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Fixed-point theorems for mappings satisfying the ordered contractive condition on noncommutative spaces

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Abstract

In the paper, we introduce noncommutative Banach spaces which generalize the concept of Banach spaces, and the k -ordered contractive condition; we then discuss an ordered structure and several properties on noncommutative Banach spaces. Moreover, some fixed-point theorems for mappings with the k -ordered contractive condition on noncommutative Banach spaces are presented. In addition, we investigate the existence and uniqueness of fixed points for an integral equation of Fredholm type.

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1 Introduction

The well-known fixed-point theorem of Banach [1] is a very important tool for solving existence problems in many branches of mathematics and physics. There are a large number of generalizations of the Banach contraction principle in the literature (see [2–14] and others). The theorem has been generalized in two directions. On the one side, the usual contractive condition is replaced by weakly contractive conditions. On the other side, the action spaces are replaced by metric spaces endowed with an ordered or partially ordered structure. In particular, there is much interest in obtaining the existence and uniqueness of fixed points for self-maps by altering the action spaces. In this direction, Dhage *et al.* [8] addressed a new category of fixed-point problems for a self-map with the help of ordered Banach spaces. Further improvements in those spaces were found in [14]. In recent years, Ran and Reurings [13], O'Regan and Petruşel [12] and others started the investigations concerning a fixed-point theory in ordered metric spaces. Later, many authors followed this concept by introducing and investigating the different types of contractive mappings, *e.g.*, in [6] Caballero *et al.* considered contractive-like mappings in ordered metric spaces and applied their results in ordinary differential equations. Some interesting fixed-point theorems concerning partially ordered metric spaces can also be found in [3, 5].

The results obtained by Huang and Zhang [9] have become of interest for many scholars. They reconsidered the Banach contraction principle by initiating a new concept of cone metric spaces. Recently, also, the existence of fixed points for the given contractive type mappings in partially ordered cone metric spaces was investigated (see [4, 10]).

The purpose of this paper is to present some fixed-point theorems for mappings satisfying the ordered contractive condition in the context of noncommutative metric spaces which are noncommutative sense of those in [9].

The paper is organized as follows: we firstly introduce a noncommutative Banach space and the k -ordered contractive condition, and then discuss the ordered structure and several properties on noncommutative Banach spaces. Moreover, some fixed-point theorems on this space are established. Finally, we investigate the existence and uniqueness of fixed points for integral equation of Fredholm type.

Throughout this paper, the letters \mathbb{R} , \mathbb{R}^+ , \mathbb{N} will denote the sets of all real numbers, nonnegative real numbers and natural numbers, respectively.

To begin with, we introduce some definitions and properties which will be used later.

Definition 1.1 Let E be a group with a unit e and suppose that there exists a metric d on E such that (E, d) is a complete metric space. E is said to be a noncommutative Banach space if the following conditions hold:

- (1) for any $x, y, z \in E$, we have $d(xz, yz) = d(x, y)$;
- (2) there exists a binary continuous operation

$$F : \mathbb{R} \times E \rightarrow E, \quad (\alpha, x) \mapsto x^\alpha$$

such that $F(-1, x) = x^{-1}$ is exactly the inverse of x in the group E and $F(0, x) = x^0 = e$ is the unit in the group E , and that

$$F(mn, x) = F(m, F(n, x)), \quad F(m + n, x) = F(m, x)F(n, x)$$

for $m, n \in \mathbb{R}$, $x \in E$;

- (3) for any $x \in E$, there exists a constant $M_x > 0$ such that

$$d(x^\alpha, e) \leq M_x |\alpha|, \quad \forall \alpha \in \mathbb{R}.$$

In particular, if there exists a constant $M > 0$ such that $d(x^\alpha, e) \leq M|\alpha|$ for $x \in E$, $\alpha \in \mathbb{R}$, then E is said to be uniformly bounded.

Let E be a uniformly bounded noncommutative Banach space. Taking $\alpha = 1$, we conclude that $d(x, e) \leq M$, which together with the triangular inequality yields $d(x, y) \leq 2M$. This shows E is bounded.

