Wang *Journal of Inequalities and Applications* 2012, **2012**:154
http://www.journalofinequalitiesandapplications.com/content/2012/1/154

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A class of retarded nonlinear integral inequalities and its application in nonlinear differential-integral equation

Wu-Sheng Wang*

*Correspondence:
wang4896@126.com
Department of Mathematics, Hechi University, Yizhou, Guangxi 546300, P.R. China

Abstract
In this paper, we discuss a class of retarded nonlinear integral inequalities and give an upper bound estimation of an unknown function by the integral inequality technique. This estimation can be used as a tool in the study of differential-integral equations with the initial conditions.

MSC: 26D10; 26D15; 26D20; 34A12; 34A40

Keywords: integral inequality; analysis technique; retarded differential-integral equation; estimation

1 Introduction

Gronwall-Bellman inequalities [1, 2] can be used as important tools in the study of existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of differential equations, integral equations, and integral-differential equations. There can be found a lot of generalizations of Gronwall-Bellman inequalities in various cases from literature (e.g., [3–13]).

Lemma 1 (Abdeldaim and Yakout [4]) *We assume that $u(t)$ and $f(t)$ are nonnegative real-valued continuous functions defined on $I = [0, \infty)$ and they satisfy the inequality*

$$u^{p+1}(t) \le u_0 + \left(\int_0^t f(s)u^p(s)\,ds\right)^2 + 2\int_0^t f(s)u^p(s)\left[u^p(s) + \int_0^s f(\lambda)u^p(\lambda)\,d\lambda\right]ds, \tag{1.1}$$

for all $t \in I$, where $u_0 > 0$ and $p \in (0,1)$ are constants. Then

$$u(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1}\int_0^t f(s)D_2(s)\,ds, \quad \forall t \in I, \tag{1.2}$$

where

$$D_2(t) = \beta(t)\left[u_0^{\frac{1-p}{1+p}} + 2\frac{1-p}{1+p}\int_0^t f(s)\exp\left(-2\frac{1-p}{p}\int_0^s f(\lambda)\,d\lambda\right)ds\right]^{\frac{p}{1-p}}, \tag{1.3}$$

and $\beta(t) = \exp(2\int_0^t f(s)\,ds)$, for all $t \in I$.

In this paper, we discuss a class of retarded nonlinear integral inequalities and give an upper bound estimation of an unknown function by the integral inequality technique.

2 Main result

In this section, we discuss some retarded integral inequalities of Gronwall-Bellman type. Throughout this paper, let $I = [0, \infty)$.

Theorem 1 *Suppose $\alpha \in C^1(I, I)$ is increasing function with $\alpha(t) \le t$, $\alpha(0) = 0$, $\forall t \in I$. We assume that $u(t)$ and $f(t)$ are nonnegative real-valued continuous functions defined on I, and they satisfy the inequality*

$$\begin{aligned} u^{p+1}(t) \le{}& u_0 + \left(\int_0^{\alpha(t)} f(s)u^p(s)\,ds\right)^2 \\ &+ 2\int_0^{\alpha(t)} f(s)u^p(s)\left[u^p(s) + \int_0^s f(\lambda)u^p(\lambda)\,d\lambda\right] ds, \quad \forall t \in I, \end{aligned} \tag{2.1}$$

where $u_0 > 0$ and $p \in (0,1)$ are constants. Then

$$u(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1}\int_0^{\alpha(t)} f(s)\theta_1\big(\alpha^{-1}(s)\big)\,ds, \quad \forall t \in I, \tag{2.2}$$

where

$$\theta_1(t) = \beta_1(t)\left[u_0^{\frac{1-p}{1+p}} + 2\frac{1-p}{1+p}\int_0^{\alpha(t)} f(s)\exp\left(-2\frac{1-p}{p}\int_0^s f(\lambda)\,d\lambda\right) ds\right]^{\frac{p}{1-p}}, \tag{2.3}$$

and $\beta_1(t) = \exp(2\int_0^{\alpha(t)} f(s)\,ds)$, for all $t \in I$.

