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Estimates for Marcinkiewicz commutators with Lipschitz functions under nondoubling measures

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Abstract

Under the assumption that μ is a nondoubling measure on $\mathbb{R}^d$ satisfying the growth condition, the author proves that the commutator $\mathcal{M}_b$ generated by the Marcinkiewicz integral operator and the Lipschitz function is bounded from the Hardy space $H^{1,\infty,0}_{\mathrm{fin}}(\mu)$ into $L^q(\mu)$ for $1/q = 1 - \beta/n$ with the kernel satisfying a certain Hörmander-type condition. Moreover, the author shows that for $p = n/\beta$, $\mathcal{M}_b$ is bounded from the Morrey space $\mathcal{M}^p_q(\mu)$ into RBMO(μ), from $L^{n/\beta}(\mu)$ into RBMO(μ) and from $\mathcal{M}^p_q(\mu)$ into $\mathrm{Lip}_{(\beta-\frac{n}{p})}(\mu)$, respectively.

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1 Introduction

In recent years, harmonic analysis on spaces with nondoubling measures has become a very active research topic. There has been significant progress in the study of boundedness for singular integrals on these spaces; see [1–8]. Among a long list of research papers, some of them [9–11] are on the Marcinkiewicz integral operators. The motivation for developing the analysis with nondoubling measures and some important examples of nondoubling measures can be found in [12].

We recall that a nonnegative Radon measure μ on $\mathbb{R}^d$ is said to be a nondoubling measure if there is a positive constant C_0 such that for all $x \in \mathbb{R}^d$ and all $r > 0$ it satisfies:

$$\mu\big(B(x,r)\big) \le C_0 r^n, \tag{1.1}$$

where n is a positive constant and $0 < n \le d$, $B(x,r)$ is the open ball centered at x and having radius r.

Let $K(x,y)$ be a locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$. Assume that there exists a constant $C > 0$ such that for any $x, y \in \mathbb{R}^d$ with $x \neq y$,

$$\big|K(x,y)\big| \le C|x-y|^{-(n-1)}, \tag{1.2}$$

and for any $x, y, y' \in \mathbb{R}^d$,

$$\int_{|x-y|\ge 2|y-y'|} \big[\big|K(x,y) - K\big(x,y'\big)\big| + \big|K(y,x) - K\big(y',x\big)\big|\big] \frac{1}{|x-y|}\, d\mu(x) \le C. \tag{1.3}$$

The Marcinkiewicz integral $\mathcal{M}$ associated to the kernel $K(x,y)$ and the measure μ as in (1.1) is defined by

$$\mathcal{M}(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} K(x,y)f(y)\,d\mu(y)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \quad x \in \mathbb{R}^d. \tag{1.4}$$

Let $b \in L_{\mathrm{loc}}(\mu)$, the Marcinkiewicz commutator $\mathcal{M}_b$ is formally defined by

$$\mathcal{M}_b(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} \bigl[b(x) - b(y)\bigr]K(x,y)f(y)\,d\mu(y)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \quad x \in \mathbb{R}^d. \tag{1.5}$$

If μ is the d-dimensional Lebesgue measure in $\mathbb{R}^d$, and

$$K(x,y) = \frac{\Omega(x-y)}{|x-y|^{d-1}}$$

with Ω homogeneous of degree zero and $\Omega \in \mathrm{Lip}_\alpha(S^{d-1})$ for some $\alpha \in (0,1]$, then it is easy to verify that $K(x,y)$ satisfies (1.2) and (1.3), and $\mathcal{M}$ in (1.4) is just the higher dimensional Marcinkiewicz integral $\mathcal{M}_\Omega$ defined by Stein in [13], which is important in classical harmonic analysis and is a focus of active research; see [14–20]. Particularly, we should mention the work of Torchinsky and Wang [21], where they established the $L^p(\mathbb{R}^d)$ boundedness for the commutator generated by the Marcinkiewicz integral $\mathcal{M}_\Omega$ and $\mathrm{BMO}(\mathbb{R}^d)$ function with $p \in (1,\infty)$. However, it is also worth to study the different behavior of another type commutator generated by the Marcinkiewicz integral $\mathcal{M}_\Omega$ and $\mathrm{Lip}_\beta(\mathbb{R}^d)$ function, which was recently studied by Mo and Lu in [22] when Ω is homogeneous of degree zero and satisfies the cancellation condition. They obtained its boundedness from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ for $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$.

When μ satisfies growth condition (1.1), $\mathcal{M}$ as in (1.4) was first introduced by Hu *et al.* in [9], where the boundedness of such an operator in $L^p(\mu)$ with $1 < p < \infty$ and the Hardy space $H^1(\mu)$ were established under the assumption that $\mathcal{M}$ is bounded on $L^2(\mu)$ with the kernel $K(x,y)$ satisfying (1.2) and (1.3). Moreover, they got the same estimates for the commutator $\mathcal{M}_b$ defined as (1.5) with $b \in \mathrm{RBMO}(\mu)$ when the kernel $K(x,y)$ satisfies (1.2) and (1.6), which is slightly stronger than (1.3) and is defined as follows:

$$\begin{aligned}\sup_{\substack{y,y'\in\mathbb{R}^d, l>0,\\ |y-y'|\le l}} \sum_{k=1}^\infty k \int_{2^k l<|x-y|\le 2^{k+1}l} \bigl[\bigl|K(x,y) - K\bigl(x,y'\bigr)\bigr| \\ + \bigl|K(y,x) - K\bigl(y',x\bigr)\bigr|\bigr] \frac{1}{|x-y|}\,d\mu(x) \le C.\end{aligned} \tag{1.6}$$

However, in our problem, we discover that the kernels should satisfy some other kind of smoothness to replace condition (1.6).

Definition 1.1 Let $1 \leq s < \infty$, $0 < \varepsilon < 1$. We say that the kernel K satisfies a Hörmander-type condition if there exist $c_s > 1$ and $C_s > 0$ such that for any $x \in \mathbb{R}^d$ and $l > c_s|x|$,

$$\sup_{\substack{l>0, y, y' \in \mathbb{R}^d \\ |y-y'| \leq l}} \sum_{k=1}^{\infty} 2^{k\varepsilon} \left(2^k l\right)^n \left(\frac{1}{(2^k l)^n} \int_{2^k l < |x-y| \leq 2^{k+1} l} \left[\left(\left| K(x,y) - K\left(x,y'\right) \right| \right. \right. \right.$$
$$\left. \left. \left. + \left| K(y,x) - K\left(y',x\right) \right| \right) \frac{1}{|x-y|} \right]^s d\mu(x) \right)^{1/s} \leq C_s. \tag{1.7}$$

Directly, one can see that condition (1.7) can be rewritten as

$$\sup_{\substack{l>0, y, y' \in \mathbb{R}^d \\ |y-y'| \leq l}} \sum_{k=1}^{\infty} 2^{k\varepsilon} \left(2^k l\right)^{(n/s'-1)} \left(\int_{2^k l < |x-y| \leq 2^{k+1} l} \left[\left(\left| K(x,y) - K\left(x,y'\right) \right| \right. \right. \right.$$
$$\left. \left. \left. + \left| K(y,x) - K\left(y',x\right) \right| \right) \right]^s d\mu(x) \right)^{1/s} \leq C_s. \tag{1.7$'$}$$

We note that this kind of smoothness was not new. Condition (1.7′) is similar to the Hörmander-type condition which allows that the integral operator can be controlled by a maximal operator in doubling measure spaces, and also useful in the research of Schrödinger operators; see [23–25] for details. We denote by $\mathcal{H}^s$ the class of kernels satisfying this condition. It is clear that these classes are nested,

$$\mathcal{H}^{s_2} \subset \mathcal{H}^{s_1} \subset \mathcal{H}^1, \quad 1 < s_1 < s_2 < \infty.$$

We should point out that $\mathcal{H}^1$ is not condition (1.6).