Example 1.1 All Banach spaces are noncommutative Banach spaces. Let $(X, \|\cdot\|)$ be a Banach space, then $(X, +)$ is a group with a unit θ , and there exists a metric d induced by the norm $\|\cdot\|$ such that (X, d) is a complete metric space. Firstly, the metric d satisfies $d(x + z, y + z) = d(x, y)$ for $x, y, z \in X$. Secondly, there exists a binary continuous mapping $F : \mathbb{R} \times X \rightarrow X$, $(k, x) \mapsto kx$ satisfying the condition (2) in Definition 1.1. Finally, for any $x \in X$, there exists a constant $M_x = \|x\| + 1 > 0$ such that $d(kx, \theta) = \|kx\| = |k|\|x\| \leq |k|M_x$, $\forall k \in \mathbb{R}$. According to the definition, X is a noncommutative Banach space.

Example 1.2 Let \mathbb{R}^n be the standard n -dimensional vector space over \mathbb{R} . Define

$$d(x, y) = \sum_{i=1}^n \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

for $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. Clearly, \mathbb{R}^n is a complete metric space. In order to verify that \mathbb{R}^n is a noncommutative Banach space. It suffices to show that for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, there exists a constant $M_x > 0$ such that $d(kx, \theta) \leq M_x |k|$, for $k \in \mathbb{R}$, where θ is unit in \mathbb{R}^n . Indeed, choose $M_x = \{1, |x_1|, |x_2|, \dots, |x_n|\}$, and we get $d(kx, \theta) \leq M_x |k|$.

Example 1.3 Suppose that H is a Hilbert space and $U(H)$ is the unitary group of H . Put

$$d(S, T) = \|ST^{-1} - I\| = \|S - T\| \quad \forall S, T \in U(H),$$

then $U(H)$ is a complete metric space as a subset of $B(H)$, where $B(H)$ denotes the algebra of all bounded linear operators on H . Moreover, for any $T \in U(H)$ and $\alpha \in \mathbb{R}$, set

$$T^\alpha = \int_0^{2\pi} e^{i\alpha\theta} dE_\theta,$$

where E_θ means the spectral measure associated with the operator T . Notice that

$$d(T^\alpha, I) = \left\| \int_0^{2\pi} (e^{i\alpha\theta} - 1) dE_\theta \right\| \leq \sup_{\theta \in [0, 2\pi]} |e^{i\alpha\theta} - 1| \leq 2\pi |\alpha|$$

for $\alpha \in \mathbb{R}$. Then $U(H)$ is a uniformly bounded noncommutative Banach space.

Definition 1.2 Let E be a noncommutative Banach space. P is a subset of E satisfying the following conditions:

- (1) P is nonempty, closed, and $P \neq \{e\}$;
- (2) $x, y \in P$ and $\alpha, \beta \in \mathbb{R}^+$ implies $x^\alpha y^\beta \in P$;
- (3) $P \cap P^{-1} = \{e\}$ where $P^{-1} = \{x^{-1} : x \in P\}$.

Then P is called a cone in E .

Given a cone P in a noncommutative Banach space E , a relation can be introduced as follows:

$$x \lesssim y \iff y^\beta x^{-\beta} \in P \quad \text{for all } \beta \in [0, 1].$$

One can show that ' \lesssim ' is a partial ordering in E with respect to P . In fact:

- (i) For $x \in E, x^\beta x^{-\beta} = x^{\beta-\beta} = x^0 = e \in P$ for all $\beta \in [0, 1]$. This implies that $x \lesssim x$.
- (ii) If $x \lesssim y$ and $y \lesssim x$, then $y^\beta x^{-\beta} \in P$ and $(y^\beta x^{-\beta})^{-1} = x^\beta y^{-\beta} \in P$ for all $\beta \in [0, 1]$. By $P \cap P^{-1} = \{e\}$, we get $y^\beta = x^\beta$, which implies that $y = x$.
- (iii) If $x \lesssim y$ and $y \lesssim z$, then $y^\beta x^{-\beta} \in P$ and $z^\beta y^{-\beta} \in P$ for all $\beta \in [0, 1]$, which together with the condition (2) in Definition 1.2 can infer $z^\beta x^{-\beta} \in P$. This shows that $x \lesssim z$.