Remark 1 *If $\alpha(t) = t$, then Theorem 1 reduces Lemma 1.*

Proof Let $z_1^{p+1}(t)$ denote the function on the right-hand side of (2.1), which is a positive and nondecreasing function on I with $z_1(0) = u_0^{\frac{1}{p+1}}$. Then (2.1) is equivalent to

$$u(t) \le z_1(t), \qquad u\big(\alpha(t)\big) \le z_1\big(\alpha(t)\big), \quad \forall t \in I. \tag{2.4}$$

Differentiating $z_1^{p+1}(t)$ with respect to t, using (2.4) we have

$$\begin{aligned} (p+1)z_1^p(t)\frac{dz_1(t)}{dt} ={}& 2\alpha'(t)f\big(\alpha(t)\big)u^p\big(\alpha(t)\big)\int_0^{\alpha(t)} f(s)u^p(s)\,ds \\ &+ 2\alpha'(t)f\big(\alpha(t)\big)u^p\big(\alpha(t)\big)\left[u^p\big(\alpha(t)\big) + \int_0^{\alpha(t)} f(\lambda)u^p(\lambda)\,d\lambda\right] \\ \le{}& 2\alpha'(t)f\big(\alpha(t)\big)z_1^p(t)\left[z_1^p(t) + 2\int_0^{\alpha(t)} f(\lambda)z_1^p(\lambda)\,d\lambda\right], \quad \forall t \in I. \end{aligned} \tag{2.5}$$

Since $z_1^p(t) > 0$, from (2.5) we have

$$\frac{dz_1(t)}{dt} \le \frac{2}{p+1}\alpha'(t)f\big(\alpha(t)\big)Y_1(t), \quad \forall t \in I, \tag{2.6}$$

where

$$Y_1(t) := z_1^p(t) + 2\int_0^{\alpha(t)} f(\lambda) z_1^p(\lambda)\,d\lambda, \quad \forall t \in I. \tag{2.7}$$

Then $Y_1(t)$ is a positive and nondecreasing function on I with $Y_1(0) = u_0^{p/(p+1)}$ and

$$z_1(t) \le Y_1(t)^{1/p}. \tag{2.8}$$

Differentiating $Y_1(t)$ with respect to t, and using (2.6), (2.7) and (2.8), we get

$$\begin{aligned} \frac{dY_1(t)}{dt} &\le \frac{2p}{p+1}\alpha'(t) f\bigl(\alpha(t)\bigr) z_1^{p-1}(t) Y_1(t) + 2\alpha'(t) f\bigl(\alpha(t)\bigr) z_1^p\bigl(\alpha(t)\bigr) \\ &\le \frac{2p}{p+1}\alpha'(t) f\bigl(\alpha(t)\bigr) Y_1^{\frac{2p-1}{p}}(t) + 2\alpha'(t) f\bigl(\alpha(t)\bigr) Y_1(t), \quad \forall t \in I. \end{aligned} \tag{2.9}$$

From (2.9), we have

$$Y_1^{\frac{1-2p}{p}}(t)\frac{dY_1(t)}{dt} - 2\alpha'(t) f\bigl(\alpha(t)\bigr) Y_1^{\frac{1-p}{p}}(t) \le \frac{2p}{p+1}\alpha'(t) f\bigl(\alpha(t)\bigr), \quad \forall t \in I. \tag{2.10}$$

Let $S_1(t) = Y_1^{\frac{1-p}{p}}(t)$, then $S_1(0) = u_0^{\frac{1-p}{p+1}}$, from (2.10) we obtain

$$\frac{dS_1(t)}{dt} - 2\frac{1-p}{p}\alpha'(t) f\bigl(\alpha(t)\bigr) S_1(t) \le 2\frac{1-p}{p+1}\alpha'(t) f\bigl(\alpha(t)\bigr), \quad \forall t \in I. \tag{2.11}$$

Consider the ordinary differential equation

$$\begin{cases} \dfrac{dS_2(t)}{dt} - 2\dfrac{1-p}{p}\alpha'(t) f\bigl(\alpha(t)\bigr) S_2(t) = 2\dfrac{1-p}{p+1}\alpha'(t) f\bigl(\alpha(t)\bigr), \quad \forall t \in I, \\ S_2(0) = u_0^{\frac{1-p}{p+1}}. \end{cases} \tag{2.12}$$

