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Some new generalizations of nonempty intersection theorems without convexity assumptions and essential stability of their solution set with applications

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Abstract

As a generalization of the KKM theorem in (Yang and Pu in J. Optim. Theory Appl. 154:17-29 2012), we propose some new nonempty intersection theorems for an infinite family of set-valued mappings without convexity assumptions, and consider generic stability and essential components of solutions of a nonempty intersection theorem for an infinite family of set-valued mappings without convexity assumptions. This paper is an attempt to establish analogue results for the class of equilibria removing convexity assumptions. As applications, we deduce the corresponding results for Ky Fan's points, Nash equilibrium and variational relations.

MSC: 49J53; 49J40

Keywords: nonempty intersection theorem; convexity assumptions; essential stability

1 Introduction

The celebrated KKM theorem introduced in 1929 [1] has been extended with various applications to optimization-related problems for many decades. Fan [2] and Browder [3] gave a version of Hausdorff topological vector spaces for this problem. As a generalization of the KKM theorem, Guillerme [4] proved an intersection theorem for an infinite family of set-valued mappings, where index is any set. Moreover, Hou [5] proposed an intersection theorem for an infinite family of set-valued mappings, which was defined on non-compact spaces. Ding [6] introduced product FC-spaces to generalize the KKM theorem, and established the existence of equilibrium for generalized multi-objective games in FC-spaces, where the number of players was finite or infinite, and all payoffs were all set-valued mappings. Recently, Lin [7] brought forward systems of nonempty intersection theorems, and established the existence of solutions of systems of quasi-KKM problems, systems of quasi-variational inclusions and systems of quasi-variational inclusions, as particular cases.

Convexity assumptions or some convexity of mappings played an important role in [1–7]. But, in many works on the theory, some authors replaced the convexity of mappings by more general concepts. For example, two important concepts were marked by the seminal papers of Fan [8, 9] for removing the concavity/quasi-concavity assumptions of functions, and Nishimura and Friedman [10] abandoned concavity completely. Later extensions of the theory were due to Forgo for CF-concavity, Kim and Lee for $\mathcal{C}$-concavity, Hou for $\mathcal{C}$-quasi-concavity, and others; see [11–15]. Moreover, Pu and Yang [16, 17] studied the KKM theorem without convex hull and variational relation problems without the KKM property.

The method of essential solutions has been widely used in various fields [18–30]. The notation of an essential solution for fixed points was first introduced in [18]. For a fixed point x of a mapping f, if each mapping sufficiently near to f has a fixed point arbitrarily near to x, x is said to be essential. However, it is not true that any continuous mapping has at least one essential fixed point, even though the space has the fixed point property. Instead of considering the essential solution, Kinoshita [19] introduced the notion of essential components of the set of fixed points and proved that, for any continuous mapping of the Hilbert cube into itself, there exists at least one essential component of the set of its fixed points. Kohlberg and Mertens [20] introduced the notions of stable set and essential components of Nash equilibria, and proved that every finite n-person noncooperative game has at least one essential connected component of the set of its Nash equilibrium points. Later, Yu and Xiang [21] brought forward the notion of essential components of the set of Ky Fan's points, and deduced that every infinite n-person noncooperative game with concave payoff functions has at least one essential component of the set of its equilibrium points.

Motivated and inspired by research works mentioned above, we propose some new nonempty intersection theorems for an infinite family of set-valued mappings without convexity assumptions. Furthermore, we study the notion of essential stability of solutions of a nonempty intersection theorem without convexity assumptions.

2 Nonempty intersection theorem without convexity assumptions

We recall first some definitions and known results concerning set-valued mappings.

Definition 2.1 Let X, Y be two Hausdorff topological spaces. A set-valued mapping $F: X \rightrightarrows Y$ is said to be:

(1) upper semicontinuous at $x \in X$ if, for any open subset O of Y with $O \supset F(x)$, there exists an open neighborhood $U(x)$ of x such that $O \supset F(x')$ for any $x' \in U(x)$;
(2) upper semicontinuous on X if F is upper semicontinuous at each $x \in X$;
(3) lower semicontinuous at $x \in X$ if, for any open subset O of Y with $O \cap F(x) \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $O \cap F(x') \neq \emptyset$ for any $x' \in U(x)$;
(4) lower semicontinuous on X if F is lower semicontinuous at each $x \in X$;
(5) closed if $\mathrm{Graph}(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is a closed subset of $X \times Y$.

Definition 2.2 ([31]) Let X be a nonempty subset of a Hausdorff topological space E. X has the fixed point property, if and only if, every continuous mapping $f: X \longrightarrow X$ has a fixed point.

Throughout this paper, let $K(X)$ ($CK(X)$) stand for the set of nonempty compact (convex) subsets of X, and

$$\Delta_n = \left\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0\right\},$$

$$J(\lambda) = \left\{i \in \{1, \dots, n\} \mid \lambda_i > 0\right\}, \quad \forall \lambda \in \Delta_n.$$

A new nonempty intersection theorem for an infinite family of set-valued mappings without convexity assumptions is obtained.

Theorem 2.1 *Let I be any index set. For each $i \in I$, let X_i be a nonempty and compact subset of a Hausdorff topological space E_i, let $X = \prod_{i\in I} X_i$ have the fixed point property, and let $F_i : X_i \rightrightarrows X$ be a set-valued mapping. Assume that*:

(i) *for any $y_i \in X_i$, $F_i(y_i)$ is closed in X;*
(ii) *for any finite subset $N = \{y^1, \dots, y^n\}$ of X, there exists a continuous mapping $\phi_N : \Delta_n \longrightarrow X$ such that, for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta_n$, there exists $j \in J(\lambda)$ for which $\phi_N(\lambda) \in F_i(y^j_i)$ for each $i \in I$.*

Then

$$\bigcap_{i\in I} \bigcap_{y_i\in X_i} F_i(y_i) \neq \emptyset.$$

Proof Define the set-valued mapping $F : X \rightrightarrows X$ by

$$F(x) = \bigcap_{i\in I} F_i(x_i), \quad \forall x = (x_i)_{i\in I} \in X.$$

Let $x \in X$ be arbitrarily fixed. By (ii), for the set $N_1 = \{x\}$, there exists a continuous mapping $\phi_{N_1} : \Delta_1 \longrightarrow X$ such that $\phi_{N_1}(1) \in F(x)$, hence F has nonempty closed values.