In [11], by supposing that the kernel K satisfies (1.2) and (1.3), the authors studied the commutator $\mathcal{M}_b$ in the case of $b \in \mathrm{Lip}_\beta(\mu)$ and established that it is bounded from $L^p(\mu)$ into $L^q(\mu)$ for $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. Furthermore, when condition (1.3) is replaced by (1.7), $\mathcal{M}_b$ is bounded from $L^p(\mu)$ into $\mathrm{Lip}_{\beta-n/p}(\mu)$ for some $0 < \beta < 1/2$ and $n/\beta < p < \infty$, from $L^{n/\beta}(\mu)$ into $\mathrm{RBMO}(\mu)$ for some $0 < \beta < 1$ and $n/\beta < p < \infty$, respectively.

The purpose of this paper is to get some estimates for the commutator $\mathcal{M}_b$ with the kernel K satisfying (1.2) and (1.7) on the Hardy-type space and $\mathrm{RBMO}(\mu)$ spaces. To be precise, we establish the boundedness of $\mathcal{M}_b$ in $H^{1,\infty}_{\mathrm{fin}}(\mu)$ for $1/q = 1 - \beta/n$ in Section 2. In Section 3, we prove that $\mathcal{M}_b$ is bounded from $\mathrm{RBMO}(\mu)$ to the Morrey space $\mathcal{M}^p_q(\mu)$, from $\mathrm{RBMO}(\mu)$ to $L^{n/\beta}(\mu)$ for $p = n/\beta$.

Before stating our result, we need to recall some necessary notation and definitions. For a cube $Q \subset \mathbb{R}^d$, we mean a closed cube whose sides are parallel to the coordinate axes. We denote its center and its side length by x_Q and $\ell(Q)$, respectively. Let $\alpha > 1$, αQ denote the cube with the same center as Q and $\ell(\alpha Q) = \alpha \ell(Q)$. Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$S_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{[\ell(2^k Q)]^n},$$

where $N_{Q,R}$ is the smallest positive integer k such that $\ell(2^k Q) \geq \ell(R)$. The concept $S_{Q,R}$ was introduced in [1], where some useful properties of $S_{Q,R}$ can be found.

The following characterization of the Lipschitz space $\mathrm{Lip}_\beta(\mu)$ for $0 < \beta \le 1$ in [26] plays a key role in the proof of theorems.

Lemma 1.1 *For a function* $b \in L^1_{\mathrm{loc}}(\mu)$, *conditions* I, II *and* III *below are equivalent.*

(I) *There is a constant* $C_1 \ge 0$ *such that*

$$\left|b(x) - b(y)\right| \le C_1|x - y|^\beta$$

for μ*-almost every* x *and* y *in the support of* μ.

(II) *There exist some constant* $C_2 \ge 0$ *and a collection of numbers* b_Q *such that these two properties hold: for any cube* Q,

$$\frac{1}{\mu(2Q)} \int_Q \left|b(x) - b_Q\right| d\mu(x) \le C_2 \ell(Q)^\beta, \tag{1.8}$$

and for any cube R *such that* $Q \subset R$ *and* $\ell(R) \le 2\ell(Q)$,

$$\left|m_Q(b) - m_R(b)\right| \le C_2 \ell(Q)^\beta. \tag{1.9}$$

(III) *For any given* p, $1 \le p \le \infty$, *there is a constant* $C(p) \ge 0$ *such that for every cube* Q, *we have*

$$\left[\frac{1}{\mu(Q)} \int_Q \left|b(x) - m_Q(b)\right|^p d\mu(x)\right]^{1/p} \le C(p) \ell(Q)^\beta, \tag{1.10}$$

where, and in the sequel,

$$m_Q(b) = \frac{1}{\mu(Q)} \int_Q b(y)\, d\mu(y),$$

and also for any cube R *such that* $Q \subset R$ *and* $\ell(R) \le 2\ell(Q)$,

$$\left|m_Q(b) - m_R(b)\right| \le C(p) \ell(Q)^\beta.$$

In addition, the quantities $\inf\{C_1\}$, $\inf\{C_2\}$ *and* $\inf\{C(p)\}$ *with a fixed* p *are equivalent and denoted by* $\|b\|_{\mathrm{Lip}_\beta}$.

Remark 1.1 Lemma 1.1 is a slight variant of Theorem 2.3 in [26]. To be precise, if we replace all balls in Theorem 2.3 of [26] by cubes, we then obtain Lemma 1.1.

Remark 1.2 For $0 < \beta \le 1$, (1.9) is equivalent to

$$|b_Q - b_R| \le C_2' S_{Q,R} \ell(R)^\beta \tag{1.11}$$

for any two cubes $Q \subset R$ with $\ell(R) \le 2\ell(Q)$; see Remark 2.7 in [26]. Note that for $\beta = 0$ (1.9) and (1.10) is just the space $\mathrm{RBMO}(\mu)$ of Tolsa; see [27]. Therefore, the space $\mathrm{Lip}_\beta(\mu)$ for $0 \le \beta \le 1$ can be seen as a member of a family containing $\mathrm{RBMO}(\mu)$.

We also need the following lemma for the $L^p(\mu)$-boundedness of $\mathcal{M}_b$, which was proved in [11].

Lemma 1.2 *Let* $b \in \mathrm{Lip}_\beta(\mu)$, $0 < \beta \le 1$. *Suppose that* $K(x,y)$ *satisfies* (1.2) *and* (1.3) *and that* $\mathcal{M}_b$ *is as in* (1.5). *If* $\mathcal{M}$ *is bounded on* $L^2(\mu)$, *then there exists a positive constant* $C > 0$ *such that for all bounded functions* f *with compact support*,

$$\left\|\mathcal{M}_b(f)\right\|_{L^q(\mu)} \le C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^p(\mu)}, \tag{1.12}$$

where $1 < p < n/\beta$ *and* $1/q = 1/p - \beta/n$.

Throughout this paper, we use the constant C with subscripts to indicate its dependence on the parameters. We denote simply by $A \lesssim B$ if there exists a constant $C > 0$ such that $A \le CB$; and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. For a μ-measurable set E, χ_E denotes its characteristic function. For any $p \in [1,\infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.