Definition 1.3 A cone $P \subseteq E$ is called normal if there is a number $N > 0$ such that

$$e \lesssim x \lesssim y \Rightarrow d(x, e) \leq Nd(y, e), \quad \forall x, y \in E.$$

The least positive number N satisfying the above is called the normal constant of P . It is clear that $N \geq 1$.

Remark 1.1 Let $x \in P, \alpha \in \mathbb{R}$, then the following relation holds:

$$\begin{cases} x \lesssim x^\alpha, & \alpha \geq 1, \\ x^\alpha \lesssim x, & \alpha < 1. \end{cases}$$

Indeed, for any $\beta \in [0, 1]$, if $\alpha \geq 1$, then $(x^\alpha)^\beta x^{-\beta} = x^{(\alpha-1)\beta} \in P$ since P is a cone, thus we can get $x \lesssim x^\alpha$; if $\alpha < 1$, then $x^\beta (x^\alpha)^{-\beta} = x^{(1-\alpha)\beta} \in P$, which means $x^\alpha \lesssim x$.

Definition 1.4 For $u, v \in E$, if either $u \lesssim v$ or $v \lesssim u$ holds, we say that u and v are comparable, denoted

$$\vee(u, v) = \begin{cases} u, & v \lesssim u, \\ v, & u \lesssim v. \end{cases}$$

From the above definitions, we have the following properties.

Lemma 1.1 Suppose that P is a cone in E . For $u, v \in E$, we have:

- (1) Set $u \lesssim v$, then $u^\alpha \lesssim v^\alpha$ holds for any $0 \leq \alpha \leq 1$.
- (2) If u and v are comparable, then uv^{-1} and vu^{-1} are comparable, and furthermore $e \lesssim \vee(uv^{-1}, vu^{-1})$.
- (3) If u and v are comparable, then $d(\vee(uv^{-1}, vu^{-1}), e) = d(u, v)$.
- (4) (compatibility) Let $\{u_n\}, \{v_n\} \subset E$, u_n and v_n be comparable for all $n \in \mathbb{N}$. If $u_n \rightarrow u_0, v_n \rightarrow v_0$, then u_0 and v_0 are comparable.

Proof (1) Let $u \lesssim v$, we have $v^\beta u^{-\beta} \in P$ for all $\beta \in [0, 1]$. Since $\alpha\beta \in [0, 1]$ for any $\alpha \in [0, 1]$, we see $v^{\alpha\beta} u^{-\alpha\beta} \in P$, which implies that $u^\alpha \lesssim v^\alpha$.

(2) Without loss of generality, one can suppose that $u \lesssim v$, which means $vu^{-1} \in P$. Using Remark 1.1, one can see $uv^{-1} \lesssim vu^{-1}$. Furthermore $(vu^{-1})^\beta e^{-\beta} = (vu^{-1})^\beta \in P$ for all $\beta \in [0, 1]$, which implies $e \lesssim vu^{-1}$, and therefore $e \lesssim \vee(uv^{-1}, vu^{-1})$.

(3) Assume that $u \lesssim v$, then $\vee(uv^{-1}, vu^{-1}) = vu^{-1}$. It follows immediately from Definition 1.1 that

$$d(\vee(uv^{-1}, vu^{-1}), e) = d(vu^{-1}, e) = d(v, u) = d(u, v).$$

(4) Since u_n and v_n are comparable for all n , we can suppose that there exist two subsequences u_{n_k} and v_{n_k} such that $u_{n_k} \lesssim v_{n_k}$ for all k . Note that for all $\beta \in [0, 1]$

$$\begin{aligned} d(v_{n_k}^\beta u_{n_k}^{-\beta}, v_0^\beta u_0^{-\beta}) &\leq d(v_{n_k}^\beta u_{n_k}^{-\beta}, v_0^\beta u_{n_k}^{-\beta}) + d(v_0^\beta u_{n_k}^{-\beta}, v_0^\beta u_0^{-\beta}) \\ &= d(v_{n_k}^\beta, v_0^\beta) + d(u_{n_k}^{-\beta}, u_0^{-\beta}). \end{aligned}$$

It follows directly that

$$d(v_{n_k}^\beta u_{n_k}^{-\beta}, v_0^\beta u_0^{-\beta}) \rightarrow 0.$$

As P is closed, one obtains $v_0^\beta u_0^{-\beta} \in P$. This says that $u_0 \lesssim v_0$. □

2 Fixed-point theorems on noncommutative Banach spaces

From now on, we always suppose that E is a noncommutative Banach space with a partial ordering \lesssim induced by a normal cone P with the normal constant N . And some fixed-point theorems for mappings on E satisfying the ordered contractive condition will be presented. Let us begin with the following theorem.

