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Estimates of singular integrals and multilinear commutators in weighted Morrey spaces

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Abstract

Suppose T is a singular integral operator whose kernel is a variable kernel with mixed homogeneity; the purpose of this paper is to study the continuity of the operator in weighted Morrey spaces $L^{p,\kappa}(\omega)$, $1 \le p < \infty$, $0 < \kappa < 1$. A special attention is paid also to the multilinear commutator of this operator with BMO function.

MSC: 42B20; 42B35

Keywords: mixed homogeneity; multilinear commutators; weighted Morrey spaces; BMO; Orlicz maximal operator

1 Introduction

Let $K(x,\xi):\mathbb{R}^n \times \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ be a variable kernel. The singular integral operator is defined by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x,x-y)f(y)\,dy \tag{1.1}$$

and its multilinear commutator with the BMO function

$$[\vec{b},T]f(x) = \int_{\mathbb{R}^n} \prod_{i=1}^{N} \bigl(b_i(x)-b_i(y)\bigr)K(x,x-y)f(y)\,dy, \tag{1.2}$$

where $\vec{b} = (b_1,\dots,b_n)$, $b_i \in BMO$, $1 \le i \le N$. The variable kernel $K(x,\xi)$ depends on some parameter x and possesses 'good' properties with respect to the second variable ξ, which was firstly introduced by Fabes and Rievière in [1]. They generalized the classical Calderón and Zygmund kernel and the parabolic kernel studied by Jones in [2]. By introducing a new metric ρ, Fabes and Rievière studied (1.1) in $L^p(\mathbb{R}^n)$, where $\mathbb{R}^n$ was endowed with the topology induced by ρ and defined by ellipsoids.

By using this metric ρ, Softova in [3] obtained that the integral operator (1.1) and commutator were continuous in generalized Morrey spaces $L^{p,\omega}(\mathbb{R}^n)$, $1 < p < \infty$, ω satisfying suitable conditions.

The multilinear commutator was introduced by Pérez and González [4] who proved the weighted Lebesgue estimates. Xu in [5] also showed that the multilinear commutators (1.2) were continuous in $L^{p,\omega}(\mathbb{R}^n)$, $1 < p < \infty$.

The weighted Morrey spaces $L^{p,\kappa}(w)$ were introduced by Komori and Shirai [6]. Moreover, they showed some classical integral operators and corresponding commutators were

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bounded in weighted Morrey spaces. Recently, Wang [7–9] obtained that some other kind of integral operators (*e.g.*, Bochner-Riesz operator, Marcinkiewicz operators *etc.*) and commutators were also bounded in weighted Morrey spaces. He Sha [10] showed that multilinear operators were bounded on weighted Morrey spaces with the symbol of $b \in \mathrm{Lip}(\beta)$. The main purpose of this paper is to discuss the continuity of the singular integral operator whose kernel is a variable kernel with mixed homogeneity and its multilinear commutator in the weighted Morrey spaces $L^{p,\kappa}(\omega)$, $1 < p < \infty$, $0 < \kappa < 1$, where the weight function ω is A_p weight. Furthermore, we shall give the weighted weak type estimate of theses operators in the weighted Morrey spaces $L^{1,\kappa}(\omega)$, $0 < \kappa < 1$. Our main results are stated as follows.

Theorem 1.1 *Let $1 < p < \infty$, $0 < \kappa < 1$. If $w \in A_p$, then there exists a constant $C > 0$ such that*

$$\|Tf\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

When $p = 1$, for any $\lambda > 0$ and ellipsoid $\mathcal{E}$, there exists a constant $C > 0$ such that

$$\lambda w\bigl(\bigl\{x \in \mathcal{E} : \bigl|Tf(x)\bigr| > \lambda\bigr\}\bigr) \leq C\|f\|_{L^{1,\kappa}(w)}.$$

If $K(x,\xi)$ is a constant kernel and a metric ρ is Euclidean one, this result is just Theorem 3.3 in [6].

Theorem 1.2 *Let $1 < p < \infty$, $0 < \kappa < 1$. If $b_i \in BMO(\mathbb{R}^n)$, $1 \leq i \leq N$, $w \in A_p$, then there exists a constant $C > 0$ such that*

$$\bigl\|[\vec{b}, T]f\bigr\|_{L^{p,\kappa}(w)} \leq C\|\vec{b}\|\|f\|_{L^{p,\kappa}(w)},$$

where $\|\vec{b}\| = \prod_{i=1}^{N}\|b_i\|_*$. When $p = 1$, for any $\lambda > 0$ and ellipsoid $\mathcal{E}$, then there exists a constant $C > 0$ such that

$$\lambda w\bigl(\bigl\{x \in \mathcal{E} : \bigl|[\vec{b}, T]f(x)\bigr| > \lambda\bigr\}\bigr) \leq C\|\vec{b}\|\|f\|_{L^{\Phi,\kappa}(w)},$$

where $\Phi(t) = t\log^N(e+t)$ and $\|f\|_{L^{\Phi,\kappa}(w)} = \|\Phi(|f|)\|_{L^{1,\kappa}(w)}$.

In what follows, we denote by C positive constants which are independent of the main parameters but may vary from line to line.

2 Some notations and lemmas

In this section, we introduce some basic definitions and lemmas needed for the proof of the main results.