2 Boundedness of $\mathcal{M}_b$ in Hardy spaces

This section is devoted to the behavior of the commutator $\mathcal{M}_b$ in Hardy spaces. In order to define the Hardy space $H^1(\mu)$, Tolsa introduced the 'grand' maximal operator M_Φ in [27].

Definition 2.1 Given $f \in L^1_{\mathrm{loc}}(\mu)$, we define

$$M_\Phi f(x) = \sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d} f\varphi\, d\mu\right|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \le 1$,
(ii) $0 \le \varphi(y) \le \frac{1}{|y-x|^n}$ for all $y \in \mathbb{R}^d$,
(iii) $|\varphi'(y)| \le \frac{1}{|y-x|^{n+1}}$ for all $y \in \mathbb{R}^d$.

Based on Theorem 1.2 in [27], we can define the Hardy space $H^1(\mu)$ as follows; see also [1].

Definition 2.2 The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$ satisfying that $\int_{\mathbb{R}^d} f\, d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Moreover, we define the norm of $f \in H^1(\mu)$ by

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}.$$

We recall the atomic Hardy space $H^{1,\infty,0}_{\mathrm{atb}}(\mu)$ as follows.

Definition 2.3 Let $\rho > 1$. A function $h \in L^1_{\mathrm{loc}}(\mu)$ is called an atomic block if

(1) there exists some cube R such that $\operatorname{supp} h \subset R$,
(2) $\int_{\mathbb{R}^d} h(x)\, d\mu(x) = 0$,
(3) for $i = 1, 2$, there are functions a_i supported on cubes $Q_i \subset R$ and numbers $\lambda_i \in \mathbb{R}$ such that $h = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_i\|_{L^\infty(\mu)} \le \left[\mu(\rho Q_i) S_{Q_i,R}\right]^{-1}.$$

Then we define

$$|h|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)} = |\lambda_1| + |\lambda_2|.$$

Define $H^{1,\infty,0}_{\mathrm{atb}}(\mu)$ and $H^{1,\infty,0}_{\mathrm{fin}}(\mu)$ as follows:

$$\|f\|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)} = \inf\left\{\sum_{j}^{\infty} |h_j|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)} : f = \sum_{j=1}^{\infty} h_j, \{h_j\}_{j\in\mathbb{N}} \text{ are } (1,\infty,0)\text{-atoms}\right\}$$

and

$$\|f\|_{H^{1,\infty,0}_{\mathrm{fin}}(\mu)} = \inf\left\{\sum_{j=1}^{k} |h_j|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)} : f = \sum_{j=1}^{k} h_j, \{h_j\}_{j=1}^{k} \text{ are } (1,\infty,0)\text{-atoms}\right\},$$

where the infimum is taken over all possible decompositions of f in atomic blocks, $H^{1,\infty,0}_{\mathrm{fin}}(\mu)$ is the set of all finite linear combinations of $(1,\infty,0)$-atoms.

Remark 2.1 It was proved in [1] that for each $\rho > 1$, the atomic Hardy space $H^{1,\infty,0}_{\mathrm{atb}}(\mu)$ is independent of the choice of ρ.

To establish the boundedness of operators in Hardy-type spaces on $\mathbb{R}^n$, one usually appeals to the atomic decomposition characterization (see [28, 29]) of these spaces, which means that a function or distribution in Hardy-type spaces can be represented as a linear combination of atoms. Then the boundedness of linear operators in Hardy-type spaces can be deduced from their behavior on atoms in principle. However, Meyer [30] (see also [31]) gave an example of $f \in H^1(\mathbb{R}^n)$ whose norm cannot be achieved by its finite atomic decompositions via $(1,\infty,0)$-atoms. Based on this fact, Bownik [31] (Theorem 2) constructed a surprising example of a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all $(1,\infty,0)$-atoms into bounded scalars, but yet cannot extend to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$.

Recently, in [32], a boundedness criterion was established via Lusin function characterizations of Hardy spaces on $\mathbb{R}^n$ as follows: a sublinear operator T extends to a bounded sublinear operator from Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1]$ to some quasi-Banach space B if and only if T maps all $(p,2,s)$-atoms into uniformly bounded elements of B for some $s \geq [n(1/p-1)]$. Here and in what follows $[t]$ means the integer part of real t. This result shows the structural difference between atomic characterization of $H^p(\mathbb{R}^n)$ via $(p,2,s)$-atoms and (p,∞,s)-atoms. On the other hand, Meda *et al.* [33] independently obtained some similar results by grand maximal function characterizations of Hardy spaces on $\mathbb{R}^n$. In fact, let $p \in (0,1]$, $p < q \in [1,\infty]$ and integer $s \geq [n(1/p-1)]$, and let $H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ be the set of all finite linear combinations of (p,q,s)-atoms. Denote by $C(\mathbb{R}^n)$ the set of all continuous functions. For any $f \in H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$, when $q < \infty$ or $f \in H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ when $q = \infty$, Meda *et al.* in [33] proved that $f \in H^p(\mathbb{R}^n)$ can be achieved by a finite atomic decomposition via (p,q,s)-atom when $q < \infty$ or continuous (p,q,s)-atom when $q = \infty$; from this, they further deduced that if T is a linear operator and maps all $(1,q,0)$-atoms with $q \in (1,\infty)$ or all continuous $(1,q,0)$-atoms with $q = \infty$ into uniformly bounded elements of some Banach space B, then T uniquely extends to a bounded linear operator from $H^1(\mathbb{R}^n)$ to B which coincides with T on these $(1,q,0)$-atoms.

According to the theory of Meda *et al.* [33], we get the result as follows.

Theorem 2.1 *Let $0 < \beta \le 1$, $b \in \mathrm{Lip}_\beta(\mu)$ and $1/q = 1 - \beta/n$. Suppose that K satisfies* (1.2) *and $\mathcal{H}^q$ condition. If $f \in H^{1,\infty,0}_{\mathrm{fin}}(\mu)$, then $\mathcal{M}_b$ is bounded from the Hardy space into the Lebesgue space, namely, there exists a positive constant C such that*

$$\left\|\mathcal{M}_b(f)\right\|_{L^q(\mu)} \le C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{H^{1,\infty,0}_{\mathrm{fin}}(\mu)}.$$