Theorem 2.1 *Let $A : E \rightarrow E$ be a continuous mapping and suppose that the following two assertions hold.*

- (1) *There exists a constant $k \in (0, 1)$ such that for all $u, v \in E$, if u and v are comparable, then Au and Av are comparable and furthermore*

$$\vee(Av(Au)^{-1}, Au(Av)^{-1}) \lesssim \vee(vu^{-1}, uv^{-1})^k.$$

In this case, we say A satisfies the k -ordered contractive condition.

- (2) *There exists $x_0 \in E$ such that x_0 and Ax_0 are comparable.*

Then A has a fixed point x^ which is unique in the comparable sense. Namely, if $Ay^* = y^*$ and y^* and x^* are comparable, then $y^* = x^*$. Moreover,*

$$d(x^*, x_0) \leq \frac{1}{1-k} NM_{(Ax_0)x_0^{-1}}.$$

Proof Define a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq E$ by the formula $x_n = A^n x_0$, $n \in \mathbb{N}$. The proof can be divided into three steps.

Step I. $\{x_n\}$ is a Cauchy sequence.

Now, since x_0 and $Ax_0 = x_1$ are comparable, and A satisfies the k -ordered contractive condition, we obtain x_1 and x_2 , x_2 and x_3 , ..., x_n and x_{n+1} are comparable. Notice

$$\vee(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}) \lesssim \vee(x_{n-1} x_n^{-1}, x_n x_{n-1}^{-1})^k.$$

Inductively the following holds:

$$\vee(x_1 x_2^{-1}, x_2 x_1^{-1}) \lesssim \vee(x_0 x_1^{-1}, x_1 x_0^{-1})^k.$$

By the above for all $n \in \mathbb{N}$ and Lemma 1.1(1), we obtain

$$e \lesssim \vee(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}) \lesssim \vee(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1})^k \lesssim \dots \lesssim \vee(x_1 x_0^{-1}, x_0 x_1^{-1})^{k^n}.$$

From the above it is easy to conclude that

$$d(\vee(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}), e) \leq Nd(\vee(x_1 x_0^{-1}, x_0 x_1^{-1})^{k^n}, e),$$

which together with Definition 1.1 yields

$$d(\vee(x_{n+1}x_n^{-1}, x_nx_{n+1}^{-1}), e) \leq NM_{x_1x_0^{-1}}k^n.$$

Finally, by Lemma 1.1(3), this implies

$$d(x_n, x_{n+1}) \leq NM_{x_1x_0^{-1}}k^n \rightarrow 0,$$

which shows $\{x_n\}$ is a Cauchy sequence. The completeness of E implies that there exists $x^* \in E$ such that $\lim_{n \rightarrow \infty} x_n = x^*$.

Step II. x^* is a fixed point of A .

Suppose that $\lim_{n \rightarrow \infty} x_n = x^*$, which together with A is continuous, we get

$$Ax^* = A \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} Ax_n = x^*,$$

which shows x^* is a fixed point of A .

Step III. The uniqueness of the fixed point of A in the comparable sense.

Let us consider $Ay^* = y^*$, and let x^* and y^* be comparable. Without loss of generality, set $y^* \lesssim x^*$, which shows $x^*y^{*-1} \in P$. By the condition (1),

$$\vee(Ay^*(Ax^*)^{-1}, Ax^*(Ay^*)^{-1}) \lesssim \vee(y^*x^{*-1}, x^*y^{*-1})^k.$$

That is

$$\vee(y^*x^{*-1}, x^*y^{*-1}) \lesssim \vee(y^*x^{*-1}, x^*y^{*-1})^k.$$

In addition, we know $\vee(y^*x^{*-1}, x^*y^{*-1}) = x^*y^{*-1}$. Then $x^*y^{*-1} \lesssim (x^*y^{*-1})^k$, which implies $(x^*y^{*-1})^{k-1} \in P$. On the other hand, we have $(x^*y^{*-1})^{1-k} \in P$. Now, from the definition of a cone, we have $x^*y^{*-1} = e$, and then $y^* = x^*$.