The solution of Equation (2.12) is

$$\begin{aligned} S_2(t) = {} & \exp\biggl(\int_0^{\alpha(t)} 2\frac{1-p}{p} f(s)\,ds\biggr)\biggl(u_0^{\frac{1-p}{p+1}} \\ & + \int_0^{\alpha(t)} 2\frac{1-p}{p+1} f(s)\exp\biggl(-\int_0^s 2\frac{1-p}{p} f(\tau)\,d\tau\biggr)ds\biggr), \end{aligned} \tag{2.13}$$

for all $t \in I$. By (2.11), (2.12) and (2.13), we obtain

$$Y_1(t) = S_1^{\frac{p}{1-p}}(t) \le S_2^{\frac{p}{1-p}}(t) = \theta_1(t), \quad \forall t \in I, \tag{2.14}$$

where $\theta_1(t)$ as defined in (2.3). From (2.6) and (2.14), we have

$$\frac{dz_1(t)}{dt} \le \frac{2}{p+1}\alpha'(t) f\bigl(\alpha(t)\bigr)\theta_1(t), \quad \forall t \in I.$$

By taking $t = s$ in the above inequality and integrating it from 0 to t, we get

$$u(t) \le z_1(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1}\int_0^{\alpha(t)} f(s)\theta_1\big(\alpha^{-1}(s)\big)\,ds, \quad \forall t \in I.$$

The estimation (2.2) of the unknown function in the inequality (2.1) is obtained. □

Theorem 2 *Suppose $\alpha \in C^1(I,I)$ is increasing function with $\alpha(t) \le t$, $\alpha(0) = 0$, $\forall t \in I$. We assume that $u(t)$ and $f(t)$ are nonnegative real-valued continuous functions defined on I and satisfy the inequality*

$$\begin{aligned} u^{p+1}(t) \le{}& u_0 + \left(\int_0^{\alpha(t)} f(s)u^p(s)\,ds\right)^2 \\ &+ 2\int_0^{\alpha(t)} f(s)u^p(s)\left[u(s) + \int_0^s f(\lambda)u(\lambda)\,d\lambda\right]ds, \quad \forall t \in I, \end{aligned} \tag{2.15}$$

where $u_0 > 0$ and $p \in (0,1)$ are constants. Then

$$u(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1}\int_0^{\alpha(t)} f(s)\theta_2\big(\alpha^{-1}(s)\big)\,ds, \quad \forall t \in I, \tag{2.16}$$

where

$$\theta_2(t) = \beta_2(t)\left[u_0^{\frac{1-p}{p+1}} + \int_0^{\alpha(t)} (1-p)f(s)\exp\left(-\int_0^s \frac{(1-p)(p+3)}{p+1}f(\tau)\,d\tau\right)ds\right]^{\frac{1}{1-p}}, \tag{2.17}$$

and $\beta_2(t) = \exp(\int_0^{\alpha(t)} \frac{p+3}{p+1}f(s)\,ds)$, for all $t \in I$.

Proof Let $z_2^{p+1}(t)$ denote the function on the right-hand side of (2.15), which is a positive and nondecreasing function on I with $z_2(0) = u_0^{\frac{1}{p+1}}$. Then (2.15) is equivalent to

$$u(t) \le z_2(t), \qquad u\big(\alpha(t)\big) \le z_2\big(\alpha(t)\big), \quad \forall t \in I. \tag{2.18}$$

Differentiating $z_2^{p+1}(t)$ with respect to t, using (2.18) we have

$$\begin{aligned} (p+1)z_2^p(t)\frac{dz_2(t)}{dt} ={}& 2\alpha'(t)f\big(\alpha(t)\big)u^p\big(\alpha(t)\big)\int_0^{\alpha(t)} f(s)u^p(s)\,ds \\ &+ 2\alpha'(t)f\big(\alpha(t)\big)u^p\big(\alpha(t)\big)\left[u\big(\alpha(t)\big) + \int_0^{\alpha(t)} f(\lambda)u(\lambda)\,d\lambda\right] \\ \le{}& 2\alpha'(t)f\big(\alpha(t)\big)z_2^p(t)\left[z_2(t) + \int_0^{\alpha(t)} f(\lambda)z_2(\lambda)\,d\lambda\right. \\ &\left. + \int_0^{\alpha(t)} f(\lambda)z_2^p(\lambda)\,d\lambda\right], \quad \forall t \in I. \end{aligned} \tag{2.19}$$