Let $\alpha_1, \ldots, \alpha_n$ be real numbers, $\alpha_i \geq 1$ and $|\alpha| = \sum_{i=1}^{n}\alpha_i$. Following Fabes and Rivière [1], there exists a function ρ such that $\rho(x - y)$ defines a distance between any two points $x, y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$ endowed with the metric ρ results in a homogeneous metric space [1, 3]. The balls with respect to $\rho(x)$ centered at the origin and of radius r are the ellipsoids

$$\mathcal{E}_r(0) = \left\{x \in \mathbb{R}^n : \frac{x_1^2}{r^{2\alpha_1}} + \cdots + \frac{x_n^2}{r^{2\alpha_n}} < 1\right\}$$

with Lebesgue measure $|\mathcal{E}_r| = C(n)r^{|\alpha|}$. It is easy to see that the unit sphere with respect to this metric coincides with the unit sphere Σ_n with respect to the Euclidean one.

Definition 2.1 The function $K(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is called a variable kernel with mixed homogeneity if

(i) for every fixed x, the function $K(x,\cdot)$ is a constant kernel satisfying
  (1) $K(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
  (2) for any $\mu > 0$, $\alpha_i \geq 1$, $|\alpha| = \sum_{i=1}^n \alpha_i$

$$K(x, \mu^{\alpha_1}\xi_1, \ldots, \mu^{\alpha_n}\xi_n) = \mu^{-|\alpha|}K(x,\xi);$$

  (3) $\int_{\Sigma_n} K(x,\xi)\,d\sigma_\xi = 0$ and $\int_{\Sigma_n} |K(x,\xi)|\,d\sigma_\xi < \infty$;

(ii) for every multiindex β, $\sup_{\xi\in\Sigma_n} |D_\xi^\beta K(x,\xi)| \leq C(\beta)$ independent of x.

In the case $\alpha_i = 1$, $1 \leq i \leq n$, Definition 2.1 gives rise to the classical Calderón-Zygmund kernel. On the other hand, when $\alpha_i = 1$, $1 \leq i \leq n-1$ and $\alpha_n \geq 1$, we obtain the kernel studied by Jones in [2] and discussed in [1].

Definition 2.2 Let $1 \leq p < \infty$, $0 < \kappa < 1$ and w be a weight function. Then a weighted Morrey space is defined by

$$L^{p,\kappa}(w) := \{f \in L^1_{\mathrm{loc}}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_{\mathcal{E}} \left(\frac{1}{w(\mathcal{E})^\kappa} \int_{\mathcal{E}} |f(x)|^p w(x)\,dx \right)^{1/p},$$

the supremum is taken over all ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$.

Definition 2.3 For the function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and any ellipsoid $\mathcal{E}$, b is called a *BMO* function if

$$\|b\|_* = \sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(x) - b_{\mathcal{E}}|\,dx < \infty,$$

where $b_{\mathcal{E}} = \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} b(y)\,dy$. The quantity $\|b\|_*$ is a norm in the *BMO* modulo constant function under which *BMO* results in a Banach space (see [11]).

Definition 2.4 Let $1 < p < \infty$. For any locally integrable function w and ellipsoid $\mathcal{E}$, if

$$\left(\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} w(x)\,dx \right) \left(\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} w(x)^{\frac{1}{1-p}}\,dx \right)^{p-1} < \infty$$

holds, then w belongs to the Muckenhoupt class A_p. We denote $A_\infty = \bigcup_{1<p<\infty} A_p$.

When $p = 1$, $w \in A_1$ if there exists $C > 1$ such that

$$Mw(x) \leq Cw(x)$$

for almost every $x \in \mathbb{R}^n$.

Remark 2.5 Given a weight function $w \in A_p$, $1 \le p \le \infty$, it also satisfies the doubling condition Δ_2: for any ellipsoid $\mathcal{E}$, there exists a constant $C > 0$ such that $w(2\mathcal{E}) \le Cw(\mathcal{E})$.

In fact, $w \in \Delta_2$, we have the following inequality.

Lemma A [6, 12] *Suppose $w \in \Delta_2$, there exists a constant $D > 1$ such that*

$$w(2\mathcal{E}) \ge Dw(\mathcal{E})$$

for any ellipsoid $\mathcal{E}$.

Lemma B [13] *Suppose $w \in A_\infty$, then the norm of $BMO(w)$ is equivalent to the norm of $BMO(\mathbb{R}^n)$, where*

$$BMO(w) = \left\{ b : \|b\|_{*,w} = \sup_{\mathcal{E}} \frac{1}{w(\mathcal{E})} \int_{\mathcal{E}} \left| b(x) - b_{\mathcal{E},w} \right| w(x)\,dx < \infty \right\},$$

where $b_{\mathcal{E},w} = \frac{1}{w(\mathcal{E})} \int_{\mathcal{E}} b(x)w(x)\,dx$.

Lemma C [14] *Let the ellipsoid $\mathcal{E} = \mathcal{E}(x_0, r)$ centered at x_0 with side length of r. For any positive integer i, $2^i\mathcal{E}$ denotes the ellipsoid centered at x_0 with side length of $2^i r$, we have the inequality*

$$|b_{2^i\mathcal{E}} - b_{\mathcal{E}}| \le Ci\|b\|_*,$$

where $b_{\mathcal{E},w} = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} b(x)w(x)\,dx$.

Lemma D [6] *Suppose $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$, if $\bar{T}$ is the classical Calderón-Zygmund operator with a constant kernel, then the operator $\bar{T}$ is bounded on $L^{p,\kappa}(w)$.*

If $p = 1$, $0 < \kappa < 1$ and $w \in A_1$, then there exists a constant $C > 0$ such that

$$\lambda w\left(\left\{x \in \mathcal{E} : \left|\bar{T}f(x)\right| > \lambda\right\}\right) \le C\|f\|_{L^{1,\kappa}(w)} w(\mathcal{E})^\kappa$$

for all $\lambda > 0$ and any ellipsoid $\mathcal{E}$.