Furthermore,

$$d(x^*, x_0) \leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} NM_{(Ax_0)x_0^{-1}}k^n = \frac{1}{1-k} NM_{(Ax_0)x_0^{-1}}. \quad \square$$

Remark 2.1 Assume in addition that E satisfies the condition:

- (3) For $u, v \in E$, if they are not comparable, then there exists $w \in E$ such that u and w , v and w are comparable, respectively.

Then A has a unique fixed point. Note that condition (3) is always valid if E is a lattice.

It suffices to show the uniqueness of the fixed point of A . Suppose that x^*, y^* are fixed points of A . Claim that x^* and y^* are comparable. If not, there exists $z \in E$ such that x^* and z, y^* and z are comparable, respectively. Since A satisfies the k -ordered contractive condition, then $A^{n+1}x^*$ and $A^{n+1}z, A^{n+1}y^*$ and $A^{n+1}z$ are comparable for any $n \in \mathbb{N}$, respectively. Also,

$$e \lesssim \vee(A^{n+1}x^*(A^{n+1}z)^{-1}, A^{n+1}z(A^{n+1}x^*)^{-1}) \lesssim \vee(x^*z^{-1}, z(x^*)^{-1})^{k^n},$$

and

$$e \lesssim \vee(A^{n+1}y^*(A^{n+1}z)^{-1}, A^{n+1}z(A^{n+1}y^*)^{-1}) \lesssim \vee(y^*z^{-1}, z(y^*)^{-1})^{k^n}.$$

Then

$$\begin{aligned} d(A^{n+1}x^*, A^{n+1}z) &= d(\vee(A^{n+1}x^*(A^{n+1}z)^{-1}, A^{n+1}z(A^{n+1}x^*)^{-1}), e) \\ &\leq Nd(\vee(x^*z^{-1}, z(x^*)^{-1})^{k^n}, e) \\ &\leq Nk^n M_{x^*z^{-1}}, \end{aligned}$$

and similarly

$$d(A^{n+1}y^*, A^{n+1}z) \leq Nk^n M_{y^*z^{-1}}.$$

The triangular inequality tells us that

$$\begin{aligned} d(x^*, y^*) &= d(A^{n+1}x^*, A^{n+1}y^*) \\ &\leq d(A^{n+1}x^*, A^{n+1}z) + d(A^{n+1}y^*, A^{n+1}z) \\ &\leq Nk^n (M_{x^*z^{-1}} + M_{y^*z^{-1}}) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, which implies $x^* = y^*$. This is a contradiction. Therefore, x^* and y^* are comparable. By Theorem 2.1, $x^* = y^*$.

Corollary 2.1 *Let E be a uniformly bounded noncommutative Banach space, P a normal cone with the normal constant N . For $c \in \mathbb{R}^+$, $x_0 \in E$, set $B(x_0, c) = \{x \in E : d(x, x_0) \leq c\}$. Suppose that a continuous mapping $A : E \rightarrow E$ satisfies the k -ordered contractive condition and $d(Ax_0, x_0) \leq c - kNM$. Also, x_0 and $x \in B(x_0, c)$ are comparable. Then there exists a unique fixed point in $B(x_0, c)$ in the comparable sense.*

Proof It suffices to show that $Ax \in B(x_0, c)$ for any $x \in B(x_0, c)$.

