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Hybrid methods for accretive variational inequalities involving pseudocontractions in Banach spaces

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available at the end of the article**Abstract**

We use strongly pseudocontractions to regularize a class of accretive variational inequalities in Banach spaces, where the accretive operators are complements of pseudocontractions and the solutions are sought in the set of fixed points of another pseudocontraction. In this paper, we consider an implicit scheme that can be used to find a solution of a class of accretive variational inequalities.

Our results improve and generalize some recent results of Yao et al. (*Fixed Point Theory Appl*, doi:10.1155/2011/180534, 2011) and Lu et al. (*Nonlinear Anal*, 71(3-4), 1032-1041, 2009).

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1. Introduction

Throughout this paper, we always assume that E is a real Banach space, $\langle \cdot, \cdot \rangle$ is the dual pair between E and E^* , and 2^E denotes the family of all the nonempty subsets of E . Let C be a nonempty closed convex subset of E and $T : C \rightarrow E$ be a nonlinear mapping. Denote by $\text{Fix}(T)$ the set of fixed points of T , that is, $\text{Fix}(T) = \{x \in C : Tx = x\}$. The generalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|, \|f^*\| = \|x\|\}, \quad \forall x \in E.$$

In the sequel, we shall denote the single-valued duality mapping by j . When $\{x_n\}$ is a sequence in E , $x_n \rightarrow x$ ($x_n \rightharpoonup x$, $x_n \boxtimes x$) will denote strong (respectively, weak and weak*) convergence of the sequence $\{x_n\}$ to x .

A mapping T with domain $D(T)$ and range $R(T)$ in E is called pseudocontractive if the inequality

$$\|x - y\| \leq \|x - y + t((I - T)x - (I - T)y)\| \quad (1.1)$$

holds for each $x, y \in D(T)$ and for all $t > 0$. As a result of [1], it follows from (1.1) that T is pseudocontractive if and only if there exists $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2$ for any $x, y \in D(T)$. T is called strongly pseudocontractive if there exist $j(x - y) \in J(x - y)$ and $\beta \in (0, 1)$ such that $\langle Tx - Ty, j(x - y) \rangle \leq \beta \|x - y\|^2$ for any $x, y \in D(T)$. T is called Lipschitzian if there exists $L \geq 0$ such that $\|Tx - Ty\|$

$\leq L\|x - y\|, \forall x, y \in D(T)$. If $L = 1$, then T is called nonexpansive, and it is called contraction if $L \in [0, 1)$.

Let $E = H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Recall that $T : C \rightarrow H$ is called monotone if $\langle Tx - Ty, x - y \rangle \geq 0, \forall x, y \in C$. A variational inequality problem, denoted by $VI(T, C)$, is to find a point x^* with the property

$$x^* \in C, \text{ such that } \langle Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

If the mapping T is a monotone operator, then we say that $VI(T, C)$ is monotone.

In [2], Lu et al. considered the following type of monotone variational inequality problem in Hilbert spaces (denoted by $VI(1.2)$)

$$\text{find } x^* \in \text{Fix}(T) \text{ such that } \langle (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T), \quad (1.2)$$

where $T, S : C \rightarrow C$ are nonexpansive mappings and $\text{Fix}(T) \neq \emptyset$. Let W denote the set of solutions of the $VI(1.2)$.

Very recently, Yao et al. [3] considered $VI(1.2)$ in Hilbert spaces when $T, S : C \rightarrow C$ are pseudocontractions.

In this paper, we consider the following variational inequality problem in Banach spaces (denoted by $VI(1.3)$)

$$\text{find } x^* \in \text{Fix}(T) \text{ such that } \langle (I - S)x^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in \text{Fix}(T), \quad (1.3)$$

where $T, S : C \rightarrow C$ are pseudocontractions. Let Ω denote the set of solutions of the $VI(1.3)$ and assume that Ω is nonempty. Since $I - S$ is accretive, then we say $VI(1.3)$ is an accretive variational inequality.