Since $z_2^p(t) > 0$, we have

$$\frac{dz_2(t)}{dt} \le \frac{2}{p+1}\alpha'(t)f\big(\alpha(t)\big)Y_2(t), \quad \forall t \in I, \tag{2.20}$$

where

$$Y_2(t) := z_2(t) + \int_0^{\alpha(t)} f(\lambda) z_2(\lambda)\, d\lambda + \int_0^{\alpha(t)} f(\lambda) z_2^p(\lambda)\, d\lambda, \quad \forall t \in I. \tag{2.21}$$

Then $Y_2(t)$ is a positive and nondecreasing function on I with $Y_2(0) = z_2(0) = u_0^{1/(p+1)}$ and

$$z_2(t) \le Y_2(t). \tag{2.22}$$

Differentiating $Y_2(t)$ with respect to t, and using (2.20), (2.21) and (2.22), we get

$$\begin{aligned} \frac{dY_2(t)}{dt} &\le \frac{2}{p+1}\alpha'(t) f\big(\alpha(t)\big) Y_2(t) + \alpha'(t) f\big(\alpha(t)\big) z_2\big(\alpha(t)\big) + \alpha'(t) f\big(\alpha(t)\big) z_2^p\big(\alpha(t)\big) \\ &\le \frac{p+3}{p+1}\alpha'(t) f\big(\alpha(t)\big) Y_2(t) + \alpha'(t) f\big(\alpha(t)\big) Y_2^p(t), \quad \forall t \in I. \end{aligned} \tag{2.23}$$

From (2.23), we have

$$Y_2^{-p}(t)\frac{dY_2(t)}{dt} - \frac{p+3}{p+1}\alpha'(t) f\big(\alpha(t)\big) Y_2^{1-p}(t) \le \alpha'(t) f\big(\alpha(t)\big), \quad \forall t \in I. \tag{2.24}$$

Let $S_3(t) = Y_2^{1-p}(t)$, then $S_3(0) = u_0^{\frac{1-p}{p+1}}$, from (2.24) we obtain

$$\frac{dS_3(t)}{dt} - \frac{(1-p)(p+3)}{p+1}\alpha'(t) f\big(\alpha(t)\big) S_3(t) \le (1-p)\alpha'(t) f\big(\alpha(t)\big), \quad \forall t \in I. \tag{2.25}$$

Consider the ordinary differential equation

$$\begin{cases} \dfrac{dS_4(t)}{dt} - \dfrac{(1-p)(p+3)}{p+1}\alpha'(t) f\big(\alpha(t)\big) S_4(t) = (1-p)\alpha'(t) f\big(\alpha(t)\big), \quad \forall t \in I, \\ S_4(0) = u_0^{\frac{1-p}{p+1}}. \end{cases} \tag{2.26}$$

The solution of Equation (2.26) is

$$\begin{aligned} S_4(t) = {} & \exp\left(\int_0^{\alpha(t)} \frac{(1-p)(p+3)}{p+1} f(s)\, ds\right)\Bigg(u_0^{\frac{1-p}{p+1}} \\ & + \int_0^{\alpha(t)} (1-p) f(s) \exp\left(-\int_0^s \frac{(1-p)(p+3)}{p+1} f(\tau)\, d\tau\right) ds\Bigg), \end{aligned} \tag{2.27}$$

for all $t \in I$. By (2.25), (2.26) and (2.27), we obtain

$$Y_2(t) = S_3^{\frac{1}{1-p}}(t) \le S_4^{\frac{1}{1-p}}(t) = \theta_2(t), \quad \forall t \in I, \tag{2.28}$$

where $\theta_2(t)$ as defined in (2.17). From (2.20) and (2.28), we have

$$\frac{dz_2(t)}{dt} \le \frac{2}{p+1}\alpha'(t) f\big(\alpha(t)\big)\theta_2(t), \quad \forall t \in I.$$