For any $x \in B(x_0, c)$, the triangular inequality gives

$$d(x_0, Ax) \leq d(x_0, Ax_0) + d(Ax_0, Ax).$$

By Lemma 1.1(3),

$$d(Ax_0, Ax) = d(\vee(Ax_0(Ax)^{-1}, Ax(Ax_0)^{-1}), e).$$

Since A satisfies the k -ordered contractive condition, we have

$$\vee(Ax_0(Ax)^{-1}, Ax(Ax_0)^{-1}) \lesssim \vee(x_0x^{-1}, xx_0^{-1})^k,$$

and then

$$d(\vee(Ax_0(Ax)^{-1}, Ax(Ax_0)^{-1}), e) \leq Nd(\vee(x_0x^{-1}, xx_0^{-1})^k, e) \leq kNM.$$

Hence we get

$$d(x_0, Ax) \leq d(x_0, Ax_0) + d(Ax_0, Ax) \leq c - kNM + kNM = c,$$

from which one deduces that

$$Ax \in B(x_0, c). \quad \square$$

Corollary 2.2 *Let $A : E \rightarrow E$ be a continuous mapping and suppose that the following two assertions hold:*

- (1) *there exist $k \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that for all $u, v \in E$, if u and v are comparable, then Au and Av are comparable and furthermore*

$$\vee(A^{n_0}v(A^{n_0}u)^{-1}, A^{n_0}u(A^{n_0}v)^{-1}) \lesssim \vee(vu^{-1}, uv^{-1})^k;$$

- (2) *there exists $x_0 \in E$ such that x_0 and Ax_0 are comparable.*

Then A admits a unique fixed point x^ in the comparable sense.*

Proof By Theorem 2.1, we know A^{n_0} has a unique fixed point x^* in the comparable sense. Notice that

$$A^{n_0}(Ax^*) = A(A^{n_0}x^*) = Ax^*$$

which means Ax^* is also a fixed point of A^{n_0} . Again, since x_0 and Ax_0 are comparable, then x^* and Ax^* are comparable, which implies $Ax^* = x^*$. Since the fixed point of A is also the fixed point of A^{n_0} , the fixed point of A is unique in the comparable sense. \square

Theorem 2.2 *Let $A : E \rightarrow E$ be a mapping and suppose that the following two assertions hold:*

- (1) *there exists a constant $k \in (0, \frac{1}{2})$ such that for all $\beta \in [0, 1]$, if u and v are comparable, then Au and Av , Au and u , Av and v are comparable, and furthermore*

$$\vee(Av(Au)^{-1}, Au(Av)^{-1})^\beta \lesssim \vee(Auu^{-1}, u(Au)^{-1})^{k\beta} \vee(Avv^{-1}, v(Av)^{-1})^{k\beta};$$

- (2) *A is continuous.*

Then A admits a unique fixed point x^ in the comparable sense, and for each $x_0 \in E$, $\lim_{n \rightarrow \infty} A^n x_0 = x^*$. Moreover,*

$$d(x^*, x_0) \leq \frac{1-k}{1-2k} NM_{(Ax_0)x_0^{-1}}.$$

Proof By the reflexivity of the partial ordering ' \lesssim ' in E , $x_0 \lesssim x_0$. And since A satisfies the condition (1), then x_0 and Ax_0 , Ax_0 and A^2x_0 , ..., $A^n x_0$ and $A^{n+1}x_0$ are comparable. Put $x_n = A^n x_0$. Then for each integer $n \geq 1$, from the condition (1), we get

$$\begin{aligned} \vee(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1})^\beta &= \vee(Ax_{n-1}(Ax_n)^{-1}, Ax_n(Ax_{n-1})^{-1})^\beta \\ &\lesssim \vee(Ax_{n-1}x_{n-1}^{-1}, x_{n-1}(Ax_{n-1})^{-1})^{k\beta} \vee(Ax_n x_n^{-1}, x_n(Ax_n)^{-1})^{k\beta} \\ &= \vee(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1})^{k\beta} \vee(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1})^{k\beta}. \end{aligned}$$

By the definition of partial ordering in E with respect to P , one has

$$\vee(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1})^{k\beta} \vee (x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1})^{-(1-k)\beta} \in P.$$

That is

$$\vee(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1})^{1-k} \lesssim \vee(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1})^k.$$

From Lemma 1.1(1), one can obtain

$$\vee(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}) \lesssim \vee(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1})^{\frac{k}{1-k}}.$$

As in the proof of Theorem 2.1, $\{x_n\}$ is a Cauchy sequence and there exists $x^* \in E$ such that $\lim_{n \rightarrow \infty} x_n = x^*$. Also, x^* is a fixed point of A .