Using again (ii), we infer that, for every finite subset $N = \{x^1, \dots, x^n\}$ of X, there exists a continuous mapping $\phi_N : \Delta_n \longrightarrow X$ such that, for any $(\lambda_1, \dots, \lambda_n) \in \Delta_n$, there exists $j \in J(\lambda)$ for which $\phi_N(\lambda) \in F(x)$. Thus F satisfies all the conditions of Theorem 2.1 in [16]. Hence there exists $x^* \in X$ such that

$$x^* \in \bigcap_{y\in X} F(y) = \bigcap_{i\in I} \bigcap_{y_i\in X_i} F_i(y_i).$$

This completes the proof. □

If I is a singleton, then Theorem 2.1 collapses Theorem 2.1, the main result of [16].

Theorem 2.2 *Let X be a nonempty and compact subset of a Hausdorff topological space E, let X have the fixed point property, and let $F : X \rightrightarrows X$ be a set-valued mapping. Assume that*:

(i) *for any $y \in X$, $F(y)$ is nonempty and closed in X;*
(ii) *for any finite subset $N = \{y^1, \dots, y^n\}$ of X, there exists a continuous mapping $\phi_N : \Delta_n \longrightarrow X$ such that, for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta_n$, there exists $j \in J(\lambda)$ for which $\phi_N(\lambda) \in F(y^j)$.*

Then

$$\bigcap_{y\in X} F(y) \neq \emptyset.$$

Next, we obtain a generalized nonempty intersection theorem for an infinite family of set-valued mappings without convexity assumptions.

Theorem 2.3 *Let I be any index set. For each $i \in I$, let X_i be a nonempty and compact subset of a locally convex topological linear space E_i, let $X = \prod_{i\in I} X_i$ have the fixed point property, and let $G_i : X \rightrightarrows X_i$, $F_i : X_i \rightrightarrows X$ be two set-valued mappings. Assume that*:

(i) $\mathrm{Graph}(F_i)$ *is closed in* $X_i \times X$;

(ii) G_i *is continuous with nonempty compact values*;

(iii) *for any finite subset $N = \{y^1, \dots, y^n\}$ of X, there exists a continuous mapping $\phi_N : \Delta_n \longrightarrow X$ such that, for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta_n$, there exists $j \in J(\lambda)$ for which $\phi_N(\lambda) \in F_i(y_i^j)$ and $\phi_N(\lambda)_i \in G_i(\phi_N(\lambda))$ for each $i \in I$.*

Then there exists $x^* \in X$ such that, for each $i \in I$, $x_i^* \in G_i(x^*)$ and

$$x^* \in \bigcap_{i\in I} \bigcap_{y_i \in G_i(x^*)} F_i(y_i).$$

Proof Let E_i be the locally convex topological vector space containing X_i, and let Λ_i be a basis of open neighborhoods of E_i. For every $V_i \in \Lambda_i$, consider the set-valued mapping

$$G_{V_i}(x) = \big(G_i(x) + V_i\big) \cap X_i.$$

Since G_i is continuous with nonempty compact values, then $\Omega_i = \{x \in X : x_i \in G_i(x)\}$ is closed in X, and $G_{Vi}^{-1}(y_i)$ is open in X for any $y_i \in X_i$.

For any $V \in \prod_{i\in I} \Lambda_i$ and each $i \in I$, define the mapping $F_i^V : X_i \rightrightarrows X$ by

$$F_i^V(y_i) = \big[\big(X \backslash G_{Vi}^{-1}(y_i)\big) \cup F_i(y_i)\big] \cap \Omega_i.$$

Clearly, (i) for any $y_i \in X_i$, $F_i^V(y_i)$ is closed in X; (ii) for any finite subset $N = \{y^1, \dots, y^n\}$ of X, there exists a continuous mapping $\phi_N : \Delta_n \longrightarrow X$ such that, for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta_n$, there exists $j \in J(\lambda)$ for which $\phi_N(\lambda) \in F_i^V(y_i^j)$ for each $i \in I$. By Theorem 2.1, there exists $x^V \in X$ such that

$$x^V \in \bigcap_{i\in I} \bigcap_{y_i \in X_i} F_i^V(y_i),$$

i.e., for each $i \in I$, $x_i^V \in G_i(x^V)$ and $x^V \in F_i(y_i)$ for any $y_i \in G_{Vi}(x^V)$. Since X is compact, we may assume without loss of generality that $x^V \longrightarrow x$. Then $x_i \in G_i(x)$ for each $i \in I$.

Suppose that there are $i \in I$ and $y_i \in G_i(x)$ such that $x \notin F_i(y_i)$, *i.e.*, $(y_i, x) \notin \mathrm{Graph}(F_i)$. Since G is continuous, there is $y_i^V \in X_i$ such that $y_i^V \in G_i(x^V) \subset G_i(x^V) + V_i$. Since $\mathrm{Graph}(F_i)$ is closed in $X_i \times X$, $(y_i^V, x^V) \notin \mathrm{Graph}(F_i)$, which implies that $y_i^V \in G_{Vi}(x^V)$ and $x^V \notin F_i(y_i^V)$. It is a contradiction. Hence, for each $i \in I$, $x_i \in G_i(x)$ and

$$x \in \bigcap_{i\in I} \bigcap_{y_i \in G_i(x)} F_i(y_i).$$

□

3 Essential stability

In this section, we study the essential stability of solutions of a nonempty intersection theorem without convexity assumptions. Let I be a finite set. For each $i \in I$, let X_i be a nonempty, convex and compact subset of a normed linear space E_i, and let $\{\phi_N : \Delta_{|N|} \longrightarrow X \mid N \text{ is any finite subset of } X\}$ be a set of continuous mappings. Denote by $\mathcal{M}$ the set of

$w = (F, G)$ such that the following conditions hold: (i) for any $i \in I$, $\mathrm{Graph}(F_i)$ is closed in $X \times X_i$; (ii) G_i is continuous with nonempty convex compact values; (iii) for any finite subset $N = \{y^1, \ldots, y^n\}$ of X, there exists a continuous mapping $\phi_N : \Delta_n \longrightarrow X$ such that, for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n$, there exists $j \in J(\lambda)$ for which $\phi_N(\lambda) \in F_i(y_i^j)$ and $\phi_N(\lambda)_i \in G_i(\phi_N(\lambda))$ for each $i \in I$.