Proof of Theorem 2.1 Via Remark 2.1, without loss of generality, we may assume that $\rho = 4$ and $f = \sum h$ as a finite sum of atomic blocks defined in Definition 2.3. It is easy to see that we only need to prove the theorem for one atomic block h. Let R be a cube such that $\operatorname{supp} h \subset R$, $\int_{\mathbb{R}^d} h(x)\,d\mu(x) = 0$, and

$$h(x) = \lambda_1 a_1(x) + \lambda_2 a_2(x), \tag{2.1}$$

where λ_i for $i = 1, 2$, is a real number, $|h|_{H^{1,\infty}_{\mathrm{atb}}(\mu)} = |\lambda_1| + |\lambda_2|$, a_i for $i = 1, 2$, is a bounded function supported on some cube $Q_i \subset R$ and it satisfies

$$\|a_i\|_{L^\infty(\mu)} \le \left[\mu(4Q_i)S_{Q_i,R}\right]^{-1}. \tag{2.2}$$

Write

$$\begin{aligned}
&\left\|\mathcal{M}_b(h)\right\|_{L^q(\mu)} \\
&\quad\lesssim \left(\int_{2R}\left|\mathcal{M}_b(h)(x)\right|^q d\mu(x)\right)^{1/q} + \left(\int_{\mathbb{R}^d\setminus(2R)}\left|\mathcal{M}_b(h)(x)\right|^q d\mu(x)\right)^{1/q} \\
&\quad\lesssim \left(\int_{2R}\left|\mathcal{M}_b(h)(x)\right|^q d\mu(x)\right)^{1/q} \\
&\qquad + \left\{\int_{\mathbb{R}^d\setminus(2R)}\left(\int_0^{|x-x_R|+2\ell(R)}\left|\int_{|x-y|\le t} K(x,y)\right.\right.\right. \\
&\qquad \times \left.\left.\left[b(x)-b(y)\right]h(y)\,d\mu(y)\right|^2 \frac{dt}{t^3}\right)^{q/2} d\mu(x)\Bigg\}^{1/q} \\
&\qquad + \left\{\int_{\mathbb{R}^d\setminus(2R)}\left(\int_{|x-x_R|+2\ell(R)}^{\infty}\left|\int_{|x-y|\le t} K(x,y)\left[b(x)-b(y)\right]h(y)\,d\mu(y)\right|^2 \frac{dt}{t^3}\right)^{q/2} d\mu(x)\right\}^{1/q} \\
&\quad= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

By (2.1), we have

$$\begin{aligned}
\mathrm{I} &\le |\lambda_1|\left(\int_{2R}\left|\mathcal{M}_b(a_1)(x)\right|^q d\mu(x)\right)^{1/q} + |\lambda_2|\left(\int_{2R}\left|\mathcal{M}_b(a_2)(x)\right|^q d\mu(x)\right)^{1/q} \\
&= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

To estimate I_1, we write

$$\begin{aligned}
\mathrm{I}_1 &\le |\lambda_1|\left(\int_{2Q_1}\left|\mathcal{M}_b(a_1)(x)\right|^q d\mu(x)\right)^{1/q} + |\lambda_1|\left(\int_{2R\setminus 2Q_1}\left|\mathcal{M}_b(a_1)(x)\right|^q d\mu(x)\right)^{1/q} \\
&= \mathrm{I}_{11} + \mathrm{I}_{12}.
\end{aligned}$$

Choose p_1 and q_1 such that $1 < p_1 < n/\beta$, $1 < q < q_1$ and $1/q_1 = 1/p_1 - \beta/n$. By the Hölder inequality, the fact that $S_{Q_1,R} \geq 1$ and the $(L^{p_1}(\mu), L^{q_1}(\mu))$-boundedness of $\mathcal{M}_b$ (Lemma 1.2), we have that

$$\begin{aligned}
\mathrm{I}_{11} &\leq |\lambda_1| \left[\int_{2Q_1} \left| \mathcal{M}_b(a_1)(x) \right|^{q_1} d\mu(x) \right]^{1/q_1} \mu(2Q_1)^{1/q-1/q_1} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} |\lambda_1| \|a_1\|_{L^{p_1}(\mu)} \mu(2Q_1)^{1/q-1/q_1} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} |\lambda_1| \|a_1\|_{L^\infty(\mu)} \mu(2Q_1)^{1/p_1+1/q-1/q_1} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} |\lambda_1|.
\end{aligned}$$

Denote $N_{2Q_1,2R}$ simply by N_1. Invoking the fact that $\|a_1\|_{L^\infty(\mu)} \leq [\mu(4Q_1) S_{Q_1,R}]^{-1}$, we thus get

$$\begin{aligned}
\mathrm{I}_{12} &\lesssim |\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \int_{2^{k+1}Q_1 \setminus 2^k Q_1} \left[\int_0^\infty \left| \int_{|x-y| \leq t} \frac{[b(x)-b(y)]}{|x-y|^{n-1}} a_1(y)\, d\mu(y) \right|^2 \frac{dt}{t^3} \right]^{q/2} d\mu(x) \right\}^{1/q} \\
&\lesssim |\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \int_{2^{k+1}Q_1 \setminus 2^k Q_1} \left[\int_{Q_1} \frac{|b(x)-b(y)|}{|x-y|^{n-1}} |a_1(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} d\mu(y) \right]^q d\mu(x) \right\}^{1/q} \\
&\lesssim |\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \int_{2^{k+1}Q_1 \setminus 2^k Q_1} \left[\int_{Q_1} \frac{|b(x)-b(y)|}{|x-y|^{n}} |a_1(y)|\, d\mu(y) \right]^q d\mu(x) \right\}^{1/q} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} |\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell(2^k Q_1)^{q(\beta-n)} \int_{2^{k+1}Q_1 \setminus 2^k Q_1} \left[\int_{Q_1} |a_1(y)|\, d\mu(y) \right]^q d\mu(x) \right\}^{1/q} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} |\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell(2^k Q_1)^{q(\beta-n)} \mu(2^{(k+1)} Q_1) \|a_1\|_{L^\infty(\mu)}^q \mu(Q_1)^q \right\}^{1/q} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} |\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell(2^k Q_1)^{q(\beta-n)} \mu(4Q_1)^{-q} S_{Q_1,R}^{-q} \mu(2^{(k+1)} Q_1) \mu(Q_1)^q \right\}^{1/q} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} |\lambda_1| \left\{ S_{Q_1,R}^{-q} \sum_{k=2}^{N_1+1} \frac{\mu(2^k Q_1)}{\ell(2^k Q_1)^n} \right\}^{1/q} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} |\lambda_1|,
\end{aligned}$$

here we have used the fact that

$$\sum_{k=2}^{N_1+1} \frac{\mu(2^k Q)}{l(2^k Q)^n} \leq C S_{Q_1,R};$$

see [1, 27] for details.

The estimates for I_{11} and I_{12} give the desired estimate for I_1. A similar argument tells us that

$$\mathrm{I}_2 \lesssim \|b\|_{\mathrm{Lip}_\beta} |\lambda_2|.$$

Combining the estimates for I_1 and I_2 yields the desired estimate for I.