Definition 2.6 Let $\Phi(t) = t\log^N(t + e)$. The Orlicz maximal operator M_Φ is given by

$$M_\Phi f(x) = \sup_{x\in\mathcal{E}} \|f\|_{\Phi,\mathcal{E}} = \sup_{x\in\mathcal{E}} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \Phi\left(|f|\right)(x)\,dx.$$

From the above definition, observe that $Mf(x) \le M_\Phi f(x) \le M(\Phi(|f|))(x)$. This inequality will be relevant in our work.

Aside from the properties of an A_p weight function and a BMO function, we need some estimates of multilinear commutators. The following results were proved by Pérez and González [4].

Lemma E *Let $1 < p < \infty$ and $w \in A_p$. Suppose $b_j \in BMO(\mathbb{R}^n)$, $1 \le j \le N$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} \left|[\vec{b}, \bar{T}](f)(x)\right|^p w(x)\,dx \le C\|\vec{b}\|^p \int_{\mathbb{R}^n} \left|f(x)\right|^p w(x)\,dx.$$

Although the commutators with a BMO function are not of weak type $(1,1)$, we have the following inequality.

Lemma F *Let $w \in A_\infty$. There exists a constant $C > 0$ such that*

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w\bigl(x \in \mathbb{R}^n : \bigl|[\vec{b}, \bar{T}](f)(x)\bigr| > t\bigr) \\ \leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w\bigl(x \in \mathbb{R}^n : M_\Phi\bigl(\|\vec{b}\| f\bigr)(x) > t\bigr),$$

where $\Phi(t) = t\log^N(e+t)$.

By the above inequality, we have the following result.

Lemma G *Let $w \in A_1$. There exists a constant $C > 0$ such that, for all $\lambda > 0$,*

$$w\bigl(x \in \mathbb{R}^n : \bigl|[\vec{b}, \bar{T}](f)(x)\bigr| > \lambda\bigr) \leq C \int_{\mathbb{R}^n} \Phi\bigl(|f|\bigr)(x) w(x)\, dx,$$

where $\Phi(t) = t\log^N(e+t)$.

Finally, we need the spherical harmonics and their properties (see more detail in [1, 15, 16]). Recall that any homogeneous polynomial $P : \mathbb{R}^n \to \mathbb{R}$ of degree m that satisfies $\Delta P = 0$ is called an n-dimensional solid harmonic of degree m. Its restriction to the unit sphere Σ_n will be called an n-dimensional spherical harmonic of degree m. Denote by H_m the space of all n-dimensional spherical harmonics of degree m. In general, it results in a finite-dimensional linear space with $g_m = \dim H_m$ such that $g_0 = 1$, $g_1 = n$ and

$$g_m = C_{m+n-1}^{n-1} - C_{m+n-3}^{n-1} \leq C(n)m^{n-2}, \quad m \geq 2. \tag{2.1}$$

Furthermore, let $\{Y_{sm}\}_{s=1}^{g_m}$ be an orthonormal basis of H_m. Then $\{Y_{sm}\}_{s=1}^{g_m}{}_{m=0}^{\infty}$ is a complete orthonormal system in $L^2(\Sigma_n)$ and

$$\sup_{x\in\Sigma_n} \bigl|D_x^\beta Y_{sm}(x)\bigr| \leq C(n)m^{|\beta|+(n-2)/2}, \quad m = 1, 2, \ldots. \tag{2.2}$$

If, for instance, $\phi \in C^\infty(\Sigma_n)$, then $\Sigma_{s,m} b_{sm} Y_{sm}(x)$ is the Fourier series expansion of $\phi(x)$ with respect to $\{Y_{sm}\}_{s,m}$ ($\Sigma_{s,m}$ substitutes $\Sigma_{m=0}^{\infty}\Sigma_{s=1}^{g_m}$) and

$$b_{sm} = \int_{\Sigma_n} \phi(x) Y_{sm}(x)\, d\sigma, \quad |b_{sm}| \leq C(n,l)m^{-2l} \sup_{\substack{|\beta|=2l \\ y\in\Sigma_n}} \bigl|D_y^\beta \phi(y)\bigr|, \tag{2.3}$$

for any integer l. In particular, the expansion of ϕ into spherical harmonics converges uniformly to ϕ. For more detail, we can see [15].

3 Proof of the theorems

In this section, we shall use the complete orthonormal system in $L^2(\Sigma_n)$ and some lemmas as above to finish the theorems.

Proof of Theorem 1.1 In order to ensure the existence of the operator (1.1) in $L^{p,\kappa}(w)$, $1 \leq p < \infty$, we restrict our consideration to the function $f \in L^{p,\kappa}(w)$, for which the norm of $L^p(w)$ is finite. For the sake of convenience, we still denote these spaces by $L^{p,\kappa}(w)$. Let $x, y \in \mathbb{R}^n$ and $\bar{y} = y/\rho(y) \in \Sigma_n$. In view of the properties of the kernel K with respect to the second variable and the complete of $\{Y_{sm}(x)\}$ in $L^2(\Sigma_n)$, we get

$$\begin{aligned} K(x, x-y) &= \rho(x-y)^{-|\alpha|} K(x, \overline{x-y}) \\ &= \rho(x-y)^{-|\alpha|} \sum_{s,m} b_{sm}(x) Y_{sm}(\overline{x-y}). \end{aligned}$$