By taking $t = s$ in the above inequality and integrating it from 0 to t, we get

$$u(t) \le z_2(t) \le u_0^{\frac{1}{p+1}} + \frac{2}{p+1}\int_0^{\alpha(t)} f(s)\theta_2\bigl(\alpha^{-1}(s)\bigr)\,ds, \quad \forall t \in I.$$

The estimation (2.16) of the unknown function in the inequality (2.15) is obtained. □

Theorem 3 *Suppose $\phi_1, \phi_2, \phi_2/\phi_1, \alpha \in C^1(I,I)$ are increasing functions with $\alpha(t) \le t$, $\phi_i(t) > 0$, $\forall t > 0$, $i = 1, 2$, $\alpha(0) = 0$. We assume that $u(t)$ and $f(t)$ are nonnegative real-valued continuous functions defined on I and satisfy the inequality*

$$\begin{aligned} u(t) \le{}& u_0 + \left(\int_0^{\alpha(t)} f(s)\phi_1\bigl(u(s)\bigr)\,ds\right)^2 \\ &+ 2\int_0^{\alpha(t)} f(s)\phi_1\bigl(u(s)\bigr)\left[u(s) + \int_0^s g(\lambda)\phi_2\bigl(u(\lambda)\bigr)\,d\lambda\right]ds, \quad \forall t \in I, \end{aligned} \tag{2.29}$$

where $u_0 > 0$ is a constant. Then

$$u(t) \le \Phi_1^{-1}\left[\Phi_2^{-1}\left(\Phi_2\left(\Phi_1(u_0) + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} 2f(s)\,ds\right)\right], \quad \forall t < T_1, \tag{2.30}$$

where

$$\Phi_1(r) := \int_1^r \frac{dt}{\phi_2(t)}, \quad r > 0, \tag{2.31}$$

$$\Phi_2(r) := \int_1^r \frac{\phi_2(\Phi_1^{-1}(s))\,ds}{\phi_1(\Phi_1^{-1}(s))(\Phi_1^{-1}(s)+1)}, \quad r > 0, \tag{2.32}$$

and T_1 is the largest number such that

$$\Phi_2\left(\Phi_1(u_0) + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} 2f(s)\,ds \le \int_1^{\infty} \frac{\phi_2(\Phi_1^{-1}(s))\,ds}{\phi_1(\Phi_1^{-1}(s))(\Phi_1^{-1}(s)+1)},$$
$$\Phi_2^{-1}\left(\Phi_2\left(\Phi_1(u_0) + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} 2f(s)\,ds\right) \le \int_1^{\infty} \frac{dt}{\phi_2(t)}$$

for all $t \le T_1$.

Proof Let $z_3(t)$ denote the function on the right-hand side of (2.29), which is a positive and nondecreasing function on I with $z_3(0) = u_0$. Then (2.29) is equivalent to

$$u(t) \le z_3(t), \qquad u\bigl(\alpha(t)\bigr) \le z_3(t), \quad \forall t \in I. \tag{2.33}$$

Differentiating $z_3(t)$ with respect to t, using (2.33) we have

$$\begin{aligned}\frac{dz_3(t)}{dt} &\le 2\alpha'(t)f(\alpha(t))\phi_1(z_3(t))\int_0^{\alpha(t)} f(s)\phi_1(z_3(s))\,ds \\ &\quad + 2\alpha'(t)f(\alpha(t))\phi_1(z_3(t))\left[z_3(t) + \int_0^{\alpha(t)} g(\lambda)\phi_2(z_3(\lambda))\,d\lambda\right] ds \\ &\le 2\alpha'(t)f(\alpha(t))\phi_1(z_3(t))\left[z_3(t) + \int_0^{\alpha(t)} f(s)\phi_1(z_3(s))\,ds \right. \\ &\quad \left. + \int_0^{\alpha(t)} g(\lambda)\phi_2(z_3(\lambda))\,d\lambda\right], \quad \forall t \in I.\end{aligned} \tag{2.34}$$