For $i = 1, 2$, $y \in Q_i \subset R$, $x \in \mathbb{R}^d \setminus (2R)$, we have $|x-y| \sim |x-x_R| \sim |x-x_R| + 2\ell(R)$. By the Minkowski inequality, we get

$$\begin{aligned}
\mathrm{II} &\lesssim \left\{\int_{\mathbb{R}^d\setminus(2R)}\left[\int_{\mathbb{R}^d}\left(\int_{|x-y|}^{|x-x_R|+2\ell(R)}\frac{dt}{t^3}\right)^{1/2}\frac{|h(y)|}{|x-y|^{n-1}}\big|b(x)-b(y)\big|\,d\mu(y)\right]^q d\mu(x)\right\}^{1/q}\\
&\lesssim \int_R\left\{\int_{\mathbb{R}^d\setminus(2R)}\left[\left(\frac{1}{(|x-x_R|+2\ell(R))^2}-\frac{1}{|x-y|^2}\right)^{1/2}\right.\right.\\
&\quad\left.\left.\times\frac{|h(y)|}{|x-y|^{n-1}}\big|b(x)-b(y)\big|\right]^q d\mu(x)\right\}^{1/q}d\mu(y)\\
&\lesssim \int_R\left\{\int_{\mathbb{R}^d\setminus(2R)}\left(\frac{\ell(R)^{1/2}}{|x-y|^{3/2}}\frac{|h(y)|}{|x-y|^{n-1}}\big|b(x)-b(y)\big|\right)^q d\mu(x)\right\}^{1/q}d\mu(y)\\
&\lesssim \int_R\left\{\sum_{k=1}^{\infty}\int_{2^{(k+1)}R\setminus 2^kR}\left(\frac{\ell(R)^{1/2}}{|x-y|^{n-\beta+1/2}}\|b\|_{\mathrm{Lip}_\beta}\right)^q d\mu(x)\right\}^{1/q}\big|h(y)\big|\,d\mu(y)\\
&\lesssim \|b\|_{\mathrm{Lip}_\beta}\left(\sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^1(\mu)}\right)\left\{\sum_{k=1}^{\infty}\ell(R)^{1/2}\ell\big(2^kR\big)^{-n+\beta-1/2}\mu\big(2^{k+1}R\big)^{1/q}\right\}\\
&\lesssim \|b\|_{\mathrm{Lip}_\beta}\left(\sum_{j=1}^{2}|\lambda_j|\right).
\end{aligned}$$

For any $y \in R$, we have $t \geq |x-x_R| + 2\ell(R) \geq |x-x_R| + |y-x_R| \geq |x-y|$. It follows that

$$\begin{aligned}
\mathrm{III} &\leq \left\{\int_{\mathbb{R}^d\setminus(2R)}\left|\int_R K(x,y)\big[b(x)-b(y)\big]h(y)\,d\mu(y)\left(\int_{|x-x_R|+2\ell(R)}^{\infty}\frac{dt}{t^3}\right)^{1/2}\right|^q d\mu(x)\right\}^{1/q}\\
&\lesssim \left\{\int_{\mathbb{R}^d\setminus(2R)}\left|\int_R K(x,y)\big[b(x)-b(y)\big]h(y)\,d\mu(y)\frac{1}{|x-x_R|+2\ell(R)}\right|^q d\mu(x)\right\}^{1/q}\\
&\lesssim \left\{\int_{\mathbb{R}^d\setminus(2R)}\left|\int_R \frac{K(x,y)h(y)}{|x-x_R|+2\ell(R)}\big[b(x)-m_R(b)\big]\,d\mu(y)\right|^q d\mu(x)\right\}^{1/q}\\
&\quad+\left\{\int_{\mathbb{R}^d\setminus(2R)}\left|\int_R \frac{K(x,y)h(y)}{|x-x_R|+2\ell(R)}\big[m_R(b)-b(y)\big]\,d\mu(y)\right|^q d\mu(x)\right\}^{1/q}\\
&= \mathrm{III}_1 + \mathrm{III}_2.
\end{aligned}$$

For III_1, by the Minkowski inequality, we have

$$\begin{aligned}
\mathrm{III}_1 &= \left\{\int_{\mathbb{R}^d\setminus(2R)}\left|\big[b(x)-m_R(b)\big]\int_R\frac{K(x,y)-K(x,x_R)}{|x-x_R|+2\ell(R)}h(y)\,d\mu(y)\right|^q d\mu(x)\right\}^{1/q}\\
&\lesssim \int_R\sum_{k=1}^{m}\left(\int_{2^{k+1}R\setminus 2^kR}\left[\|b\|_{\mathrm{Lip}_\beta}|x-x_R|^\beta\frac{|K(x,y)-K(x,x_R)|}{|x-y|}\right]^q d\mu(x)\right)^{1/q}\big|h(y)\big|\,d\mu(y)\\
&\lesssim \|b\|_{\mathrm{Lip}_\beta}\int_R\big|h(y)\big|\sum_{k=1}^{m}\left(\int_{2^{k+1}R\setminus 2^kR}\left[\ell\big(2^kR\big)^\beta\frac{|K(x,y)-K(x,x_R)|}{|x-y|}\right]^q d\mu(x)\right)^{1/q}d\mu(y)\\
&\lesssim \|b\|_{\mathrm{Lip}_\beta}\int_R\big|h(y)\big|\sum_{k=1}^{m}2^{k(\beta-\varepsilon-n+n/q)}\ell(R)^{\beta-n+n/q}2^{k\varepsilon}\big[2^k\ell(R)\big]^n
\end{aligned}$$

$$\times \left(\frac{1}{[2^k\ell(R)]^n} \int_{2^k\ell(R)<|x-y|\leq 2^{k+1}\ell(R)} \left[\frac{|K(x,y)-K(x,x_R)|}{|x-y|} \right]^q d\mu(x) \right)^{1/q} d\mu(y)$$

$$\lesssim \|b\|_{\mathrm{Lip}_\beta} \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^1(\mu)}$$

$$\lesssim \|b\|_{\mathrm{Lip}_\beta} \sum_{j=1}^{2} |\lambda_j|,$$

here we used the fact that $1/q = 1 - \beta/n$ and $0 < \varepsilon \leq 1$.

We now turn to estimate III_2. Note that for any $y \in R$, $x \in \mathbb{R}^d \setminus 2R$, we have $|x-y| \sim |x-x_R| + 2\ell(R)$, so by the Minkowski inequality,

$$\begin{aligned}
\mathrm{III}_2 &\leq \int_R \sum_{k=1}^{\infty} \left(\int_{2^{k+1}R\setminus 2^kR} \left[\frac{|K(x,y)|}{|x-y|} \big| m_R(b) - b(y) \big| \right]^q d\mu(x) \right)^{1/q} \big| h(y) \big| \, d\mu(y) \\
&\lesssim \int_R \sum_{k=1}^{\infty} \left(\int_{2^{k+1}R\setminus 2^kR} \left[\frac{|m_R(b) - b(y)|}{|x-y|^n} \right]^q d\mu(x) \right)^{1/q} \big| h(y) \big| \, d\mu(y) \\
&\lesssim \int_R \sum_{k=1}^{\infty} \|b\|_{\mathrm{Lip}_\beta} \ell(R)^\beta \ell\big(2^kR\big)^{-n} \mu\big(2^{k+1}R\big)^{1/q} \sum_{j=1}^{2} |\lambda_j| \big| a_j(y) \big| \, d\mu(y) \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^1(\mu)} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} \sum_{j=1}^{2} |\lambda_j|.
\end{aligned}$$

Then

$$\mathrm{III} \lesssim \|b\|_{\mathrm{Lip}_\beta} \sum_{j=1}^{2} |\lambda_j|.$$

Combining the estimates for I, II and III yields that

$$\big\| \mathcal{M}_b(h) \big\|_{L^q(\mu)} \leq C|h|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)},$$

and this is the result of Theorem 2.1. □

3 Boundedness of $\mathcal{M}_b$ in RBMO(μ) space

In this section, we investigate the boundedness for the commutator $\mathcal{M}_b$ as in (1.5) in the space RBMO(μ) for $f \in \mathcal{M}^p_q(\mu)$ and $f \in L^{n/\beta}(\mu)$, respectively.