For solving the $VI(T, C)$, hybrid methods were studied by Yamada [4] where he assumed that T is Lipschitzian and strongly monotone. However, his methods do not apply to the $VI(1.2)$ since the mapping $I - S$ fails, in general, to be strongly monotone, though it is Lipschitzian. In fact the $VI(1.2)$ is, in general, ill-posed, and thus regularization is needed. Let $T, S : C \rightarrow C$ be nonexpansive and $f : C \rightarrow C$ be contractive. In 2006, Moudafi and Mainge [5] studied the $VI(1.2)$ by regularizing the mapping $tS + (1 - t)T$ and defined $\{x_{s,t}\}$ as follows:

$$x_{s,t} = sf(x_{s,t}) + (1 - s)[tSx_{s,t} + (1 - t)Tx_{s,t}], \quad s, t \in (0, 1). \quad (1.4)$$

Since Moudafi and Mainge's regularization depends on t , the convergence of the scheme (1.4) is more complicated. So Lu et al. [2] defined $\{x_{s,t}\}$ as follows by regularizing the mapping S :

$$x_{s,t} = s[tf(x_{s,t}) + (1 - t)Sx_{s,t}] + (1 - s)Tx_{s,t}, \quad s, t \in (0, 1). \quad (1.5)$$

Note that Lu et al.'s regularization does no longer depend on t . And their result for the regularization (1.5) is under dramatically less restrictive conditions than Moudafi and Mainge's [5].

Very recently, Yao et al. [3] extended Lu et al.'s result to a general case, i.e., in the scheme (1.5), S, T are extended to Lipschitz pseudocontractive and f is extended to strongly pseudocontractive. But in [3], after careful discussion, we observe that a continuity condition on f is necessary. So, in this paper, we modify it.

Motivated and inspired by the above work, in this paper, we use strongly pseudocontractions to regularize the ill-posed accretive $VI(1.3)$, and analyze the convergence of

the scheme (1.5). The results we obtained improve and extend the corresponding results in [2,3].

2 Preliminaries

If Banach space E admits sequentially continuous duality mapping J from weak topology from weak* topology, then by [6, Lemma 1], we get that duality mapping J is single-valued. In this case, duality mapping J is also said to be weakly sequentially continuous, i.e., for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then $J(x_n) \rightharpoonup Jx$ [6,7].

A Banach space E is said to be satisfying *Opial's condition* if for any sequence $\{x_n\}$ in E , $x_n \rightharpoonup x (n \rightarrow \infty)$ implies that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - \gamma\|, \quad \forall \gamma \in E, \quad \text{with } \gamma \neq x.$$

By [6, Lemma 1], we know that if E admits a weakly sequentially continuous duality mapping, then E satisfies *Opial's condition*.

Lemma 2.1([7]) Let C be a nonempty closed convex subset of a reflexive Banach space E , which satisfies *Opial's condition*, and suppose $T : C \rightarrow E$ is nonexpansive. Then, the mapping $I - T$ is demiclosed at zero, i.e.,

$$x_n \rightharpoonup x, \quad x_n - Tx_n \rightarrow 0 \text{ implies } x = Tx.$$

Recall that $S : C \rightarrow C$ is called accretive if $I - S$ is pseudocontractive. We denote by J_r the resolvent of S , i.e., $J_r = (I + rS)^{-1}$. It is well known that J_r is nonexpansive, single-valued and $\text{Fix}(J_r) = S^{-1}(0) = \{z \in D(S) : 0 = Sz\}$ for all $r > 0$. For more details, see [8-10].

Let $T : C \rightarrow C$ be a pseudocontractive mapping; then, $I - T$ is accretive. We denote $A = J_1 = (2I - T)^{-1}$. Then, $\text{Fix}(A) = \text{Fix}(T)$ and $A : R(2I - T) \rightarrow K$ is nonexpansive and single-valued. The following lemma can be found in [11].

Lemma 2.2([11]) Let C be a nonempty closed convex subset of a real Banach space E and $T : C \rightarrow C$ be a continuous pseudocontractive map. We denote $A = J_1 = (2I - T)^{-1}$. Then,

- (i) [12, Theorem 6] The map A is a nonexpansive self-mapping on C , i.e., for all $x, y \in C$, there hold

$$\|Ax - Ay\| \leq \|x - y\| \text{ and } Ax \in C;$$

- (ii) If $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, then $\lim_{n \rightarrow \infty} \|x_n - Ax_n\| = 0$.

We also need the following lemma.