By Theorem 2.3, for each $w = (F, G) \in \mathcal{M}$, there exists $x \in X$ such that, for each $i \in X$, $x_i \in G_i(x)$ and $x \in F_i(y_i)$ for any $y_i \in X_i$, which is called a solution of (F, G). The solution set of (F, G), denoted by $S(F, G)$, is nonempty. The solution correspondence $S : \mathcal{M} \rightrightarrows X$ is well defined. Moreover, to analyze the stability of solutions, some topological structure in the collection $\mathcal{M}$ is also needed. For each $w = (F, G), w' = (F', G') \in \mathcal{M}$, we define

$$\rho(w, w') = \sup_{i \in I} H_i\big(\mathrm{Graph}(F_i), \mathrm{Graph}(F'_i)\big) + \sup_{i \in I} \sup_{x \in X} h_i\big(G_i(x), G'_i(x)\big),$$

where H_i is the Hausdorff distance defined on $X_i \times X$, and h_i is the Hausdorff distance defined on X_i. Then $\mathcal{M}$ becomes a metric space.

Definition 3.1 Let $w \in \mathcal{M}$. An $x \in S(w)$ is said to be an essential point of $S(w)$ if, for any open neighborhood $N(x)$ of x in X, there is a positive δ such that $N(x) \cap S(w') \neq \emptyset$ for any $w' \in \mathcal{M}$ with $\rho(w, w') < \delta$. w is said to be essential if all $x \in S(w)$ is essential.

Definition 3.2 Let $w \in \mathcal{M}$. A nonempty closed subset $e(w)$ of $S(w)$ is said to be an essential set of $S(w)$ if, for any open set U, $e(w) \subset U$, there is a positive δ such that $U \cap S(w') \neq \emptyset$ for any $w' \in \mathcal{M}$ with $\rho(w, w') < \delta$.

Definition 3.3 Let $w \in \mathcal{M}$. An essential subset $m(w) \subset S(w)$ is said to be a minimal essential set of $S(w)$ if it is a minimal element of the family of essential sets ordered by set inclusion. A component $C(w)$ is said to be an essential component of $S(w)$ if $C(w)$ is essential.

Remark 3.1 It is easy to see that the problem $w \in \mathcal{M}$ is essential, if and only if, the mapping $S : \mathcal{M} \rightrightarrows X$ is lower semicontinuous at w.

First of all, let us introduce some mathematical tools for the following proof.

Lemma 3.1 ([32]) *Let X and Y be two topological spaces with Y compact. If F is a closed set-valued mapping from X to Y, then F is upper semicontinuous.*

Lemma 3.2 ([33]) *If X, Y are two metric spaces, X is complete and $F : X \rightrightarrows Y$ is upper semicontinuous with nonempty compact values, then the set of points, where F is lower semicontinuous, is a dense residual set in X.*

Lemma 3.3 ([25]) *Let (Y, ρ) be a metric space, K_1 and K_2 be two nonempty compact subsets of Y, V_1 and V_2 be two nonempty disjoint open subsets of Y. If $h(K_1, K_2) < \rho(V_1, V_2) :=$ $\inf\{\rho(x, y) \mid x \in V_1, y \in V_2\}$, then*

$$h\big(K_1, (K_1 \backslash V_2) \cup (K_2 \backslash V_1)\big) \leq h(K_1, K_2),$$

where h is the Hausdorff distance defined on Y.

Lemma 3.4 ([26]) *Let X, Y, Z be three metric spaces, $S_1 : Y \rightrightarrows X$ and $S_2 : Z \rightrightarrows X$ be two set-valued mappings. Suppose that there exists at least one essential component of $S_1(y)$ for each $y \in Y$, and there exists a continuous single-valued mapping $T : Z \longrightarrow Y$ such that $S_2(z) \supset S_1(T(z))$ for each $z \in Z$. Then there exists at least one essential component of $S_2(z)$ for each $z \in Z$.*

Lemma 3.5 ([27]) *Let C, D be two nonempty, convex and compact subsets of a linear normed space Y. Then*

$$h(C, \lambda C + \mu D) \leq h(C, D),$$

where $\lambda, \mu \geq 0$, $\lambda + \mu = 1$, and h is the Hausdorff distance defined on Y.

Theorem 3.1 *$(\mathcal{M}, \rho)$ is a complete metric space.*

Proof Let $\{w^n\}_{n=1}^{\infty}$ be any Cauchy sequence in $\mathcal{M}$, *i.e.*, for any $\varepsilon > 0$, there exists $N_0 > 0$ such that $\rho(w^n, w^m) < \varepsilon$ for any $n, m > N_0$. Then, for each $i \in I$ and $x \in X$, $\{\text{Graph}(F_i^n)\}_{n=1}^{\infty}$ and $\{G_i^n(x)\}_{n=1}^{\infty}$ are two Cauchy sequences in $K(X_i \times X)$ and $CK(X_i)$, converging to $A_i \in K(X_i \times X)$ and $G_i(x) \in CK(X_i)$. Denote $F_i(y_i) = \{x \in X : (y_i, x) \in A_i\}$. We will show that $w := ((F_i)_{i\in I}, (G_i)_{i\in I}) \in \mathcal{M}$.

(i) Clearly, $w^m \longrightarrow w$ under the metric ρ.