Firstly, we recall the definition of the Morrey space with nondoubling measure denoted by $\mathcal{M}^p_q(\mu)$, which was introduced by Sawano and Tanaka in [34–36].

Definition 3.1 Let $k > 1$ and $1 \leq q \leq p < \infty$. We define the Morrey space $\mathcal{M}^p_q(\mu)$ as

$$\mathcal{M}^p_q(\mu) := \big\{ f \in L^q_{\mathrm{loc}}(\mu) \,|\, \|f\|_{\mathcal{M}^p_q(\mu)} < \infty \big\},$$

where the norm $\|f\|_{\mathcal{M}_q^p(\mu)}$ is given by

$$\|f\|_{\mathcal{M}_q^p(\mu)} := \sup_Q \mu(kQ)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q \, d\mu \right)^{\frac{1}{q}}.$$

We should note that the parameter $k > 1$ appearing in the definition does not affect the definition of the space $\mathcal{M}_q^p(\mu)$, and the space $\mathcal{M}_q^p(\mu)$ is a Banach space with its norm; see [34]. By using the Hölder inequality to (1.5), it is easy to see that for all $1 \le q_2 \le q_1 \le p$, we have

$$L^p(\mu) = \mathcal{M}_p^p(\mu) \subset \mathcal{M}_{q_1}^p(\mu) \subset \mathcal{M}_{q_2}^p(\mu).$$

Theorem 3.1 *Let $b \in \mathrm{Lip}_\beta(\mu)$, $0 < \beta \le 1$, $1 \le q < p = \frac{n}{\beta}$. Suppose that K satisfies* (1.2) *and $\mathcal{H}^{p'}$ condition, $\mathcal{M}$ is bounded on $L^2(\mu)$ and $\mathcal{M}_b$ is defined as in* (1.5). *Then there exists a positive constant C such that for all $f \in \mathcal{M}_q^p(\mu)$,*

$$\|\mathcal{M}_b(f)\|_{\mathrm{RBMO}(\mu)} \le C\|b\|_{\mathrm{Lip}_\beta} \|f\|_{\mathcal{M}_q^p(\mu)}.$$

Theorem 3.2 *Let $b \in \mathrm{Lip}_\beta(\mu)$, $0 < \beta \le 1$ and $p = n/\beta$. Suppose that K satisfies* (1.2) *and $\mathcal{H}^{n/(n-\beta)}$ condition. If $\mathcal{M}$ is bounded on $L^2(\mu)$ and $\mathcal{M}_b$ is defined as in* (1.5), *then there is a constant $C > 0$ such that for all bounded functions f with compact support,*

$$\|\mathcal{M}_b(f)\|_{\mathrm{RBMO}(\mu)} \le C\|b\|_{\mathrm{Lip}_\beta} \|f\|_{L^{n/\beta}(\mu)}.$$

Theorem 3.3 *Let $b \in \mathrm{Lip}_\beta(\mu)$, $0 < \beta \le 1$, $1 \le q < p$ and $p > n/\beta$. Suppose that K satisfies* (1.2) *and $\mathcal{H}^{p'}$ condition, $\mathcal{M}$ is bounded on $L^2(\mu)$ and $\mathcal{M}_b$ is defined as in* (1.5). *Then there exists a positive constant C such that for all $f \in \mathcal{M}_q^p(\mu)$,*

$$\|\mathcal{M}_b(f)\|_{\mathrm{Lip}_{(\beta-\frac{n}{p})}} \le C\|b\|_{\mathrm{Lip}_\beta} \|f\|_{\mathcal{M}_q^p(\mu)}.$$

Remark 3.1 By the Minkowski inequality and the kernel condition, we get that

$$\begin{aligned}
\mathcal{M}_b(f)(x) &= \left(\int_0^\infty \left| \int_{|x-y|\le t} \bigl[b(x)-b(y)\bigr] K(x,y) f(y)\, d\mu(y) \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&= \left(\int_0^\infty \left| \frac{1}{t} \int_{|x-y|\le t} \bigl[b(x)-b(y)\bigr] K(x,y) f(y)\, d\mu(y) \right|^2 \frac{dt}{t} \right)^{1/2} \\
&= \int_{\mathbb{R}^d} \bigl| K(x,y)\bigl[b(x)-b(y)\bigr] f(y) \bigr| \left(\int_{|x-y|\le t}^\infty \frac{dt}{t^3} \right)^{1/2} d\mu(y) \\
&\lesssim \int_{\mathbb{R}^d} \frac{|[b(x)-b(y)] f(y)|}{|x-y|^{n-1}} |x-y|^{-1}\, d\mu(y) \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^{n-\beta}}\, d\mu(y) \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} I_\beta\bigl(|f|\bigr)(x),
\end{aligned}$$

where I_β is the fractional integral operator. Then $\mathcal{M}_b(f) \in L^1_{\mathrm{loc}}(\mu)$.

Remark 3.2 Theorem 3.2 can be deduced as a conclusion of Theorem 3.1 in the case of $p = q = \frac{n}{\beta}$.

Remark 3.3 Applying Lemma 1.1, a slight change in the proof of Theorem 3.1 actually shows Theorem 3.3 and we leave the details to the reader.