Replacing the kernel with its series expansion, (1.1) can be written as

$$\begin{aligned} Tf(x) &= \lim_{\epsilon \to 0} T_\epsilon f(x) \\ &= \lim_{\epsilon \to 0} \int_{\rho(x-y)>\epsilon} \sum_{s,m} b_{sm}(x) \rho(x-y)^{-|\alpha|} Y_{sm}(\overline{x-y}) f(y)\, dy. \end{aligned}$$

From the properties of (2.1)-(2.3), the series expansion $\sum_{s,m} |b_{sm}(x) Y_{sm}(\overline{x-y})| \leq C(n,\alpha) m^{3(n-2)/2-2l}$, where the integer l is preliminarily chosen greater than $(3n-4)/4$. Along with the $\rho(x-y)^{-|\alpha|} f(y) \in L^1(\mathbb{R}^n)$ for a.a. $x \in \mathbb{R}^n$, by the Fubini dominated convergence theorem, we have

$$Tf(x) = \sum_{s,m} b_{sm}(x) \lim_{\epsilon \to 0} \int_{\rho(x-y)>\epsilon} H_{sm}(x-y) f(y)\, dy = \sum_{s,m} b_{sm}(x) T_{sm} f(x),$$

where $H_{sm}(x-y) = \rho(x-y)^{-|\alpha|} Y_{sm}(\overline{x-y})$. Instead of the operators $Tf(x)$, we shall study the existence and boundedness in $L^{p,\kappa}(\omega)$ of the operators $T_{sm}f(x)$ with a kernel $H_{sm}(\cdot)$. Observe that $H_{sm}(\cdot)$ is a constant kernel and satisfies

$$\left|H_{sm}(x)\right| \leq C(n,\alpha) m^{\frac{n-2}{2}} \rho^{-|\alpha|}; \qquad \left|\nabla H_{sm}(x)\right| \leq C(n,\alpha) m^{\frac{n}{2}} \rho^{-|\alpha|-1}.$$

From Lemma D, it follows

$$\left\| T_{sm} f(x) \right\|_{L^{p,\kappa}(\omega)} \leq C(n,\alpha) m^{\frac{n}{2}} \left\| f(x) \right\|_{L^{p,\kappa}(\omega)}$$

for $1 < p < \infty$. Consequently, by the above inequality and (2.1)-(2.3), we show

$$\begin{aligned} \left\| Tf(x) \right\|_{L^{p,\kappa}(\omega)} &\leq C \sum_{s,m} \left\| b_{sm}(x) \right\|_{L^\infty} \left\| T_{sm} f(x) \right\|_{L^{p,\kappa}(\omega)} \\ &\leq C \sum_{s,m} m^{-2l+\frac{n}{2}} \left\| f(x) \right\|_{L^{p,\kappa}(\omega)} \\ &\leq C \|f\|_{L^{p,\kappa}(\omega)}, \end{aligned}$$

where the integer l is preliminary chosen greater that $l > \frac{3n}{4}$. For $p = 1$, by Lemma D, we have

$$\lambda w\bigl(\bigl\{x \in \mathcal{E} : \left|T_{sm} f(x)\right| > \lambda\bigr\}\bigr) \leq C(n,\alpha) m^{\frac{n}{2}} \|f\|_{L^{1,\kappa}(w)}$$

for any $\lambda > 0$ and ellipsoid $\mathcal{E}$. Therefore, one gets

$$\begin{aligned}\lambda w(\{x\in\mathcal{E}:|Tf(x)|>\lambda\}) &\leq C\sum_{s,m}\|b_{sm}(x)\|_{L^\infty}\lambda w(\{x\in\mathcal{E}:|T_{sm}f(x)|>\lambda\})\\ &\leq C\sum_{s,m}m^{-2l+\frac{n}{2}}\|f\|_{L^{1,\kappa}(w)}\\ &\leq C\|f(x)\|_{L^{1,\kappa}(w)},\end{aligned}$$

thus we complete the proof of Theorem 1.1. □

Next we begin with the second theorem, for which further discussion is needed.

Proof of Theorem 1.2 As above, we use the series expansion of a kernel $K(x,y)$, the operator $[\vec{b},T]f(x)$ is divided into

$$[\vec{b},T]f(x)=\sum_{s,m}b_{sm}(x)[\vec{b},T_{sm}]f(x).$$

Instead of the operator $[\vec{b},T]f(x)$, we only consider the existence and boundedness in $L^{p,\kappa}(w)$ of the operators $[\vec{b},T_{sm}]f(x)$.

Let $1<p<\infty$. For any ellipsoid $\mathcal{E}$, we only need to obtain the inequality

$$\int_{\mathcal{E}}|[\vec{b},T_{sm}]f(x)|^p w(x)\,dx\leq Cm^{\frac{mp}{2}}\|b\|^p w(\mathcal{E})^k\|f\|^p_{L^{p,\kappa}(w)}.$$

In fact, by the series expansion of a kernel $K(x,y)$, we have

$$\begin{aligned}\|[\vec{b},T]f(x)\|_{L^{p,\kappa}(\omega)} &\leq C\sum_{s,m}\|b_{sm}(x)\|_{L^\infty}\|[\vec{b},T_{sm}]f(x)\|_{L^{p,\kappa}(\omega)}\\ &\leq C\sum_{s,m}m^{-2l+\frac{n}{2}}\|f\|_{L^{p,\kappa}(\omega)}\leq C\|f\|_{L^{p,\kappa}(\omega)},\end{aligned}$$

where the integer l is chosen greater than $l>\frac{3n}{4}$. Next, fix the above ellipsoid $\mathcal{E}=\mathcal{E}(x_0,r)$ and decompose $f=f_1+f_2$, where $f_1=f\chi_{2\mathcal{E}}$, $\chi_{2\mathcal{E}}$ denotes the characteristic function of $2\mathcal{E}$, then we have

$$\begin{aligned}\int_{\mathcal{E}}|[\vec{b},T_{sm}](f)(x)|^p w(x)\,dx &\leq C\int_{\mathcal{E}}\{|[\vec{b},T_{sm}](f_1)(x)|^p+|[\vec{b},T_{sm}](f_2)(x)|^p\}w(x)\,dx\\ &=C\{I+II\}.\end{aligned}\tag{3.1}$$