(ii) Assume that $w \notin \mathcal{M}$, then there are a finite set $N = \{y^1, \dots, y^n\}$ of X and $\lambda_0 \in \Delta_n$ such that, for any $j \in J(\lambda_0)$, there is $i_0 \in I$ for which $\phi_N(\lambda_0) \notin F_{i_0}(y^j_{i_0})$, *i.e.*, $(y^j_{i_0}, \phi_N(\lambda_0)) \notin \text{Graph}(F_{i_0})$, or $\phi_N(\lambda_0)_{i_0} \notin G_{i_0}(\phi_N(\lambda_0))$. Since $w^m \longrightarrow w$ under the metric ρ, $(y^j_{i_0}, \phi_N(\lambda_0)) \notin \text{Graph}(F^m_{i_0})$, *i.e.*, $\phi_N(\lambda_0) \notin F^m_{i_0}(y^j_{i_0})$, or $\phi_N(\lambda_0)_{i_0} \notin G^m_{i_0}(\phi_N(\lambda_0))$ for enough large m, which contradicts the fact that $w^m \in \mathcal{M}$. This completes the proof. □

Theorem 3.2 *The corresponding $S : (\mathcal{M}, \rho) \rightrightarrows X$ is upper semicontinuous with nonempty compact values.*

Proof The desired conclusion follows from Lemma 3.1 as soon as we show that Graph(S) is closed. Let $\{(w^n, x^n)\}$ be a sequence in $\mathcal{M} \times X$ converging to (w, x) such that $x^n \in S(w^n)$ for any n. Then, for each $i \in I$, $x_i^n \in G_i^n(x^n)$ and $x^n \in F_i^n(y_i)$ for any $y_i \in X_i$ and any $i \in I$. Since $x^n \longrightarrow x$ and $w^n \longrightarrow w$, then $x_i \in G_i(x)$ for each $i \in I$.

Suppose that there are $i \in I$ and $y_i \in G_i(x)$ such that $x \notin F_i(y_i)$, then there exists a sequence $\{y_i^n\}$ of X_i such that $y_i^n \longrightarrow y_i$ and $y_i^n \in G_i^n(x^n)$. Since $w^n \longrightarrow w$, $x^n \longrightarrow x$ and $y_i^n \longrightarrow y_i$, $(y_i^n, x^n) \notin \text{Graph}(F_i^n)$ for enough large n, which implies $y_i^n \in G_i^n(x^n)$ and $x^n \notin F_i^n(y_i^n)$. It is a contradiction. Hence, for each $i \in I$, $x_i \in G_i(x)$ and $x \in F_i(y_i)$ for any $y_i \in G_i(x)$. This completes the proof. □

Theorem 3.3 *There exists a dense residual subset $\mathcal{G}$ of $\mathcal{M}$ such that for each $w \in \mathcal{G}$, w is essential. In other words, there are most of the problems, whose solutions are all essential.*

Proof Since $(\mathcal{M}, \rho)$ is complete, and $S : \mathcal{M} \rightrightarrows X$ is upper semicontinuous with nonempty compact values, by Lemma 3.2, there is a dense residual subset $\mathcal{G}$ of $\mathcal{M}$, where w is lower semicontinuous. Hence w is essential for each $w \in \mathcal{G}$. □

Theorem 3.4 *For each $w \in \mathcal{M}$, there exists at least one minimal essential subset of $S(w)$.*

Proof Since $S : \mathcal{M} \rightrightarrows X$ is upper semicontinuous with nonempty compact values, then, for each open set $O \supset S(w)$, there exists $\delta > 0$ such that $O \supset S(w')$ for any $w' \in \mathcal{M}$ with $\rho(w, w') < \delta$. Hence $S(w)$ is an essential set of itself.

Let Θ denote the family of all essential sets of $S(w)$ ordered by set inclusion. Then Θ is nonempty and every decreasing chain of elements in Θ has a lower bound (because by the compactness the intersection is in Θ); therefore, by Zorn's lemma, Θ has a minimal element, and it is a minimal essential set of $S(w)$. □

Theorem 3.5 *For each $w \in \mathcal{M}$, every minimal essential subset of $S(w)$ is connected.*

Proof For each $w \in \mathcal{M}$, let $m(w) \subset S(w)$ be a minimal essential subset of $S(w)$. Suppose that $m(w)$ was not connected, then there exist two non-empty compact subsets $c_1(w)$, $c_2(w)$ with $m(w) = c_1(w) \cup c_2(w)$, and there exist two disjoint open subsets V_1, V_2 in X such that $V_1 \supset c_1(w)$, $V_2 \supset c_2(w)$. Since $m(w)$ is a minimal essential set of $S(w)$, neither $c_1(w)$ nor $c_2(w)$ is essential. There exist two open sets $O_1 \supset c_1(w)$, $O_2 \supset c_2(w)$ such that for any $\delta > 0$, there exist $w^1, w^2 \in \mathcal{M}$ with

$$\rho(w, w^1) < \delta, \qquad \rho(w, w^2) < \delta, \qquad S(w^1) \cap O_1 = \emptyset, \qquad S(w^2) \cap O_2 = \emptyset.$$

Here, we choose two open sets W_1, W_2 such that

$$c_1(w) \subset W_1 \subset \overline{W}_1 \subset O_1 \cap V_1, \qquad c_2(w) \subset W_2 \subset \overline{W}_2 \subset O_2 \cap V_2,$$

and, for each $i \in I$, denote $M_i^1 = X_i \times W_1$, $M_i^2 = X_i \times W_2$, which are open in $X_i \times X$, and $\inf\{d(a, b) \mid a \in M_i^1, b \in M_i^2, i \in I\} = \varepsilon > 0$.