Proof of Theorem 3.1 For any cubes Q and R in $\mathbb{R}^d$ such that $Q \subset R$ satisfies $\ell(R) \le 2\ell(Q)$, let

$$a_Q = m_Q\bigl[\mathcal{M}_b(f\chi_{\mathbb{R}^d\setminus\frac{3}{2}Q})\bigr]$$

and

$$a_R = m_R\bigl[\mathcal{M}_b(f\chi_{\mathbb{R}^d\setminus\frac{3}{2}R})\bigr].$$

It is easy to see that a_Q and a_R are real numbers. By Lemma 1.1, we need to show that for some fixed $r > q$ there exists a constant $C > 0$ such that

$$\left(\frac{1}{\mu(2Q)}\int_Q\bigl|\mathcal{M}_b(f)(x) - a_Q\bigr|^r\,d\mu(x)\right)^{1/r} \lesssim \|b\|_{\mathrm{Lip}_\beta}\|f\|_{\mathcal{M}^p_q(\mu)} \tag{3.1}$$

and

$$|a_Q - a_R| \lesssim \|b\|_{\mathrm{Lip}_\beta}\|f\|_{\mathcal{M}^p_q(\mu)}. \tag{3.2}$$

Let us first prove estimate (3.1). For a fixed cube Q and $x \in Q$, decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\frac{3}{2}Q}$ and $f_2 = f - f_1$. Write that

$$\begin{aligned}
&\frac{1}{\mu(2Q)}\int_Q\bigl|\mathcal{M}_b(f)(x) - a_Q\bigr|^r\,d\mu(x)\\
&\quad\le \frac{1}{\mu(2Q)}\int_Q\bigl|\mathcal{M}_b(f_1)(x)\bigr|^r\,d\mu(x) + \frac{1}{\mu(2Q)}\int_Q\bigl|\mathcal{M}_b(f_2)(x) - a_Q\bigr|^r\,d\mu(x)\\
&\quad= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

For $1/r = 1/q - \beta/n$ and $p = n/\beta$, it follows that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \frac{1}{\mu(2Q)}\left[\int_Q\bigl|\mathcal{M}_b(f_1)(x)\bigr|^r\,d\mu(x)\right]^{1/r}\\
&\lesssim \frac{1}{\mu(2Q)}\|b\|^r_{\mathrm{Lip}_\beta}\left(\int_{\frac{3}{2}Q}\bigl|f(x)^q\bigr|\,d\mu(x)\right)^{r/q}\\
&\lesssim \frac{1}{\mu(2Q)}\|b\|^r_{\mathrm{Lip}_\beta}\left\{\left(\mu(2Q)^{1/p-1/q}\int_{\frac{3}{2}Q}\bigl|f(x)\bigr|^q\,d\mu(x)\right)^{1/q}\right\}^r\mu(2Q)^{(1/p-1/q)r}\\
&\lesssim \|b\|^r_{\mathrm{Lip}_\beta}\|f\|^r_{\mathcal{M}^p_q(\mu)}\mu(2Q)^{(1/q-1/p)r-1}\\
&\lesssim \|b\|^r_{\mathrm{Lip}_\beta}\|f\|^r_{\mathcal{M}^p_q(\mu)}\mu(2Q)^{(1/r+\beta/n-1/p)r-1}\\
&\lesssim \|b\|^r_{\mathrm{Lip}_\beta}\|f\|^r_{\mathcal{M}^p_q(\mu)}.
\end{aligned}$$

In order to estimate the term I_2, set

$$\mathrm{D}_1(x,y)=\left(\int_0^\infty\left[\int_{|x-z|\le t<|y-z|}\left|K(x,z)\right|\left|b(z)-m_Q(b)\right|\left|f_2(z)\right|d\mu(z)\right]^2\frac{dt}{t^3}\right)^{1/2},$$

$$\mathrm{D}_2(x,y)=\left(\int_0^\infty\left[\int_{|y-z|\le t<|x-z|}\left|K(y,z)\right|\left|b(z)-m_Q(b)\right|\left|f_2(z)\right|d\mu(z)\right]^2\frac{dt}{t^3}\right)^{1/2}$$

and

$$\mathrm{D}_3(x,y)=\left(\int_0^\infty\left[\int_{\substack{|y-z|\le t\\|x-z|\le t}}\left|K(x,z)-K(y,z)\right|\left|b(z)-m_Q(b)\right|\left|f_2(z)\right|d\mu(z)\right]^2\frac{dt}{t^3}\right)^{1/2}.$$

It is easy to get that for any $x,y\in Q$,

$$\begin{aligned}
&\left|\mathcal{M}_b(f_2)(x)-\mathcal{M}_b(f_2)(y)\right|\\
&\quad=\left|\left(\int_0^\infty\left|\int_{|x-z|\le t}\left[b(x)-b(z)\right]K(x,z)f_2(z)\,d\mu(z)\right|^2\frac{dt}{t^3}\right)^{1/2}\right.\\
&\qquad\left.-\left(\int_0^\infty\left|\int_{|y-z|\le t}\left[b(y)-b(z)\right]K(y,z)f_2(z)\,d\mu(z)\right|^2\frac{dt}{t^3}\right)^{1/2}\right|\\
&\quad\lesssim\sum_{j=1}^3\mathrm{D}_j(x,y).
\end{aligned}$$

For $\mathrm{D}_1(x,y)$, since $x,y\in Q$, $z\in\frac{3}{2}Q$, we thus get

$$\begin{aligned}
\mathrm{D}_1&\le\left(\int_0^\infty\left[\int_{|x-z|\le t<|y-z|}\frac{|b(z)-m_Q(b)|}{|x-z|^{n-1}}\left|f_2(z)\right|d\mu(z)\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\lesssim\int_{|x-z|<|y-z|}\frac{|b(z)-m_Q(b)|}{|x-z|^{n-1}}\left|f_2(z)\right|\left[\int_{|y-z|}^{|x-z|}\frac{dt}{t^3}\right]^{1/2}d\mu(z)\\
&\lesssim\int_{|x-z|<|y-z|}\frac{|b(z)-m_Q(b)|}{|x-z|^{n-1}}\left|f_2(z)\right|\frac{\ell(Q)^{1/2}}{|x-z|^{3/2}}\,d\mu(z)\\
&\lesssim\ell(Q)^{1/2}\int_{\mathbb{R}^d\setminus\frac{3}{2}Q}\frac{|b(z)-m_Q(b)|}{|x-z|^{n+1/2}}\left|f(z)\right|d\mu(z)\\
&\lesssim\ell(Q)^{1/2}\sum_{k=1}^\infty\int_{\frac{3}{2}2^kQ\setminus\frac{3}{2}2^{k-1}Q}\frac{|b(z)-m_Q(b)|}{|x-z|^{n+1/2}}\left|f(z)\right|d\mu(z)\\
&\lesssim\ell(Q)^{1/2}\sum_{k=1}^\infty\frac{1}{\ell(\frac{3}{2}2^kQ)^{n+1/2}}\int_{\frac{3}{2}2^kQ}\left|b(z)-m_Q(b)\right|\left|f(z)\right|d\mu(z)\\
&\lesssim\sum_{k=1}^\infty2^{-k/2}\frac{1}{\ell(2^kQ)^n}\left(\int_{\frac{3}{2}2^kQ}\left|b(z)-m_Q(b)\right|^{q'}d\mu(z)\right)^{1/q'}\left(\int_{\frac{3}{2}2^kQ}\left|f(z)\right|^q d\mu(z)\right)^{1/q}\\
&\lesssim\|b\|_{\mathrm{Lip}_\beta}\|f\|_{\mathcal{M}_q^p(\mu)}\sum_{k=1}^\infty k2^{-k/2}\ell\left(2^kQ\right)^{\beta-n}\mu\left(\frac{3}{2}2^kQ\right)^{1-1/q}\mu\left(2^{k+2}Q\right)^{1/q-1/p}
\end{aligned}$$

$$\lesssim \|b\|_{\mathrm{Lip}_\beta} \|f\|_{\mathcal{M}_q^p(\mu)} \sum_{k=1}^{\infty} k2^{-k/2}$$
$$\lesssim \|b\|_{\mathrm{Lip}_\beta} \|f\|_{\mathcal{M}_q^p(\mu)},$$

here we used the Minkowski inequality, $\beta = n/p$, (1.10) of Lemma 1.1 and the fact that

$$\left|b(z) - m_Q(b)\right| \lesssim \ell\left(2^k Q\right)^{\beta} \|b\|_{\mathrm{Lip}_\beta} \quad \text{for } z \in \mathbb{R}^d \setminus \frac{3}{2}Q.$$