By using Lemma E, we get

$$\begin{aligned}I &\leq\int_{\mathbb{R}^n}|[\vec{b},T_{sm}](f_1)(x)|^p w(x)\,dx\\ &\leq Cm^{\frac{np}{2}}\|\vec{b}\|^p w(\mathcal{E})^\kappa\|f\|^p_{L^{p,\kappa}(w)}.\end{aligned}\tag{3.2}$$

For the term II, without loss of generality, we can assume $N = 2$. Thus, the operator $[\vec{b}, T_{sm}]$ can be divided into four parts,

$$\begin{aligned}
[\vec{b}, T_{sm}]f_2(x) &= \big(b_1(x) - \lambda_1\big)\big(b_2(x) - \lambda_2\big) \int_{\mathbb{R}^n} H_{sm}(x-y)f_2(y)\,dy \\
&\quad + \int_{\mathbb{R}^n} H_{sm}(x-y)\big(b_1(y) - \lambda_1\big)\big(b_2(y) - \lambda_2\big)f_2(y)\,dy \\
&\quad - \big(b_1(x) - \lambda_1\big) \int_{\mathbb{R}^n} H_{sm}(x-y)\big(b_2(y) - \lambda_2\big)f_2(y)\,dy \\
&\quad - \big(b_2(x) - \lambda_2\big) \int_{\mathbb{R}^n} H_{sm}(x-y)\big(b_1(y) - \lambda_1\big)f_2(y)\,dy \\
&= II_1(x) + II_2(x) + II_3(x) + II_4(x),
\end{aligned} \tag{3.3}$$

where $\lambda_i = (b_i)_{\mathcal{E},w} = \frac{1}{w(\mathcal{E})}\int_{\mathcal{E}} b_i(x)w(x)\,dx$, $i = 1, 2$. For the term $II_1(x)$, observing that $x \in \mathcal{E}$ and $y \in \mathbb{R}^n \backslash 2\mathcal{E}$, we have $\rho(x_0 - y) \leq C\rho(x-y)$. Thus, it yields

$$\begin{aligned}
\int_{\mathcal{E}} \big|II_1(x)\big|^p w(x)\,dx &\leq Cm^{\frac{np}{2}} \int_{\mathcal{E}} \big|\big(b_1(x) - (b_1)_{\mathcal{E},w}\big)\big(b_2(x) - (b_2)_{\mathcal{E},w}\big)\big|^p w(x)\,dx \\
&\quad \times \left(\int_{\mathbb{R}^n \backslash 2\mathcal{E}} \frac{|f(y)|}{\rho(x_0 - y)^{|\alpha|}}\,dy\right)^p \\
&\leq Cm^{\frac{np}{2}} w(\mathcal{E}) \left(\frac{1}{w(\mathcal{E})} \int_{\mathcal{E}} \big|b_1(x) - (b_1)_{\mathcal{E},w}\big|^{2p} w(x)\,dx\right)^{\frac{1}{2}} \\
&\quad \times \left(\frac{1}{w(\mathcal{E})} \int_{\mathcal{E}} \big|b_2(x) - (b_2)_{\mathcal{E},w}\big|^{2p} w(x)\,dx\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{j=1}^{\infty} \int_{2^{j+1}\mathcal{E} \backslash 2^j \mathcal{E}} \frac{|f(y)|}{\rho(x_0 - y)^{|\alpha|}}\,dy\right)^p \\
&\leq Cm^{\frac{np}{2}} \|b_1\|_*^p \|b_2\|_*^p w(\mathcal{E}) \left(\sum_{j=1}^{\infty} \frac{1}{|2^j\mathcal{E}|} \left(\int_{2^{j+1}\mathcal{E}} \big|f(y)\big|^p w(y)\,dy\right)^{\frac{1}{p}}\right. \\
&\quad \times \left.\left(\int_{2^{j+1}\mathcal{E}} w(y)^{-\frac{p'}{p}}\,dx\right)^{\frac{1}{p'}}\right)^p,
\end{aligned}$$

since $w \in A_p$, and by the definition of a weighted Morrey space, we get

$$\begin{aligned}
\int_{\mathcal{E}} \big|II_1(x)\big|^p w(x)\,dx &\leq Cm^{\frac{np}{2}} \|\vec{b}\|^p w(\mathcal{E}) \left(\sum_{j=1}^{\infty} w\big(2^{j+1}\mathcal{E}\big)^{\frac{-1}{p}} \left(\int_{2^{j+1}\mathcal{E}} \big|f(y)\big|^p w(y)\,dy\right)^{\frac{1}{p}}\right)^p \\
&\leq Cm^{\frac{np}{2}} \|\vec{b}\|^p w(\mathcal{E}) \left(\sum_{j=1}^{\infty} w\big(2^{j+1}\mathcal{E}\big)^{\frac{\kappa-1}{p}} \|f\|_{L^{p,\kappa}(w)}\right)^p \\
&\leq Cm^{\frac{np}{2}} \|\vec{b}\|^p w(\mathcal{E}) \left(\sum_{j=1}^{\infty} D^{j\frac{\kappa-1}{p}} w(\mathcal{E})^{\frac{\kappa-1}{p}} \|f\|_{L^{p,\kappa}(w)}\right)^p \\
&\leq Cm^{\frac{np}{2}} \|\vec{b}\|^p w(\mathcal{E})^{\kappa} \|f\|_{L^{p,\kappa}(w)}^p.
\end{aligned} \tag{3.4}$$

The third inequality is obtained by Lemma A.