Since $m(w) \subset W_1 \cup W_2$ and it is essential, there exists $0 < \delta^* < \varepsilon$ such that $S(w') \cap (W_1 \cup W_2) \neq \emptyset$ for any $w' \in \mathcal{M}$ with $\rho(w, w') < \delta^*$. Since $m(w)$ is the minimal essential set, neither $c_1(w)$ nor $c_2(w)$ is essential. Then, for $\frac{\delta^*}{8} > 0$, there exist two $w^1, w^2 \in \mathcal{M}$ such that

$$S(w^1) \cap W_1 = \emptyset, \qquad S(w^2) \cap W_2 = \emptyset, \qquad \rho(w^1, w) < \frac{\delta^*}{8}, \qquad \rho(w^2, w) < \frac{\delta^*}{8}.$$

Thus $\rho(w^1, w^2) < \frac{\delta^*}{4}$.

We define $w' = ((F_i')_{i \in I}, (G_i')_{i \in I})$ by

$$G_i'(x) = \lambda(x) G_i^1(x) + \mu(x) G_i^2(x),$$
$$A_i = [\operatorname{Graph}(F_i^1) \backslash M_i^2] \cup [\operatorname{Graph}(F_i^2) \backslash M_i^1],$$
$$F_i'(y_i) = \{x \in X \mid (y_i, x) \in A_i\},$$

where

$$\lambda(x) = \frac{d(x, \overline{W}_2)}{d(x, \overline{W}_1) + d(x, \overline{W}_2)}, \quad \forall x \in X,$$
$$\mu(x) = \frac{d(x, \overline{W}_1)}{d(x, \overline{W}_1) + d(x, \overline{W}_2)}, \quad \forall x \in X.$$

Now we will show that $w' \in \mathcal{M}$ and $\rho(w, w') < \delta^*$.

(i) Clearly, $\mathrm{Graph}(F'_i)$ is closed in $X_i \times X$ for any $i \in I$.

(ii) G'_i is continuous with nonempty convex compact values.

(iii) Assume that $w' \notin \mathcal{M}$, then there are a finite subset $N = \{y^1, \dots, y^n\}$ of X and $\lambda_0 \in \Delta_n$ such that, for any $j \in J(\lambda_0)$, there is $i_0 \in I$ for which $\phi_N(\lambda_0) \notin F'_{i_0}(y^j_{i_0})$, or $\phi_N(\lambda_0)_{i_0} \notin G'_{i_0}(\phi_N(\lambda_0))$. Since $\phi_N(\lambda_0)_{i_0} \in G^1_{i_0}(\phi_N(\lambda_0))$ and $\phi_N(\lambda_0)_{i_0} \in G^2_{i_0}(\phi_N(\lambda_0))$,

$$\phi_N(\lambda_0)_{i_0} = \lambda\big(\phi_N(\lambda_0)\big)\phi_N(\lambda_0)_{i_0} + \mu\big(\phi_N(\lambda_0)\big)\phi_N(\lambda_0)_{i_0} \in G'_{i_0}\big(\phi_N(\lambda_0)\big).$$

Hence $\phi_N(\lambda_0)_{i_0} \notin G'_{i_0}(\phi_N(\lambda_0))$ is false.

Since $W_1 \cap W_2 = \emptyset$, $\phi_N(\lambda_0) \notin W_1$ or $\phi_N(\lambda_0) \notin W_2$. Without loss of generality, we may assume that $\phi_N(\lambda_0) \notin W_1$. Since

$$\phi_N(\lambda_0) \notin F'_{i_0}\big(y^j_{i_0}\big) = \big(F^1_{i_0}\big(y^j_{i_0}\big)\backslash W_2\big) \cup \big(F^2_{i_0}\big(y^j_{i_0}\big)\backslash W_1\big),$$

then $\phi_N(\lambda_0) \notin F^2_{i_0}(y^j_{i_0})\backslash W_1$. Therefore $\phi_N(\lambda_0) \notin F^2_{i_0}(y^j_{i_0})$, which contradicts the fact that $w^2 \in \mathcal{M}$. Hence $w' \in \mathcal{M}$.

(iv) By Lemma 3.3 and Lemma 3.5,

$$\begin{aligned}
\rho\big(w', w\big) &= \sup_{i\in I, x\in X} h_i\big(G_i(x), G'_i(x)\big) + \sup_{i\in I} H_i\big(\mathrm{Graph}(F_i), \mathrm{Graph}\big(F'_i\big)\big) \\
&\leq \sup_{i\in I, x\in X} h_i\big(G_i(x), G^1_i(x)\big) + \sup_{i\in I, x\in X} h_i\big(G^1_i(x), G'_i(x)\big) \\
&\quad + \sup_{i\in I} H_i\big(\mathrm{Graph}(F_i), \mathrm{Graph}\big(F^1_i\big)\big) + \sup_{i\in I} H_i\big(\mathrm{Graph}\big(F^1_i\big), \mathrm{Graph}\big(F'_i\big)\big) \\
&\leq \left(\frac{1}{8} + \frac{1}{4} + \frac{1}{8} + \frac{1}{4}\right)\delta^* \\
&= \frac{3}{4}\delta^*.
\end{aligned}$$

Hence $\rho(w', w) < \delta^*$.

Since $(S(w') \cap W_1) \cup (S(w') \cap W_2) = S(W') \cap (W_1 \cup W_2) \neq \emptyset$, we assume $S(w') \cap W_1 \neq \emptyset$ without loss of generality, *i.e.*, there exists $x \in X$ such that $x \in S(w') \cap W_1$. It follows from the definition of w' that $x \in S(w^1)$, which contradicts the fact that $S(w^1) \cap W_1 = \emptyset$. This completes the proof. □

Theorem 3.6 *For each $w \in \mathcal{M}$, there exists at least one essential component of $S(w)$.*

Proof By Theorem 3.5, there exists at least one connected minimal essential subset $m(w)$ of $S(w)$. Thus, there is a component C of $S(w)$ such that $m(w) \subset C$. It is obvious that C is essential. Hence C is an essential component of $S(w)$. □

Denote by $\mathcal{M}'$ the set of F, when I is a singleton and $G(x) = X$. The following results are obtained.