By a similar argument, it follows that

$$\mathrm{D}_2 \lesssim \|b\|_{\mathrm{Lip}_\beta} \|f\|_{\mathcal{M}_q^p(\mu)}.$$

Finally, by the condition $\mathcal{H}^{p'}$, which the kernel K satisfies, and the fact that $\beta = n/p$, applying the Minkowski inequality, we have

$$\begin{aligned}
\mathrm{D}_3(x,y) &= \left(\int_0^\infty \left[\int_{\substack{|y-z|\le t\\|x-z|\le t}} \left|K(x,z) - K(y,z)\right|\left|b(z) - m_Q(b)\right|\left|f_2(z)\right| d\mu(z)\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\lesssim \int_{\mathbb{R}^d} \left|K(x,z) - K(y,z)\right|\left|b(z) - m_Q(b)\right|\left|f_2(z)\right| \left[\int_{\substack{|y-z|\le t\\|x-z|\le t}} \frac{dt}{t^3}\right]^{1/2} d\mu(z) \\
&\lesssim \sum_{k=1}^{\infty} \int_{\frac{3}{2}2^k Q \setminus \frac{3}{2}2^{k-1}Q} \left|K(x,z) - K(y,z)\right|\left|b(z) - m_Q(b)\right| \frac{|f(z)|}{|y-z|} d\mu(z) \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^{\infty} \ell\left(2^k Q\right)^{\beta} \int_{\frac{3}{2}2^k Q \setminus \frac{3}{2}2^{k-1}Q} \left|K(x,z) - K(y,z)\right| \frac{|f(z)|}{|y-z|} d\mu(z) \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} \|f\|_{\mathcal{M}_q^p(\mu)} \sum_{k=1}^{\infty} \ell\left(2^k Q\right)^{\beta - n/p} \ell\left(2^k Q\right)^{n/q} \\
&\quad \times \left(\int_{\frac{3}{2}2^k Q \setminus \frac{3}{2}2^{k-1}Q} \left[\left|K(x,z) - K(y,z)\right| \frac{1}{|y-z|}\right]^{q'} d\mu(z)\right)^{1/q'} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} \|f\|_{\mathcal{M}_q^p(\mu)} \sum_{k=1}^{\infty} \ell\left(2^k Q\right)^{n} \\
&\quad \times \left(\frac{1}{\ell(2^k Q)^n} \int_{\frac{3}{2}2^k Q \setminus \frac{3}{2}2^{k-1}Q} \left[\left|K(x,z) - K(y,z)\right| \frac{1}{|y-z|}\right]^{q'} d\mu(z)\right)^{1/q'} \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} \|f\|_{\mathcal{M}_q^p(\mu)}.
\end{aligned}$$

Combining these estimates, we conclude that

$$\mathrm{I}_2 \lesssim \|b\|_{\mathrm{Lip}_\beta}^r \|f\|_{\mathcal{M}_q^p(\mu)}^r,$$

and so estimate (3.1) is proved. □

We proceed to show (3.2). For any cubes $Q \subset R$ with $x \in Q$, where Q is arbitrary and R is a doubling cube with $\ell(R) \leq \ell(Q)$, denote $N_{Q,R} + 1$ simply by N. Write

$$\begin{aligned}|a_Q - a_R| &\leq \bigl|m_R\bigl[\mathcal{M}_b(f\chi_{\mathbb{R}^d\setminus 2^N Q})\bigr] - m_Q\bigl[\mathcal{M}_b(f\chi_{\mathbb{R}^d\setminus 2^N Q})\bigr]\bigr| \\ &\quad + \bigl|m_Q\bigl[\mathcal{M}_b(f\chi_{2^N Q\setminus \frac{3}{2}Q})\bigr]\bigr| + \bigl|m_R\bigl[\mathcal{M}_b(f\chi_{2^N Q\setminus \frac{3}{2}R})\bigr]\bigr| \\ &= \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3.\end{aligned}$$

As in the estimate for the term I_2, we have

$$\mathrm{E}_1 \lesssim \|b\|_{\mathrm{Lip}_\beta}\|f\|_{\mathcal{M}_q^p(\mu)}.$$

We conclude from $y \in R$, $z \in 2^N Q \setminus \frac{3}{2}Q$ that

$$\begin{aligned}\mathcal{M}_b(f\chi_{2^N Q\setminus \frac{3}{2}R})(y) &\lesssim \int_{2^N Q\setminus \frac{3}{2}R} \bigl|K(y,z)\bigl(b(y)-b(z)\bigr)f(z)\bigr| \biggl(\int_{|y-z|}^\infty \frac{dt}{t^3}\biggr)^{1/2} d\mu(z) \\ &\lesssim \int_{2^N Q\setminus \frac{3}{2}R} \frac{|b(y)-b(z)|}{|y-z|^n}\bigl|f(z)\bigr|\,d\mu(z) \\ &\lesssim \|b\|_{\mathrm{Lip}_\beta} \int_{2^N Q\setminus \frac{3}{2}R} \frac{|f(z)|}{|y-z|^{n-\beta}}\,d\mu(z) \\ &\lesssim \|b\|_{\mathrm{Lip}_\beta}\ell(R)^{\beta-n}\biggl(\int_{2^N Q}\bigl|f(z)\bigr|^q\,d\mu(z)\biggr)^{1/q} \mu\bigl(2^N Q\bigr)^{1/q'} \\ &\lesssim \|b\|_{\mathrm{Lip}_\beta}\|f\|_{\mathcal{M}_q^p(\mu)}\ell\bigl(2^N Q\bigr)^{\beta-n+n-n/q+n/q-n/p} \\ &\lesssim \|b\|_{\mathrm{Lip}_\beta}\|f\|_{\mathcal{M}_q^p(\mu)}.\end{aligned}$$

Taking mean over $y \in R$, we obtain

$$\mathrm{E}_3 \lesssim \|b\|_{\mathrm{Lip}_\beta}\|f\|_{\mathcal{M}_q^p(\mu)}.$$

Analysis similar to that in the estimate for E_3 shows that

$$\mathrm{E}_2 \lesssim \|b\|_{\mathrm{Lip}_\beta}\|f\|_{\mathcal{M}_q^p(\mu)}.$$

Finally, we get (3.2) and this is precisely the assertion of Theorem 3.1.

Competing interests
The author declares that they have no competing interests.

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