For $II_2(x)$, note that $\lambda_i = (b_i)_{\mathcal{E},w} = \frac{1}{w(\mathcal{E})}\int_{\mathcal{E}} b_i(x)w(x)\,dx$, $i = 1,2$. By Hölder's inequality and $\rho(x_0 - y) \le C\rho(x - y)$, we get

$$\begin{aligned}
\int_{\mathcal{E}} \bigl|II_2(x)\bigr|^p w(x)\,dx &\le Cm^{\frac{np}{2}} w(\mathcal{E}) \biggl(\int_{\mathbb{R}^n\setminus 2\mathcal{E}} \frac{|(b_1(y)-(b_1)_{\mathcal{E},w})(b_2(y)-(b_2)_{\mathcal{E},w})|}{\rho(x_0-y)^{|\alpha|}} \bigl|f(y)\bigr|\,dy\biggr)^p \\
&\le Cm^{\frac{np}{2}} w(\mathcal{E}) \Biggl(\sum_{j=1}^{\infty} \frac{1}{|2^j\mathcal{E}|} \int_{2^{j+1}\mathcal{E}\setminus 2^j\mathcal{E}} \bigl|\bigl(b_1(y)-(b_1)_{\mathcal{E},w}\bigr) \\
&\quad \times \bigl(b_2(y)-(b_2)_{\mathcal{E},w}\bigr)\bigr|\bigl|f(y)\bigr|\,dy\Biggr)^p \\
&\le Cm^{\frac{np}{2}} w(\mathcal{E}) \Biggl(\sum_{j=1}^{\infty} \frac{1}{|2^j\mathcal{E}|} \biggl(\int_{2^{j+1}\mathcal{E}} \bigl|f(y)\bigr|^p w(y)\,dy\biggr)^{\frac{1}{p}} \\
&\quad \times \biggl(\int_{2^{j+1}\mathcal{E}} \bigl|\bigl(b_1(y)-(b_1)_{\mathcal{E},w}\bigr)\bigr|^{2p'} w(y)^{-\frac{p'}{p}}\,dy\biggr)^{\frac{1}{2p'}} \\
&\quad \times \biggl(\int_{2^{j+1}\mathcal{E}} \bigl|\bigl(b_2(y)-(b_2)_{\mathcal{E},w}\bigr)\bigr|^{2p'} w(y)^{-\frac{p'}{p}}\,dy\biggr)^{\frac{1}{2p'}}\Biggr)^p.
\end{aligned}$$

Indeed, by Lemma B we know $BMO(\mathbb{R}^n)$ is equivalent to $BMO(w)$, $w \in A_\infty$. Let $W = w^{-\frac{p'}{p}} \in A_{p'} \subset A_\infty$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1,2$. For any ellipsoid $\mathcal{E}$, by using Lemma B and Lemma C, we show

$$\biggl(\frac{1}{W(2^{j+1}\mathcal{E})}\int_{2^{j+1}\mathcal{E}} \bigl|b_i(y)-(b_i)_{\mathcal{E},w}\bigr|^{2p'} W(y)\,dy\biggr)^{\frac{1}{2p'}} \le Cj\|b_i\|_*.$$

Thus, since $w \in A_p$, it yields

$$\begin{aligned}
\int_{\mathcal{E}} \bigl|II_2(x)\bigr|^p w(x)\,dx &\le Cm^{\frac{np}{2}} w(\mathcal{E}) \Biggl(\sum_{j=1}^{\infty} \frac{j^2}{|2^j\mathcal{E}|} \|b_1\|_*\|b_2\|_* W\bigl(2^{j+1}\mathcal{E}\bigr)^{\frac{1}{p'}} \\
&\quad \times \biggl(\int_{2^{j+1}\mathcal{E}} \bigl|f(y)\bigr|^p w(y)\,dy\biggr)^{\frac{1}{p}}\Biggr)^p \\
&\le Cm^{\frac{np}{2}} w(\mathcal{E}) \|\vec{b}\|^p \|f\|^p_{L^{p,\kappa}(w)} \Biggl(\sum_{j=1}^{\infty} \frac{j^2}{w(2^{j+1}\mathcal{E})^{\frac{1-\kappa}{p}}}\Biggr)^p \\
&\le Cm^{\frac{np}{2}} w(\mathcal{E})^{\kappa} \|\vec{b}\|^p \|f\|^p_{L^{p,\kappa}(w)}.
\end{aligned} \tag{3.5}$$

The last inequality is obtained by Lemma A and the D'Alembert judge method of positive series.