Theorem 3.7 *There exists a dense residual subset Φ of $\mathcal{M}'$ such that, for each $F \in \Phi$, F is essential.*

Theorem 3.8 *For each $F \in \mathcal{M}'$, there exists at least one minimal essential subset of $S(F)$.*

Theorem 3.9 *For each $F \in \mathcal{M}'$, every minimal essential subset of $S(F)$ is connected.*

Theorem 3.10 *For each $F \in \mathcal{M}'$, there exists at least one essential component of $S(F)$.*

Remark 3.2 Theorems 3.7-3.10 are generalizations of the results of [25], where convexity assumptions of KKM mappings are necessary.

Remark 3.3 Khanh and Quan [34] obtained generic stability and essential components of generalized KKM points. Thus it is worth comparing the results in Section 3 of this paper with the results of [34].

(i) This paper is a multiplied version of the KKM theorem.

(ii) If (1) I is a singleton, and $G(x) = X$ for any $x \in X$ in this paper; (2) in [34], $X = Y = Z$ is a nonempty and compact subset of a metric space, and holds the fixed property; (3) in [34], T is the identity mapping, Theorems 3.1-3.6 coincide with Section 3 and Section 4 of [34].

4 Application (I): Ky Fan's points

To discuss the essential components of Ky Fan's points without convexity assumptions, we need the following definitions.

Definition 4.1 ([15]) Let X be a Hausdorff topological space, let $\{\phi_N : \Delta_{|N|} \longrightarrow X \mid N$ is any finite subset of $X\}$ be a set of continuous mappings. A function $f : X \times X \longrightarrow \mathbb{R}$ is said to be $\mathcal{C}$-quasi-concave on X if, for any finite subset $N = \{x_1, \ldots, x_n\}$ of X, one has $f(\phi_N(\lambda), \phi_N(\lambda)) \geq \min_{i \in J(\lambda)} f(x_i, \phi_N(\lambda))$ for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n$.

Definition 4.2 Let X be a nonempty and compact subset of a metric space having the fixed point property, and let $\{\phi_N : \Delta_{|N|} \longrightarrow X \mid N$ is any finite subset of $X\}$ be a set of continuous mappings. Denote by Ω_1 the set of all functions $\varphi : X \times X \longrightarrow \mathbb{R}$ such that the following conditions hold: (i) for each fixed $y \in X$, $x \longrightarrow \varphi(x, y)$ is lower semicontinuous; (ii) for each fixed $x \in X$, $y \longrightarrow \varphi(x, y)$ is $\mathcal{C}$-quasi-concave on X; (iii) $\varphi(x, x) \leq 0$ for all $x \in X$.

For each $\varphi \in \Omega_1$, we denote $S_1(\varphi) = \{x \in X \mid \varphi(x, y) \leq 0, \forall y \in X\}$, which is nonempty and compact (see [16]). Furthermore, points in $S_1(\varphi)$ are called Ky Fan's points of φ (see [21]). The solution mapping $S_1 : \Omega_1 \rightrightarrows X$ is well defined. For each $\varphi \in \Omega_1$, we define the corresponding $F_\varphi : X \rightrightarrows X$ by

$$F_\varphi(y) = \{x \in X \mid \varphi(x, y) \leq 0\}, \quad \forall y \in X.$$

Clearly, $F_\varphi \in \mathcal{M}$ for each $\varphi \in \Omega_1$. It is easy to see that the single-valued mapping $T_1 : \Omega_1 \longrightarrow \mathcal{M}$ by $T_1(\varphi) = F_\varphi$ is isometric. Furthermore, $S_1(\varphi) = S(F_\varphi) = S(T_1(\varphi))$. For any $\varphi, \varphi' \in \Omega_1$, define the distance on Ω_1 by $\rho_1(\varphi, \varphi') = \rho(F_\varphi, F_{\varphi'})$.

Theorem 4.1 *For each $\varphi \in \Omega_1$, there exists at least one essential component of $S_1(\varphi)$.*

Proof Since $T_1 : \Omega_1 \longrightarrow \mathcal{M}$ is an isometric mapping, it is continuous. Since there exists at least one essential component of $S(F)$ for each $F \in \mathcal{M}$, by Lemma 3.4, there exists at least one essential component of $S_1(\varphi)$ for each $\varphi \in \Omega_1$. □

Remark 4.1 In [21], X is a nonempty, convex and compact subset of a normed linear space. Denote by Ω_1' the set of all functions $\varphi : X \times X \longrightarrow \mathbb{R}$ such that the following conditions hold: (1) for each fixed $y \in X$, $x \longrightarrow \varphi(x,y)$ is lower semicontinuous; (2) for each fixed $x \in X$, $y \longrightarrow \varphi(x,y)$ is concave; (3) $\varphi(x,x) \leq 0$ for all $x \in X$; (4) $\sup_{(x,y)\in X\times X} |\varphi(x,y)| < +\infty$. Clearly, $\Omega_1' \subset \Omega_1$. For each $\varphi \in \Omega_1$, for each fixed $x \in X$, $y \longrightarrow \varphi(x,y)$ is $\mathcal{C}$-quasi-concave, not only concave, and $\sup_{(x,y)\in X\times X} |\varphi(x,y)| < +\infty$ is unnecessary. In [21], the notion of essential components is based on the metric ρ_1', which is defined by

$$\rho_1'(\varphi,\psi) = \sup_{(x,y)\in X\times X} \bigl|\varphi(x,y) - \psi(x,y)\bigr|, \quad \forall \varphi,\psi \in \Omega_1'.$$

Next we will explain that the metric ρ_1 is neither stronger nor weaker than ρ_1' even in the same space Ω_1'.

Example 4.1 Let $X = [0,1]$, $\varphi(x,y) = 0$ for all $(x,y) \in X \times X$. Then $\varphi \in \Omega_1'$ and $F_\varphi(y) = [0,1]$ for all $y \in X$.

(1) For each n, we define $\varphi^n(x,y) = 1$ for all $(x,y) \in X \times X$. Then $\varphi^n \in \Omega_1'$, $F_{\varphi^n}(y) = [0,1]$ for all $y \in X$, and $\rho_1'(\varphi^n,\varphi) = 1$, $\rho_1(\varphi^n,\varphi) = 0$. Then $\varphi^n \longrightarrow \varphi$ under the metric ρ_1, while $\varphi^n \not\longrightarrow \varphi$ under the metric ρ_1'.