Let

$$Y_3(t) := z_3(t) + \int_0^{\alpha(t)} f(s)\phi_1(z_3(s))\,ds + \int_0^{\alpha(t)} g(\lambda)\phi_2(z_3(\lambda))\,d\lambda, \quad \forall t \in I. \tag{2.35}$$

Then $Y_3(t)$ is a positive and nondecreasing function on I with $Y_3(0) = z_3(0) = u_0$ and

$$z_3(t) \le Y_3(t). \tag{2.36}$$

Differentiating $Y_3(t)$ with respect to t, and using (2.34), (2.35) and (2.36), we get

$$\begin{aligned}\frac{dY_3(t)}{dt} &\le 2\alpha'(t)f(\alpha(t))\phi_1(z_3(t))Y_3(t) + \alpha'(t)f(\alpha(t))\phi_1(z_3(t)) \\ &\quad + \alpha'(t)g(\alpha(t))\phi_2(z_3(t)) \\ &\le 2\alpha'(t)f(\alpha(t))\phi_1(Y_3(t))(Y_3(t)+1) + \alpha'(t)g(\alpha(t))\phi_2(Y_3(t)),\end{aligned} \tag{2.37}$$

for all $t \in I$. Since $\phi_2(Y_3(t)) > 0$, $\forall t > 0$, from (2.37) we have

$$\frac{dY_3(t)}{\phi_2(Y_3(t))} \le 2\alpha'(t)f(\alpha(t))\frac{\phi_1(Y_3(t))(Y_3(t)+1)}{\phi_2(Y_3(t))} + \alpha'(t)g(\alpha(t)), \quad \forall t \in I.$$

By taking $t = s$ in the above inequality and integrating it from 0 to t, we get

$$\begin{aligned}\Phi_1(Y_3(t)) &\le \Phi_1(Y_3(0)) + \int_0^t 2\alpha'(s)f(\alpha(s))\frac{\phi_1(Y_3(s))(Y_3(s)+1)}{\phi_2(Y_3(s))}\,ds \\ &\quad + \int_0^t \alpha'(s)g(\alpha(s))\,ds,\end{aligned} \tag{2.38}$$

for all $t \in I$, where Φ_1 is defined by (2.31). From (2.38), we have

$$\begin{aligned}\Phi_1(Y_3(t)) &\le \Phi_1(Y_3(0)) + \int_0^T \alpha'(s)g(\alpha(s))\,ds \\ &\quad + \int_0^t 2\alpha'(s)f(\alpha(s))\frac{\phi_1(Y_3(s))(Y_3(s)+1)}{\phi_2(Y_3(s))}\,ds,\end{aligned} \tag{2.39}$$

for all $t < T$, where $0 < T < T_1$ is chosen arbitrarily. Let $Y_4(t)$ denote the function on the right-hand side of (2.39), which is a positive and nondecreasing function on I with $Y_4(0) =$

$\Phi_1(u_0) + \int_0^T \alpha'(s)g(\alpha(s))\,ds$ and

$$Y_3(t) \le \Phi_1^{-1}\big(Y_4(t)\big), \quad \forall t < T. \tag{2.40}$$

Differentiating $Y_4(t)$ with respect to t, using the hypothesis on ϕ_2/ϕ_1, from (2.40) we have

$$\begin{aligned}\frac{dY_4(t)}{dt} &\le 2\alpha'(t)f\big(\alpha(t)\big)\frac{\phi_1(Y_3(t))(Y_3(t)+1)}{\phi_2(Y_3(t))}\\ &\le 2\alpha'(t)f\big(\alpha(t)\big)\frac{\phi_1(\Phi_1^{-1}(Y_4(t)))(\Phi_1^{-1}(Y_4(t))+1)}{\phi_2(\Phi_1^{-1}(Y_4(t)))}, \quad \forall t < T.\end{aligned} \tag{2.41}$$

By the definition of Φ_2 in (2.32), from (2.41) we obtain

$$\begin{aligned}\Phi_2\big(Y_4(t)\big) &\le \Phi_2\big(Y_4(0)\big) + \int_0^t 2\alpha'(s)f\big(\alpha(s)\big)\,ds\\ &\le \Phi_2\left(\Phi_1(u_0) + \int_0^{\alpha(T)} g(s)\,ds\right) + \int_0^{\alpha(t)} 2f(s)\,ds, \quad \forall t < T.\end{aligned} \tag{2.42}$$