Lemma 2.3 Let C be a nonempty closed convex subset of a real Banach space E . Assume that $F : C \rightarrow E$ is accretive and weakly continuous along segments; that is $F(x + ty) \rightarrow F(x)$ as $t \rightarrow 0$. Then, the variational inequality

$$x^* \in C, \quad \langle Fx^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C \tag{2.1}$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, j(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{2.2}$$

Proof (2.1) \Rightarrow (2.2) Since F is accretive, we have

$$\langle Fx - Fx^*, j(x - x^*) \rangle \geq 0$$

and so

$$\langle Fx, j(x - x^*) \rangle \geq \langle Fx^*, j(x - x^*) \rangle \geq 0.$$

(2.2) \Rightarrow (2.1) For any $x \in C$, put $w = tx + (1 - t)x^*$, $\forall t \in (0, 1)$. Then, $w \in C$. Taking $x = w$ in (2.2), we have

$$\langle Fw, j(w - x^*) \rangle = t \langle Fw, j(x - x^*) \rangle \geq 0,$$

i.e.,

$$\langle Fw, j(x - x^*) \rangle \geq 0.$$

Letting $t \rightarrow 0$ in the above inequality, since F is weakly continuous along segments, it follows that (2.1) holds.

3. Main Results

Let C be a nonempty closed convex subset of a real Banach space E . Let $f: C \rightarrow C$ be a Lipschitz strongly pseudocontraction and $T, S: C \rightarrow C$ be two continuous pseudocontractions. For $s, t \in (0, 1)$, we define the following mapping

$$x \mapsto W_{s,t}x := s[tf(x) + (1 - t)Sx] + (1 - s)Tx.$$

It is easy to see that the mapping $W_{s,t}: C \rightarrow C$ is a continuous strongly pseudocontractive mapping. So, by [13], $W_{s,t}$ has a unique fixed point which is denoted $x_{s,t} \in C$; that is

$$x_{s,t} = W_{s,t}x_{s,t} = s[tf(x_{s,t}) + (1 - t)Sx_{s,t}] + (1 - s)Tx_{s,t}, \quad s, t \in (0, 1). \tag{3.1}$$

Theorem 3.1 Let E be a reflexive Banach space that admits a weakly sequentially continuous duality mapping from E to E^* . Let C be a nonempty closed convex subset of E . Let $f: C \rightarrow C$ be a Lipschitz strongly pseudocontraction, $S: C \rightarrow C$ be a Lipschitz pseudocontraction, and $T: C \rightarrow C$ be a continuous pseudocontraction with $Fix(T) \neq \emptyset$. Suppose that the solution set Ω of the VI(1.3) is nonempty. Let for each $(s, t) \in (0, 1)^2$, $\{x_{s,t}\}$ be defined by (3.1). Then, for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ converges in norm, as $s \rightarrow 0$, to a point $x_t \in Fix(T)$. Moreover, as $t \rightarrow 0$, the net $\{x_t\}$ converges in norm to the unique solution x^* of the following inequality variational(denoted by VI (3.2)):

$$x^* \in \Omega, \quad \langle (I - f)x^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in \Omega. \tag{3.2}$$

Hence, for each null sequence $\{t_n\}$ in $(0,1)$, there exists another null sequence $\{s_n\}$ in $(0,1)$, such that the sequence $x_{s_n t_n} \rightarrow x^*$ in norm as $n \rightarrow \infty$.

Proof We divide our proofs into several steps as follows.

Step 1 For each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ is bounded.

For any $z \in \text{Fix}(T)$, for all $s, t \in (0, 1)$, by (3.1), we have

$$\begin{aligned} \|x_{s,t} - z\|^2 &= \langle s[tf(x_{s,t}) + (1-t)Sx_{s,t}] + (1-s)Tx_{s,t} - z, J(x_{s,t} - z) \rangle \\ &= st\langle f(x_{s,t}) - f(z), J(x_{s,t} - z) \rangle + s(1-t)\langle Sx_{s,t} - Sz, J(x_{s,t} - z) \rangle \\ &\quad + (1-s)\langle Tx_{s,t} - Tz, J(x_{s,t} - z) \rangle + st\langle f(z) - z, J(x_{s,t} - z) \rangle \\ &\quad + s(1-t)\langle Sz - z, J(x_{s,t} - z) \rangle \\ &\leq st\beta\|x_{s,t} - z\|^2 + s(1-t)\|x_{s,t} - z\|^2 + (1-s)\|x_{s,t} - z\|^2 \\ &\quad + st\|f(z) - z\| \|x_{s,t} - z\| + s(1-t)\|Sz - z\| \|x_{s,t} - z\| \\ &= (1-st(1-\beta))\|x_{s,t} - z\|^2 + s[t\|f(z) - z\| \\ &\quad + (1-t)\|Sz - z\|]\|x_{s,t} - z\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{s,t} - z\| &\leq \frac{t\|f(z) - z\|}{t(1-\beta)} + \frac{(1-t)\|Sz - z\|}{t(1-\beta)} \\ &\leq \frac{1}{t(1-\beta)} \max\{\|f(z) - z\|, \|Sz - z\|\}. \end{aligned}$$