(2) For each n, we define

$$\varphi^n(x,y) = \frac{1}{n}x - \frac{1}{n}y, \quad \forall (x,y) \in X \times X.$$

Then $\varphi^n \in \Omega_1'$ and

$$F_{\varphi^n}(y) = [0,y], \qquad F_\varphi(y) = [0,1], \quad \forall y \in X.$$

Hence

$$\rho_1'\bigl(\varphi^n,\varphi\bigr) \leq \frac{2}{n} \longrightarrow 0, \qquad \rho_1\bigl(\varphi^n,\varphi\bigr) = \sup_{y\in X} h\bigl(F_{\varphi^n}(y), F_\varphi(y)\bigr) > 0.$$

Then $\varphi^n \longrightarrow \varphi$ under the metric ρ_1', while $\varphi^n \not\longrightarrow \varphi$ under the metric ρ_1.

5 Application (II): Nash equilibrium

An n-person non-cooperative game Γ is a tuple (I, X_i, f_i), where $I = \{1,\ldots,n\}$, the ith player has a strategy set X_i, and $f_i : \prod_{i\in I} X_i \longrightarrow \mathbb{R}$ is his payoff function. Denote $X = \prod_{i\in I} X_i$, $X_{-i} = \prod_{j\in I\setminus\{i\}} X_j$, $x_{-i} = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in X_{-i}$, $x = (x_i,x_{-i}) \in X$. A point $x^* = (x_i^*, x_{-i}^*) \in X$ is said to be a Nash equilibrium point if, for each $i \in I$, $f_i(x_i^*, x_{-i}^*) = \max_{u_i\in X_i} f_i(u_i, x_{-i}^*)$. Denote by $S_2(\Gamma)$ the set of Nash equilibrium points of Γ.

Definition 5.1 Denote by Ω_2 the set of all games such that the following conditions hold: (i) for each $i \in I$, X_i is a nonempty and compact subset of a metric space E_i, $X = \prod_{i\in I} X_i$ has the fixed point property, and $\{\phi_N : \Delta_{|N|} \longrightarrow X \mid N \text{ is any finite subset of } X\}$ is a set of continuous mappings; (ii) for each $i \in I$, f_i is upper semicontinuous on X, and $f_i(x_i,\cdot)$ is lower semicontinuous on X_{-i} for any $x_i \in X_i$ (iii) for any finite subset $N = \{y^1,\ldots,y^n\}$ of X and any $\lambda = (\lambda_1,\ldots,\lambda_n) \in \Delta_n$, there exists $j \in J(\lambda)$ such that $f_i(\phi_N(\lambda)) \geq f_i(y_i^j, (\phi_N(\lambda))_{-i})$, $\forall i \in I$.

Theorem 5.1 *For any* $\Gamma \in \Omega_2$, $S_2(\Gamma) \neq \emptyset$.

Proof For any $\Gamma \in \Omega_2$ and any $i \in I$, define the corresponding $F_i^\Gamma : X_i \rightrightarrows X$ by

$$F_i^\Gamma(y_i) = \{x \in X \mid f_i(y_i, x_{-i}) \leq f_i(x)\}.$$

Clearly, $F^\Gamma = (F_i^\Gamma)_{i\in I}$ satisfies all the conditions of Theorem 2.1. Hence $S_2(\Gamma) = S(F^\Gamma) \neq \emptyset$. □

Clearly, $F^\Gamma \in \mathcal{M}$ for each $\Gamma \in \Omega_2$. It is easy to see that the single-valued mapping $T_2 : \Omega_2 \longrightarrow \mathcal{M}$ by $T_2(\Gamma) = F^\Gamma$ is isometric. Furthermore, $S_2(\Gamma) = S(F^\Gamma) = S(T_2(\Gamma))$. For any $\Gamma, \Gamma' \in \Omega_2$, define the distance on Ω_2 by $\rho_2(\Gamma, \Gamma') = \rho(F^\Gamma, F^{\Gamma'})$.

Theorem 5.2 *For each* $\Gamma \in \Omega_2$, *there exists at least one essential component of* $S_2(\Gamma)$.

Proof Since $T_2 : \Omega_2 \longrightarrow \mathcal{M}$ is an isometric mapping, it is continuous. Since, there exists at least one essential component of $S(F)$ for each $F \in \mathcal{M}$, by Lemma 3.4, there exists at least one essential component of $S_2(\Gamma)$ for each $\Gamma \in \Omega_2$. □

Remark 5.1 In [21], denote by Ω_2' the set of games such that the following conditions hold: (1) for any $i \in I$, X_i is a nonempty, compact and convex subset of a normed linear space; (2) $\sum_{i=1}^n f_i$ is upper semicontinuous on X; (3) for any $i \in I$ and any $u_i \in X_i$, $f_i(u_i, \cdot)$ is lower semicontinuous on X_{-i}; (4) for any $u_{-i} \in X_{-i}$, $f_i(\cdot, u_{-i})$ is concave on X; (5) $\sup_{x\in X} \sum_{i=1}^n |f_i(x)| < +\infty$. Clearly, $\Omega_2' \neq \Omega_2$. Games in Ω_2' have concave and uniform bounded payoffs, which are invalid for games in Ω_2. In [21], the notion of essential components is based on the metric ρ_2', which is defined by

$$\rho_2'(\Gamma, \Gamma') = \sup_{x\in X} \sum_{i=1}^n |f_i(x) - f_i'(x)|, \quad \forall \Gamma, \Gamma' \in \Omega_2'.$$

It is neither stronger nor weaker than ρ_2 even in the same space $\Omega_2 \cap \Omega_2'$.