Let $t = T$, from (2.42) we have

$$\Phi_2\big(Y_4(T)\big) \le \Phi_2\left(\Phi_1(u_0) + \int_0^{\alpha(T)} g(s)\,ds\right) + \int_0^{\alpha(T)} 2f(s)\,ds. \tag{2.43}$$

Since $0 < T < T_1$ is chosen arbitrarily, from (2.33), (2.36), (2.40) and (2.43), we have

$$u(t) \le \Phi_1^{-1}\left[\Phi_2^{-1}\left(\Phi_2\left(\Phi_1(u_0) + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} 2f(s)\,ds\right)\right], \quad \forall t < T_1.$$

This proves (2.30). □

3 Application

In this section, we apply our Theorem 3 to the following differential-integral equation

$$\begin{cases}\dfrac{dx(t)}{dt} = F\big(t, x(\alpha(t))\big) + H\big(t, x(\alpha(t))\big), & \forall t \in I,\\ x(0) = x_0,\end{cases} \tag{3.1}$$

where $F \in C(I \times I, \mathbf{R})$, $H \in C(I^3, \mathbf{R})$, $|x_0| > 0$ is a constant satisfying the following conditions

$$\big|F\big(t, x(t)\big)\big| \le f^2(t)\phi_1^2\big(|x(t)|\big), \tag{3.2}$$

$$\big|H\big(t, x(t)\big)\big| \le 2f(t)\phi_1\big(|x(t)|\big)\left(|x(t)| + \int_0^t g(s)\phi_2\big(|x(s)|\big)\,ds\right), \tag{3.3}$$

where f, g is nonnegative real-valued continuous function defined on I.

Corollary 1 *Consider the nonlinear system (3.1) and suppose that F, H satisfy the conditions (3.2) and (3.3), and* $\phi_1, \phi_2, \phi_2/\phi_1, \alpha \in C^1(I, I)$ *are increasing functions with* $\alpha(t) \le t$,

$\phi_i(t) > 0$, $\forall t > 0$, $i = 1, 2$, $\alpha(0) = 0$. *Then all solutions of Equation (3.1) exist on I and satisfy the following estimation*

$$|x(t)| \le \Phi_1^{-1}\left[\Phi_2^{-1}\left(\Phi_2\left(\Phi_1(|x_0|) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right) + \int_0^{\alpha(t)} 2\frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right)\right], \quad (3.4)$$

for all $t < T_2$, *where*

$$\Phi_1(r) := \int_1^r \frac{dt}{\phi_2(t)}, \quad r > 0,$$
$$\Phi_2(r) := \int_1^r \frac{\phi_2(\Phi_1^{-1}(s))\,ds}{\phi_1(\Phi_1^{-1}(s))(\Phi_1^{-1}(s)+1)}, \quad r > 0,$$

and T_2 *is the largest number such that*

$$\Phi_2\left(\Phi_1(|x_0|) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right) + \int_0^{\alpha(t)} 2\frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds$$
$$\le \int_1^\infty \frac{\phi_2(\Phi_1^{-1}(s))\,ds}{\phi_1(\Phi_1^{-1}(s))(\Phi_1^{-1}(s)+1)},$$
$$\Phi_2^{-1}\left(\Phi_2\left(\Phi_1(|x_0|) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right) + \int_0^{\alpha(t)} 2\frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right) \le \int_1^\infty \frac{dt}{\phi_2(t)}$$

for all $t \le T_2$.