Hence, for each $t \in (0, 1)$, $\{x_{s,t}\}$ is bounded. Furthermore, by the Lipschitz continuity of f and S , we obtain $\{f(x_{s,t})\}$ and $\{Sx_{s,t}\}$, which are both bounded for each $t \in (0, 1)$. From (3.1), we have

$$\|Tx_{s,t}\| \leq \frac{1}{1-s}\|x_{s,t}\| + \frac{s}{1-s}\|tf(x_{s,t}) + (1-t)Sx_{s,t}\|.$$

So $\{Tx_{s,t}\}$ is also bounded as $s \rightarrow 0$ for each $t \in (0, 1)$.

Step 2 $x_{s,t} \rightarrow x_t \in \text{Fix}(T)$ as $s \rightarrow 0$.

From (3.1), for each $t \in (0, 1)$, we get

$$x_{s,t} - Tx_{s,t} = s[tf(x_{s,t}) + (1-t)Sx_{s,t} - Tx_{s,t}] \rightarrow 0, \quad \text{as } s \rightarrow 0. \tag{3.3}$$

It follows from (3.1) that

$$\begin{aligned} \|x_{s,t} - z\|^2 &= st\langle f(x_{s,t}) - f(z), J(x_{s,t} - z) \rangle + s(1-t)\langle Sx_{s,t} - Sz, J(x_{s,t} - z) \rangle \\ &\quad + (1-s)\langle Tx_{s,t} - Tz, J(x_{s,t} - z) \rangle + st\langle f(z) - z, J(x_{s,t} - z) \rangle \\ &\quad + s(1-t)\langle Sz - z, J(x_{s,t} - z) \rangle \\ &\leq (1-st(1-\beta))\|x_{s,t} - z\|^2 + st\langle f(z) - z, J(x_{s,t} - z) \rangle \\ &\quad + s(1-t)\langle Sz - z, J(x_{s,t} - z) \rangle. \end{aligned}$$

It turns out that

$$\|x_{s,t} - z\|^2 \leq \frac{1}{t(1-\beta)} \langle tf(z) + (1-t)Sz - z, J(x_{s,t} - z) \rangle, \quad \forall z \in \text{Fix}(T).$$

Assume that $\{s_n\} \subset (0, 1)$ is such that $s_n \rightarrow 0 (n \rightarrow \infty)$, by the above inequality we have

$$\|x_{s_n,t} - z\|^2 \leq \frac{1}{t(1-\beta)} \langle tf(z) + (1-t)Sz - z, J(x_{s_n,t} - z) \rangle, \quad \forall z \in \text{Fix}(T). \tag{3.4}$$

Since $\{x_{s_n,t}\}$ is bounded, without loss of generality, we may assume that as $s_n \rightarrow 0$, $x_{s_n,t} \rightharpoonup x_t$. Combining (3.3), Lemma 2.1 and 2.2, we obtain $x_t \in \text{Fix}(A) = \text{Fix}(T)$. Taking $z = x_t$ in (3.4), we get

$$\|x_{s_n,t} - x_t\|^2 \leq \frac{1}{t(1-\beta)} \langle tf(x_t) + (1-t)Sx_t - x_t, J(x_{s_n,t} - x_t) \rangle. \quad (3.5)$$

Since $x_{s_n,t} \rightharpoonup x_t$ and J is weakly sequentially continuous, by (3.5) as $s_n \rightarrow 0$, we obtain $x_{s_n,t} \rightarrow x_t$. This has proved that the relative norm compactness of the net $\{x_{s,t}\}$ as $s \rightarrow 0$.