It remains to be shown that x^* is a unique fixed point of A .

Suppose that there exists $y^* \in E$, $y^* \lesssim x^*$, such that $Ay^* = y^*$. Due to the condition (1), we have

$$\vee(Ay^* (Ax^*)^{-1}, Ax^* (Ay^*)^{-1}) \lesssim \vee(Ay^* y^{*-1}, y^* (Ay^*)^{-1})^k \vee (Ax^* x^{*-1}, x^* (Ax^*)^{-1})^k.$$

That is

$$\vee(y^* x^{*-1}, x^* y^{*-1}) \lesssim \vee(y^* y^{*-1}, y^* y^{*-1})^k \vee (x^* x^{*-1}, x^* x^{*-1})^k = e.$$

Since $\vee(y^* x^{*-1}, x^* y^{*-1}) = x^* y^{*-1}$. Then $x^* y^{*-1} \lesssim e$, which implies $(x^* y^{*-1})^{-1} \in P$. Now, applying Definition 1.2, we obtain the desired result.

Similar to the proof of Theorem 2.1, one can verify

$$d(x^*, x_0) \leq \frac{1-k}{1-2k} NM_{(Ax_0)x_0^{-1}}.$$

Let $u_0, v_0 \in E$, and $u_0 \lesssim v_0$. By $[u_0, v_0]$ from now on we denote the order interval $\{u \in E : u_0 \lesssim u \lesssim v_0\}$. Finally, we consider the fixed-point theorem in the order interval. \square

Theorem 2.3 *Let $[u_0, v_0]$ be the order interval in E . If $A : [u_0, v_0] \rightarrow [u_0, v_0]$ satisfies the conditions in Theorem 2.2, then A has a unique fixed point in the comparable sense.*

Proof Consider sequences $\{u_n\}$ and $\{v_n\}$ defined by $u_n = A^n u_0$ and $v_n = A^n v_0$, respectively. Then $\{u_n\}, \{v_n\} \subseteq [u_0, v_0]$. Since A satisfies Theorem 2.2(1), then for any $n \in \mathbb{N}$, u_n and v_n are comparable, and

$$e \lesssim \vee(u_n v_n^{-1}, v_n u_n^{-1}) \lesssim \vee(u_0 v_0^{-1}, v_0 u_0^{-1})^{\left(\frac{k}{1-k}\right)^n}.$$

Notice that P is a normal cone,

$$d(u_n, v_n) \leq NM_{u_0 v_0^{-1}} \left(\frac{k}{1-k}\right)^n.$$

Again $u_0 \lesssim u_1$, hence for all $n \in \mathbb{N}$, u_n and u_{n+1} are comparable. Then

$$e \lesssim \vee(u_n u_{n+1}^{-1}, u_{n+1} u_n^{-1}) \lesssim \vee(u_0 u_1^{-1}, u_1 u_0^{-1})^{\left(\frac{k}{1-k}\right)^n},$$

from which one deduces

$$d(u_n, u_{n+1}) \leq NM_{u_1 u_0^{-1}} \left(\frac{k}{1-k}\right)^n.$$

Hence, we conclude that $\{u_n\}$ is a Cauchy sequence with a limit point $u^* \in [u_0, v_0]$ for any $0 < k < \frac{1}{2}$.

Similarly, $\{v_n\}$ is a Cauchy sequence with a limit point $v^* \in [u_0, v_0]$ for any $0 < k < \frac{1}{2}$.

Now, as

$$d(u^*, v^*) = \lim_{n \rightarrow \infty} d(u_n, v_n) \leq \lim_{n \rightarrow \infty} NM_{u_0 v_0^{-1}} \left(\frac{k}{1-k}\right)^n = 0,$$

then $u^* = v^*$.

The rest of the proof is analogue to that in Theorem 2.2. □

If one checks the proof of Theorem 2.3, then one can easily obtain the following result.

Corollary 2.3 *Let $[u_0, v_0]$ be the order interval in E . If the continuous mapping $A : [u_0, v_0] \rightarrow [u_0, v_0]$ satisfies the k -ordered contractive condition, then A admits a unique fixed point in the comparable sense.*

3 Examples

We give some examples to illustrate the main result of this paper in the following.