Example 5.1 Let $I = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, $X = X_1 \times X_2$, $f_1(x_1, x_2) = 0$, $f_2(x_1, x_2) = 0$. Then $\Gamma \in \Omega_2 \cap \Omega_2'$

(1) For each n, we define $f_1^n(x_1, x_2) = 1$, $f_2^n(x_1, x_2) = 2$. Then $\Gamma^n \in \Omega_2 \cap \Omega_2'$,

$$F_i^\Gamma(y_i) = F_i^{\Gamma^n}(y_i) = [0, 1] \times [0, 1] = X, \quad \forall y \in X, i = 1, 2.$$

Then $\Gamma^n \longrightarrow \Gamma$ under the metric ρ_2, while $\Gamma^n \not\longrightarrow \Gamma$ under the metric ρ_2'.

(2) For each n, we define

$$f_1^n(x_1, x_2) = \frac{1}{n} x_1, \qquad f_2^n(x_1, x_2) = \frac{1}{n} x_1.$$

Then $\Gamma^n \in \Omega_2 \cap \Omega_2'$ and

$$F_1^{\Gamma^n}(y_1) = [y_1, 1] \times [0, 1], \qquad F_2^{\Gamma^n}(y_2) = [0, 1] \times [0, 1] \quad \forall y \in X.$$

Hence

$$\rho_2'(\Gamma^n,\Gamma) \le \frac{2}{n} \longrightarrow 0, \qquad \rho_2(\Gamma^n,\Gamma) = \sup_{i\in I}\sup_{y_i\in X_i} h\big(F_i^{\Gamma}(y_i), F_i^{\Gamma^n}(y_i)\big) > 0.$$

Then $\Gamma^n \longrightarrow \Gamma$ under the metric ρ_2', while $\Gamma^n \not\longrightarrow \Gamma$ under the metric ρ_2.

6 Application (III): variational relations

Luc [35] introduced a more general model of equilibrium problems, which is called a variational relation problem (in short, VRP). Further studies of variational relation problems were done in [36–43]. Let A, B and C be nonempty sets, $S: A \rightrightarrows A$, $T: A\times B \rightrightarrows C$ be set-valued mappings with nonempty values, and $R(a,b,c)$ be a relation linking elements $a\in A$, $b\in B$ and $c\in C$.

(VRP) Find $a^*\in A$ such that:

(i) $a^*\in S(a^*)$;

(ii) $R(a^*,b,c)$ holds for any $b\in S(a^*)$ and any $c\in T(a^*,b)$.

Definition 6.1 ([35]) Let A and B be nonempty subsets of topological spaces E_1 and E_2, respectively, and $R(a,b)$ be a relation linking $a\in A$ and $b\in B$. For each fixed $b\in B$, we say that $R(\cdot,b)$ is closed in the first variable if, for every net $\{a^\alpha\}$ converges to some a, and $R(a^\alpha,b)$ holds for any α, then the relation $R(a,b)$ holds.

Let X be a nonempty and compact subset of a metric space having the fixed point property, and let $\{\phi_N : \Delta_{|N|} \longrightarrow X \mid N \text{ is any finite subset of } X\}$ be a set of continuous mappings. Denote by Ω_3 the set of variational relations such that the following conditions hold: (i) $A := \{x\in X \mid x\in S_1(x)\}$ is closed; (ii) $S_2(x)\subset S_1(x)$ for any $x\in X$, and $S_2^{-1}(y)$ is open in X for any $y\in X$; (iii) for any fixed $y\in X$, $S_3(\cdot,y)$ is lower semicontinuous; (iv) for any fixed $y\in X$, $R(\cdot,y,\cdot)$ is closed; (v) for any finite subset $\{x_1,\dots,x_n\}$ of X and any $\lambda=(\lambda_1,\dots,\lambda_n)\in\Delta_n$, there exists $i\in J(\lambda)$ such that $R(\phi_n(\lambda),x_i,z)$ holds for any $z\in S_3(\phi_n(\lambda),x_i)$; if $x_i\in S_2(\phi_n(\lambda))$ for any $i\in J(\lambda)$, then $\phi_n(\lambda)\in S_2(\phi_n(\lambda))$.

For any $q=(S_1,S_2,S_3,R)\in\Omega_3$, denote by $V(q)$ the solution set of q, which is nonempty and compact. The solution mapping $V:\Omega_3\rightrightarrows X$ is well defined. Moreover, define the mapping $F^q : X\rightrightarrows X$ by

$$F^q(y) = \big[X\backslash S_2^{-1}(y)\big] \cup \big\{x\in X \mid x\in S_1(x) \text{ and } R(x,y,z) \text{ holds for all } z\in S_3(x,y)\big\}.$$

Clearly, $F^q\in\mathcal{M}$ for each $q\in\Omega_3$. It is easy to see that the single-valued mapping $T_3 : \Omega_3 \longrightarrow \mathcal{M}$ by $T_3(q)=F^q$ is isometric. Furthermore, $V(q)=S(F^q)=S(T_3(q))$. For any $q,q'\in\Omega_3$, define the distance on Ω_3 by $\rho_3(q,q')=\rho(F^q,F^{q'})$.

Theorem 6.1 *For each $q\in\Omega_3$, there exists at least one essential component of $V(q)$.*

Proof Since $T_3:\Omega_3\longrightarrow\mathcal{M}$ is an isometric mapping, it is continuous. Since there exists at least one essential component of $S(F)$ for each $F\in\mathcal{M}$, by Lemma 3.4, there exists at least one essential component of $V(q)$ for each $q\in\Omega_3$. □

Remark 6.1 As convexity assumptions are not necessary to variational relation problem in Ω_3, Theorem 6.1 includes properly Theorem 3.4 of [43].

7 Conclusion

As a generalization of the KKM theorem in [16], we propose a new nonempty intersection theorem for an infinite family of set-valued mappings without convexity assumptions, and study the notion of essential stability of a solution set of the nonempty intersection theorem without convexity assumptions. We show that most of problems (in the sense of Baire category) are essential and, for any problem, there exists at least one essential component of its solution set. This paper is the attempt to establish analogue results for the class of equilibria removing convexity assumptions. As applications, we deduce the corresponding results for Ky Fan's points, Nash equilibrium and variational relations.

Competing interests
The author declares that they have no competing interests.

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