Letting $n \rightarrow \infty$ in (3.4), we obtain

$$\|x_t - z\|^2 \leq \frac{1}{t(1-\beta)} \langle tf(z) + (1-t)Sz - z, J(x_t - z) \rangle, \quad \forall z \in \text{Fix}(T). \quad (3.6)$$

So, x_t is a solution of the following variational inequality:

$$x_t \in \text{Fix}(T), \langle tf(z) + (1-t)Sz - z, J(x_t - z) \rangle \geq 0, \quad \forall z \in \text{Fix}(T).$$

Letting $C = \text{Fix}(T)$, $F = t(I - f) + (1 - t)(I - S)$, by Lemma 2.3, we have the equivalent dual variational inequality:

$$x_t \in \text{Fix}(T), \langle tf(x_t) + (1-t)Sx_t - x_t, J(x_t - z) \rangle \geq 0, \quad \forall z \in \text{Fix}(T). \quad (3.7)$$

Next, we prove that for each $t \in (0, 1)$, as $s \rightarrow 0$, $\{x_{s,t}\}$ converges in norm to $x_t \in \text{Fix}(T)$. Assume $x_{s'_n,t} \rightarrow x'_t$ as $s'_n \rightarrow 0$. Similar to the above proof, we have $x'_t \in \text{Fix}(T)$, which solves the following variational inequality:

$$x'_t \in \text{Fix}(T), \langle tf(x'_t) + (1-t)Sx'_t - x'_t, J(x'_t - z) \rangle \geq 0, \quad \forall z \in \text{Fix}(T). \quad (3.8)$$

Taking $z = x'_t$ in (3.7) and $z = x_t$ in (3.8), we have

$$\langle tf(x_t) + (1-t)Sx_t - x_t, J(x_t - x'_t) \rangle \geq 0, \quad (3.9)$$

$$\langle tf(x'_t) + (1-t)Sx'_t - x'_t, J(x'_t - x_t) \rangle \geq 0. \quad (3.10)$$

Adding up (3.9) and (3.10), and since f is strongly pseudocontractive and S is pseudocontractive, we have

$$\begin{aligned} 0 &\leq t \langle (I - f)x_t - (I - f)x'_t, J(x'_t - x_t) \rangle + (1 - t) \langle (I - S)x_t - (I - S)x'_t, J(x'_t - x_t) \rangle \\ &\leq -t(1 - \beta) \|x'_t - x_t\|^2, \end{aligned}$$

which implies that $x'_t = x_t$. Hence, the net $\{x_{s,t}\}$ converges in norm to $x_t \in \text{Fix}(T)$ as $s \rightarrow 0$.

Step 3 $\{x_t\}$ is bounded.

Since $\Omega \subset \text{Fix}(T)$, for any $y \in \Omega$, taking $z = y$ in (3.7) we obtain

$$\langle tf(x_t) + (1-t)Sx_t - x_t, J(x_t - y) \rangle \geq 0. \quad (3.11)$$

Since $I - S$ is accretive, for any $y \in \Omega$, we have

$$\langle Sx_t - x_t, J(x_t - y) \rangle \leq \langle Sy - y, J(x_t - y) \rangle \leq 0. \quad (3.12)$$

Combining (3.11) and (3.12), we have

$$\langle f(x_t) - x_t, J(x_t - y) \rangle \geq 0, \quad \forall y \in \Omega, \quad (3.13)$$

i.e.,

$$\langle f(x_t) - y + y - x_t, J(x_t - y) \rangle \geq 0, \quad \forall y \in \Omega.$$

Hence,

$$\begin{aligned} \|x_t - \gamma\|^2 &\leq \langle f(x_t) - \gamma, J(x_t - \gamma) \rangle \\ &= \langle f(x_t) - f(\gamma), J(x_t - \gamma) \rangle + \langle f(\gamma) - \gamma, J(x_t - \gamma) \rangle \\ &\leq \beta \|x_t - \gamma\|^2 + \langle f(\gamma) - \gamma, J(x_t - \gamma) \rangle. \end{aligned}$$

Hence,

$$\|x_t - \gamma\|^2 \leq \frac{1}{1 - \beta} \langle f(\gamma) - \gamma, J(x_t - \gamma) \rangle, \tag{3.14}$$

which implies that

$$\|x_t - \gamma\| \leq \frac{1}{1 - \beta} \|f(\gamma) - \gamma\|.$$

So $\{x_t\}$ is bounded.