For $II_3(x)$, by the inequality $\rho(x_0 - y) \le C\rho(x - y)$ since $w \in A_p \subset A_\infty$, by Lemma B, we have

$$\begin{aligned}
&\int_{\mathcal{E}} \bigl|II_3(x)\bigr|^p w(x)\,dx \\
&\quad \le Cm^{\frac{np}{2}} \int_{\mathcal{E}} \biggl|\bigl(b_1(x)-\lambda_1\bigr)\int_{\mathbb{R}^n\setminus 2\mathcal{E}} \frac{|b_2(y)-\lambda_2|}{\rho(x-y)^{|\alpha|}} \bigl|f(y)\bigr|\,dy\biggr|^p w(x)\,dx
\end{aligned}$$

$$\leq Cm^{\frac{np}{2}} \int_{\mathcal{E}} \bigl|\bigl(b_1(x)-\lambda_1\bigr)\bigr|^p w(x)\,dx \biggl(\int_{\mathbb{R}^n\setminus 2\mathcal{E}} \frac{|b_2(y)-\lambda_2|}{\rho(x_0-y)^{|\alpha|}} \bigl|f(y)\bigr|\,dy\biggr)^p$$
$$\leq Cm^{\frac{np}{2}} w(\mathcal{E}) \|b_1\|_*^p \biggl(\int_{\mathbb{R}^n\setminus 2\mathcal{E}} \frac{|b_2(y)-\lambda_2|}{\rho(x_0-y)^{|\alpha|}} \bigl|f(y)\bigr|\,dy\biggr)^p.$$

By Hölder's inequality, Lemma B and Lemma C, we get

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus 2\mathcal{E}} \frac{|b_2(y)-\lambda_2|}{\rho(x_0-y)^{|\alpha|}} \bigl|f(y)\bigr|\,dy &\leq C\sum_{j=1}^{\infty} \frac{1}{|2^j\mathcal{E}|} \int_{2^{j+1}\mathcal{E}} \bigl|b_2(y)-\lambda_2\bigr|\bigl|f(y)\bigr|\,dy \\
&\leq C\sum_{j=1}^{\infty} \frac{1}{|2^j\mathcal{E}|} \biggl(\int_{2^{j+1}\mathcal{E}} \bigl|f(y)\bigr|^p w(y)\,dy\biggr)^{\frac{1}{p}} \\
&\quad\times \biggl(\int_{2^{j+1}\mathcal{E}} \bigl|b_2(y)-\lambda_2\bigr|^{p'} w(y)^{-\frac{p'}{p}}\,dy\biggr)^{\frac{1}{p'}} \\
&\leq C\|b_2\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} j \frac{w(2^{j+1}\mathcal{E})^{\frac{\kappa}{p}}}{|2^j\mathcal{E}|} \biggl(\int_{2^{j+1}\mathcal{E}} w(y)^{-\frac{p'}{p}}\,dy\biggr)^{\frac{1}{p'}} \\
&\leq C\|b_2\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{j}{w(2^{j+1}\mathcal{E})^{\frac{1-\kappa}{p}}},
\end{aligned}$$

indeed for $0 < \kappa < 1$, by using Lemma A, we have that

$$\sum_{j=1}^{\infty} \frac{j}{w(2^{j+1}\mathcal{E})^{\frac{1-\kappa}{p}}} \leq \sum_{j=1}^{\infty} \frac{j}{D^{(j+1)\frac{1-\kappa}{p}}} w(\mathcal{E})^{\frac{\kappa-1}{p}} \leq Cw(\mathcal{E})^{\frac{\kappa-1}{p}}.$$

Thus, we conclude

$$\int_{\mathcal{E}} \bigl|II_3(x)\bigr|^p w(x)\,dx \leq Cm^{\frac{np}{2}} w(\mathcal{E})^{\kappa} \|\vec{b}\|^p \|f\|_{L^{p,\kappa}(w)}^p. \tag{3.6}$$

In the same way, we shall get the result of $II_4(x)$

$$\int_{\mathcal{E}} \bigl|II_4(x)\bigr|^p w(x)\,dx \leq Cm^{\frac{np}{2}} w(\mathcal{E})^{\kappa} \|\vec{b}\|^p \|f\|_{L^{p,\kappa}(w)}^p. \tag{3.7}$$

Which together with (3.1)-(3.7), for $1 < p < \infty$, the proof of Theorem 1.2 is finished.

Now, we are in a position to consider the case $p = 1$. In general, the singularity of the commutator is stronger than the singular integral, and the endpoint case $p = 1$ of the commutator is not even obtained. Thus, the result for the case $p = 1$ of the multilinear commutator is interesting. We split f as above by $f = f_1 + f_2$, which yields

$$\begin{aligned}
\lambda w\bigl(\bigl\{x\in\mathcal{E} : \bigl|[\vec{b},T]f(x)\bigr| > \lambda\bigr\}\bigr) &\leq C\sum_{s,m} \|b_{sm}\|_{L^\infty} \lambda w\bigl(\bigl\{x\in\mathcal{E} : \bigl|[\vec{b},T_{sm}]f(x)\bigr| > \lambda\bigr\}\bigr) \\
&\leq C\sum_{s,m} m^{-2l}\bigl[\lambda w\bigl(\bigl\{x\in\mathcal{E} : \bigl|[\vec{b},T_{sm}]f_1(x)\bigr| > \lambda/2\bigr\}\bigr) \\
&\quad + \lambda w\bigl(\bigl\{x\in\mathcal{E} : \bigl|[\vec{b},T_{sm}]f_2(x)\bigr| > \lambda/2\bigr\}\bigr)\bigr] \\
&= C\sum_{s,m} m^{-2l}[III + IV]
\end{aligned} \tag{3.8}$$

for any ellipsoid $\mathcal{E}$, $\lambda > 0$ and integer $l > 0$. For the term III, we use Lemma G. It follows that

$$\begin{aligned} III &\leq C\int_{\mathbb{R}^n} \Phi\big(|f_1|\big)(x)w(x)\,dx \\ &\leq Cw(\mathcal{E})^\kappa \|f\|_{L^{\Phi,\kappa}(w)}. \end{aligned} \tag{3.9}$$

