j=1}^k a_j}{d} + \sum_{j=1}^k x_j \right]_{q^{\alpha d}}^n q^{d\sum_{j=1}^k (h-j)x_j} d\mu_{-q^d}(x_1)\cdots d\mu_{-q^d}(x_k). \end{aligned} \tag{22}$$

Therefore, by (22), we obtain the following theorem.

Theorem 11. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have

$$\begin{aligned} & \tilde{E}_{n,q,w}^{(\alpha)}(h,k|dx) \\ & = \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1,\cdots,a_k=0}^{d-1} w^{a_1+\cdots+a_k} q^{h\sum_{j=1}^k aj - \sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \tilde{E}_{n,q^d,w^d}^{(\alpha)}\left(h,k|x+\frac{a_1+\cdots+a_k}{d}\right). \end{aligned}$$

Let $\tilde{E}^{(\alpha)}_{n,q,w}(k,k|x) = \tilde{E}^{(\alpha)}_{n,q,w}(k|x)$. Then we get

$$(q^\alpha - 1)^n \tilde{E}^{(\alpha)}_{n,q,w}(k|x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} q^{\alpha l x} \frac{[2]_q^k}{(1+wq^{\alpha l+k})\cdots(1+wq^{\alpha l+1})},$$

and

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{-(x_1+\cdots+x_k)}[k-x+\sum_{i=1}^{k} x_i]_{q^{-\alpha}}^n q^{-\sum_{i=1}^{k}(k-i)x_i} d\mu_{-q^{-1}}(x_1)\cdots d\mu_{-q^{-1}}(x_k) \\
&= \frac{q^{\binom{k}{2}}}{(1-q^{-\alpha})^n}[2]_q^k \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{1}{(1+wq^{\alpha l+1})\cdots(1+wq^{\alpha l+k})} \\
&= (-1)^n q^{n\alpha} q^{\binom{k}{2}} \frac{[2]_q^k}{(1-q^\alpha)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{\alpha l x}}{(1+wq^{\alpha l+1})\cdots(1+wq^{\alpha l+k})} \\
&= (-1)^n q^{\alpha n+\binom{k}{2}} \tilde{E}^{(\alpha)}_{n,q,w}(k|x).
\end{aligned} \tag{23}$$

Therefore, by (23), we obtain the following theorem.

Theorem 12. For $n \in \mathbb{Z}_+$, we have

$$\tilde{E}^{(\alpha)}_{n,q^{-1},w^{-1}}(k|k-x) = (-1)^n w^k q^{\alpha n+\binom{k}{2}} \tilde{E}^{(\alpha)}_{n,q,w}(k|x).$$

Let $x = k$ in Theorem 12. Then we see that

$$\tilde{E}^{(\alpha)}_{n,q^{-1},w^{-1}}(k|0) = (-1)^n w^k q^{\alpha n+\binom{k}{2}} \tilde{E}^{(\alpha)}_{n,q,w}(k|k). \tag{24}$$

From (15), we note that

$$wq^k \tilde{E}^{(\alpha)}_{n,q,w}(k|x+1) + \tilde{E}^{(\alpha)}_{n,q,w}(k|x) = [2]_q \tilde{E}^{(\alpha)}_{n,q,w}(k-1|x). \tag{25}$$

It is easy to show that

$$(q^\alpha - 1)^n \tilde{E}^{(\alpha)}_{n,q,w}(k|0) = \sum_{l=0}^{n}\binom{n}{l}(-1)^{l+n}\frac{[2]_q^k}{(1+wq^{\alpha l+1})\cdots(1+wq^{\alpha l+k})}.$$

By simple calculation, we get

$$\begin{aligned}
&\sum_{l=0}^{n}\binom{n}{l}(q^\alpha-1)^l \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{i=1}^{k} x_k}[\sum_{i=1}^{k} x_k]_{q^\alpha}^l q^{\sum_{l=i}^{k}(k-i)x_i} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\
&= \frac{[2]_q^k}{(1+wq^{\alpha n+k})(1+wq^{\alpha n+k-1})\cdots(1+wq^{\alpha n+1})}.
\end{aligned} \tag{26}$$

From (26), we note that

$$\sum_{l=0}^{n}\binom{n}{l}(q^\alpha-1)^l \tilde{E}^{(\alpha)}_{l,q,w}(k|0) = \frac{[2]_q^k}{(1+wq^{\alpha n+k})(1+wq^{\alpha n+k-1})\cdots(1+wq^{\alpha n+1})},$$

and

$$\begin{aligned}\tilde{E}_{n,q,w}^{(\alpha)}(k|x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{\sum_{i=1}^{k} x_k} [x + \sum_{i=1}^{k} x_k]_{q^\alpha}^n q^{\sum_{i=1}^{k} (k-i)x_i} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{E}_{l,q,w}^{(\alpha)}(k|0) \\ &= \left(q^{x\alpha} \tilde{E}_{q,w}^{(\alpha)}(k|0) + [x]_{q^\alpha}\right)^n, n \in \mathbb{Z}_+,\end{aligned}$$

with the usual convention about replacing $(\tilde{E}_{q,w}^{(\alpha)}(k|0))^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}(k|0)$.

Put $x = 0$ in (25), we get

$$wq^k \tilde{E}_{n,q,w}^{(\alpha)}(k|1) + \tilde{E}_{n,q,w}^{(\alpha)}(k|0) = [2]_q \tilde{E}_{n,q}^{(\alpha)}(k-1|0).$$

Thus, we have

$$wq^k (q^\alpha \tilde{E}_{q,w}^{(\alpha)}(k|0) + 1)^n + \tilde{E}_{n,q,w}^{(\alpha)}(k|0) = [2]_q \tilde{E}_{n,q,w}^{(\alpha)}(k-1|0),$$

with the usual convention about replacing $(\tilde{E}_{q,w}^{(\alpha)}(k|0))^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}(k|0)$.

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Authors' contributions
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The authors declare that they have no competing interests.

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