Proof Integrating both sides of Equation (3.1) from 0 to t, we get

$$x(t) = x_0 + \int_0^t F\big(s, x(\alpha(s))\big)\,ds + \int_0^t H\big(s, x(\alpha(s))\big)\,ds, \quad \forall t \in I. \quad (3.5)$$

Using the conditions (3.2) and (3.3), from (3.5) we obtain

$$\begin{aligned}
\big|x(t)\big| &\le |x_0| + \int_0^t f^2(s)\phi_1^2\big(\big|x(\alpha(s))\big|\big)\,ds \\
&\quad + 2\int_0^t f(s)\phi_1\big(\big|x(\alpha(s))\big|\big)\left(\big|x(\alpha(s))\big| + \int_0^s g(\tau)\phi_2\big(\big|x(\alpha(\tau))\big|\big)\,d\tau\right) ds \\
&\le |x_0| + \left(\int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\phi_1\big(\big|x(s)\big|\big)\,ds\right)^2 \\
&\quad + 2\int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\phi_1\big(\big|x(s)\big|\big)\left(\big|x(s)\big| + \int_0^s \frac{g(\alpha^{-1}(\tau))}{\alpha'(\alpha^{-1}(\tau))}\phi_2\big(\big|x(\tau)\big|\big)\,d\tau\right) ds,
\end{aligned} \quad (3.6)$$

for all $t \in I$. Applying Theorem 3 to (3.6), we get the estimation (3.4). This completes the proof of the Corollary 1. □

Competing interests

The author declares that they have no competing interests.

Acknowledgement
The author is very grateful to the editor and the referees for their helpful comments and valuable suggestions. This research was supported by National Natural Science Foundation of China (Project No. 11161018), Guangxi Natural Science Foundation (Project No. 0991265 and 2012GXNSFAA053009), Scientific Research Foundation of the Education Department of Guangxi Province of China (Project No. 201106LX599), and the Key Discipline of Applied Mathematics of Hechi University of China (200725).

Received: 12 March 2012 Accepted: 22 June 2012 Published: 5 July 2012

References

1. Gronwall, TH: Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. Ann. Math. **20**, 292-296 (1919). doi:10.2307/1967124
2. Bellman, R: The stability of solutions of linear differential equations. Duke Math. J. **10**, 643-647 (1943). doi:10.1215/S0012-7094-43-01059-2
3. Lipovan, O: A retarded Gronwall-like inequality and its applications. J. Math. Anal. Appl. **252**, 389-401 (2000). doi:10.1006/jmaa.2000.7085
4. Abdeldaim, A, Yakout, M: On some new integral inequalities of Gronwall-Bellman-Pachpatte type. Appl. Math. Comput. **217**, 7887-7899 (2011). doi:10.1016/j.amc.2011.02.093
5. Agarwal, RP, Deng, S, Zhang, W: Generalization of a retarded Gronwall-like inequality and its applications. Appl. Math. Comput. **165**, 599-612 (2005). doi:10.1016/j.amc.2004.04.067
6. Bihari, IA: A generalization of a lemma of Bellman and its application to uniqueness problem of differential equation. Acta Math. Acad. Sci. Hung. **7**, 81-94 (1956). doi:10.1007/BF02022967
7. Pachpatte, BG: Inequalities for Differential and Integral Equations. Academic Press, London (1998)
8. Kim, YH: On some new integral inequalities for functions in one and two variables. Acta Math. Sin. **21**, 423-434 (2005). doi:10.1007/s10114-004-0463-7
9. Cheung, WS: Some new nonlinear inequalities and applications to boundary value problems. Nonlinear Anal. **64**, 2112-2128 (2006). doi:10.1016/j.na.2005.08.009
10. Wang, WS: A generalized retarded Gronwall-like inequality in two variables and applications to BVP. Appl. Math. Comput. **191**, 144-154 (2007). doi:10.1016/j.amc.2007.02.099
11. Wang, WS, Shen, C: On a generalized retarded integral inequality with two variables. J. Inequal. Appl. **2008**, Article ID 518646 (2008)
12. Wang, WS, Li, Z, Li, Y, Huang, Y: Nonlinear retarded integral inequalities with two variables and applications. J. Inequal. Appl. **2010**, Article ID 240790 (2010)
13. Wang, WS, Luo, RC, Li, Z: A new nonlinear retarded integral inequality and its application. J. Inequal. Appl. **2010**, Article ID 462163 (2010)

doi:10.1186/1029-242X-2012-154
Cite this article as: Wang: **A class of retarded nonlinear integral inequalities and its application in nonlinear differential-integral equation.** *Journal of Inequalities and Applications* 2012 **2012**:154.