For the last term IV, without loss of generality, we still suppose $N = 2$. By homogeneity, it is enough to assume $\lambda/2 = \|b_1\|_* = \|b_2\|_* = 1$, and hence we only need to prove

$$w\big(\big\{x \in \mathcal{E} : \big|[\vec{b}, T_{sm}]f_2(x)\big| > 1\big\}\big) \leq Cw(\mathcal{E})^\kappa \|f\|_{L^{\Phi,\kappa}(w)}.$$

In fact, by Lemma F, we get

$$\begin{aligned} w\big(\big\{x \in \mathcal{E} : \big|[\vec{b}, T_{sm}]f_2(x)\big| > 1\big\}\big) &\leq \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w\big(\big\{x \in \mathcal{E} : \big|[\vec{b}, T_{sm}]f_2(x)\big| > t\big\}\big) \\ &\leq C\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w\big(\big\{x \in \mathcal{E} : M_\Phi f_2(x) > t\big\}\big) \\ &= C\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w\big(\big\{x \in \mathcal{E} : M\big(\Phi|f_2|\big)(x) > t\big\}\big), \end{aligned} \tag{3.10}$$

where $\Phi(t) = t\log^N(e+t)$. We use the Fefferman-Stein maximal inequality

$$\int_{x:Mf(x)>t} \phi(t)\,dx \leq \frac{C}{t}\int_{\mathbb{R}^n} \big|f(x)\big|M\phi(x)\,dx,$$

for any functions f and $\phi \geq 0$. This yields

$$\begin{aligned} w\big(\big\{x \in \mathcal{E} : M\big(\Phi|f_2|\big)(x) > t\big\}\big) &\leq \frac{1}{t}\int_{\{x\in\mathbb{R}^n : M(\Phi|f_2|)(x)>t\}} \chi_\mathcal{E}(x)w(x)\,dx \\ &\leq \frac{C}{t}\int_{\mathbb{R}^n} \Phi\big(|f_2|\big)(x)M(w\chi_\mathcal{E})(x)\,dx \\ &= \frac{C}{t}\left(\int_{3\mathcal{E}} + \int_{\mathbb{R}^n\setminus 3\mathcal{E}}\right)\Phi\big(|f_2|\big)(x)M(w\chi_\mathcal{E})(x)\,dx \\ &= \frac{C}{t}(IV_1 + IV_2). \end{aligned} \tag{3.11}$$

For IV_1, since $w \in A_1$, it follows that

$$\begin{aligned} IV_1 &\leq C\int_{3\mathcal{E}} \Phi\big(|f|\big)(x)w(x)\,dx \\ &\leq Cw(3\mathcal{E})^\kappa \big\|\Phi\big(|f|\big)\big\|_{L^{1,\kappa}(w)} \\ &\leq Cw(\mathcal{E})^\kappa \|f\|_{L^{\Phi,\kappa}}. \end{aligned} \tag{3.12}$$

To estimate the term IV_2, we first consider the form

$$\frac{1}{|\mathcal{F}|}\int_{\mathcal{E}\cap\mathcal{F}} w(y)\,dy$$

for any $x \in \mathbb{R}^n \backslash 3\mathcal{E}$, $x \in \mathcal{F}$ and $\mathcal{F} \cap \mathcal{E} \neq \emptyset$. By simple geometric observation, we have

$$\begin{aligned}\frac{1}{|\mathcal{F}|}\int_{\mathcal{E}\cap\mathcal{F}} w(y)\,dy &\le C\frac{1}{\rho(x-x_0)^{|\alpha|}}\int_{\mathcal{E}} w(y)\,dy \\ &= \frac{C}{\rho(x-x_0)^{|\alpha|}} w(\mathcal{E}).\end{aligned}$$

Therefore, we obtain

$$M(w\chi_{\mathcal{E}})(x) \le \frac{C}{\rho(x-x_0)^{|\alpha|}} w(\mathcal{E}).$$

Since $w \in A_1$ satisfies the doubling condition and Lemma A, we estimate the term IV_2 as follows:

$$\begin{aligned}IV_2 &\le C\int_{\mathbb{R}^n\backslash 3\mathcal{E}} \frac{\Phi(|f|)(x)}{\rho(x-x_0)^{|\alpha|}} w(\mathcal{E})\,dx \\ &\le Cw(\mathcal{E})\sum_{j=1}^{\infty}\frac{1}{|3^j\mathcal{E}|}\int_{3^{j+1}\mathcal{E}} \Phi\bigl(|f|\bigr)(x)\,dx \\ &\le Cw(\mathcal{E})\bigl\|\Phi\bigl(|f|\bigr)\bigr\|_{L^{1,\kappa}(w)}\sum_{j=1}^{\infty}\frac{1}{w(3^j\mathcal{E})^{1-\kappa}} \\ &\le Cw(\mathcal{E})^{\kappa}\|f\|_{L^{\Phi,\kappa}}. \end{aligned} \tag{3.13}$$

The last inequality is similar to (3.4). Noting that $t\Phi(\frac{1}{t}) > 1$, from (3.8)-(3.11), we conclude

$$w\bigl(\bigl\{x \in \mathcal{E} : \bigl|[\vec{b}, T_{sm}]f_2(x)\bigr| > 1\bigr\}\bigr) \le Cw(\mathcal{E})^{\kappa}\|f\|_{L^{\Phi,\kappa}(w)}.$$

Thus, the proof of Theorem 1.2 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XFY conceived of the study and drafted the manuscript. XSZ participated in the discussion. All authors read and approved the final manuscript.

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