Example 3.1 Consider the space (\mathbb{R}^n, d) given in Example 1.2. Let $P = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$ be a normal cone. The partial ordering in E with respect to P is given by

$$x \lesssim y \iff x_i \leq y_i, \quad 1 \leq i \leq n,$$

then E is a lattice. It is known that the operators on \mathbb{R}^n are in one-to-one correspondence with the $n \times n$ matrices. Consider a matrix

$$A = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix},$$

where $0 < a_i < 1, i = 1, 2, \dots, n$. Choose $k \in \mathbb{R}$ such that $\max_{1 \leq i \leq n} \{a_i\} < k < 1$. Clearly, Ax and Ay are comparable if x and y are comparable. Moreover, if $x \lesssim y$, then $Ay - Ax \lesssim k(y - x)$. By Theorem 2.1 and Remark 2.1, we know that A has a unique fixed point.

Example 3.2 Consider the integral equation of Fredholm type

$$x(t) = \int_0^1 K(t, s, x(s)) ds + g(t), \quad t \in [0, 1].$$

Assume that

- (1) $K : [0, 1] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : [0, 1] \rightarrow \mathbb{R}$ are continuous;
- (2) $K(t, s, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is monotonous for $t, s \in \mathbb{R}$;
- (3) there exist a continuous function $\varphi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ and $k \in (0, 1)$ such that

$$|K(t, s, u) - K(t, s, v)| \leq k |\varphi(t, s)(u - v)|,$$

for $t, s \in [0, 1], u, v \in \mathbb{R}$.

- (4) $\sup_{t \in [0, 1]} \int_0^1 |\varphi(t, s)| ds \leq 1$;
- (5) there exists $x_0 \in C[0, 1]$ such that for any $t \in [0, 1], x_0(t) \leq \int_0^1 K(t, s, x_0(s)) ds + g(t)$ or $x_0(t) \geq \int_0^1 K(t, s, x_0(s)) ds + g(t)$.

Then the integral equation has a unique solution x^* in $C[0, 1]$.

In fact, let $E = C[0, 1]$ with the metric induced by the supremum norm, *i.e.*,

$$d(x, y) = \|x - y\| = \max\{|x(t) - y(t)| : t \in [0, 1]\}$$

for $x, y \in C[0, 1]$, and $P = \{x \in C[0, 1] : x(t) \geq 0 \text{ for any } t \in [0, 1]\}$ be a normal cone in $C[0, 1]$. The partial ordering in E induced by P is given as follows:

$$x \lesssim y \iff x(t) \leq y(t) \text{ for any } t \in [0, 1].$$

Define $A : C[0, 1] \rightarrow C[0, 1]$ by

$$A(x(t)) = \int_0^1 K(t, s, x(s)) ds + g(t), \quad t \in [0, 1].$$

From the condition (2), A is monotonous. The monotonicity condition implies that Ax and Ay are comparable if x and y are comparable. Observe that for $x, y \in E$, if x and y are comparable, then

$$\begin{aligned} |A(y(t)) - A(x(t))| &\leq \int_0^1 |K(t, s, y(s)) - K(t, s, x(s))| ds \\ &\leq k \int_0^1 |\varphi(t, s)(y(s) - x(s))| ds \\ &\leq k \int_0^1 |\varphi(t, s)| ds \|y - x\| \\ &\leq k \|y - x\|, \end{aligned}$$

for $t \in [0, 1]$, which implies that $\|Ay - Ax\| \leq k \|y - x\|$. Thus, A is continuous. Again, E is a lattice, by Theorem 2.1 and Remark 2.1, the integral equation has a unique solution x^* in $C[0, 1]$.

Observe that the functions satisfying the conditions in Example 3.2 do exist. For example,

$$K(t, s, u) = ktsu, \quad \text{where } k \in (0, 1),$$

$$g(t) \geq 0 \quad \text{or} \quad g(t) \leq 0 \quad \text{is continuous,}$$

$$\varphi(t, s) = ts,$$

$$x_0(t) = 0,$$

for $t, s \in [0, 1]$ and $u \in \mathbb{R}$.

Notice that the examples given above are in linear spaces. As to the noncommutative case, it is under consideration now.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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