Step 4 $x_t \rightarrow x^* \in \Omega$ which is a solution of variational inequality (3.2).

Since f is strongly pseudocontractive, it is easy to see that the solution of the variational inequality (3.2) is unique.

Next, we prove that $\omega_w(x_t) \subset \Omega$; namely, if (t_n) is a null sequence in $(0,1)$ such that $x_{t_n} \rightarrow x'$ as $n \rightarrow \infty$, then $x' \in \Omega$. Indeed, it follows from (3.7) that

$$\langle (I - S)x_t, J(z - x_t) \rangle \geq \frac{t}{1 - t} \langle (I - f)x_t, J(x_t - z) \rangle.$$

Since $I - S$ is accretive, from the above inequality, we have

$$\langle (I - S)z, J(z - x_t) \rangle \geq \frac{t}{1 - t} \langle (I - f)x_t, J(x_t - z) \rangle, \quad \forall z \in \text{Fix}(T). \tag{3.15}$$

Letting $t = t_n \rightarrow 0$ in (3.15), we have

$$\langle (I - S)z, J(z - x') \rangle \geq 0, \quad \forall z \in \text{Fix}(T),$$

which is equivalent to its dual variational inequality by Lemma 2.3

$$\langle (I - S)x', J(z - x') \rangle \geq 0, \quad \forall z \in \text{Fix}(T).$$

Since $\text{Fix}(T)$ is closed convex, then $\text{Fix}(T)$ is weakly closed. Thus, $x' \in \text{Fix}(T)$ by virtue of $x_t \in \text{Fix}(T)$. So, $x' \in \Omega$.

Finally, we show that $x' = x^*$, the unique solution of VI(3.2). In fact, taking $t = t_n$ and $y = x'$ in (3.14), we obtain

$$\|x_{t_n} - x'\|^2 \leq \frac{1}{1 - \beta} \langle f(x') - x', J(x_{t_n} - x') \rangle,$$

which together with $x_{t_n} \rightarrow x'$ implies that $x_{t_n} \rightarrow x'$ as $t_n \rightarrow 0$. Let $t = t_n \rightarrow 0$ in (3.13), we have

$$\langle f(x') - x', J(x' - \gamma) \rangle \geq 0, \quad \forall \gamma \in \Omega. \tag{3.16}$$

It follows from (3.16) and $x' \in \Omega$ that x' is a solution of VI(3.2). By uniqueness, we have $x' = x^*$. Therefore, $x_t \rightarrow x^*$ as $t \rightarrow 0$.

By Theorem 3.1, we have the following corollary directly.

Corollary 3.1([2, Theorem 3.3]) Let C be a nonempty closed convex subset of a real Hilbert space H . Let $f: C \rightarrow C$ be a contraction, $S, T: C \rightarrow C$ be nonexpansive with $\text{Fix}(T) \neq \emptyset$. Suppose that the solution set W of the VI(1.2) is nonempty. Let for each $(s, t) \in (0, 1)^2$, $\{x_{s,t}\}$ be defined by (3.1). Then, for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ converges in norm, as $s \rightarrow 0$, to a point $x_t \in \text{Fix}(T)$. Moreover, as $t \rightarrow 0$, the net $\{x_t\}$ converges in norm to the unique solution x^* of the following inequality variational:

$$x^* \in W, \langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in W.$$

Hence, for each null sequence $\{t_n\}$ in $(0,1)$, there exists another null sequence $\{s_n\}$ in $(0,1)$, such that the sequence $x_{s_n t_n} \rightarrow x^*$ in norm as $n \rightarrow \infty$.

Remark Theorem 3.1 improves and generalizes Theorem 3.1 of Yao et al.[3] in the following aspects:

- (i) Theorem 3.1 generalizes Theorem 3.1 in [3] from Hilbert spaces to more general Banach spaces;
- (ii) The mappings T in [3, Theorem 3.1] is weakened from Lipschitzian to continuous;
- (iii) We modify the condition of f , i.e., we suppose that f is Lipschitz strongly pseudocontractive.

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Authors' contributions

The authors discussed the idea of this manuscript. YW drafted the manuscript and RC helped to complete it. We discussed some details by Email. All authors read and approved the final manuscript.

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