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A note on entire functions and their differences

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Abstract

In this paper, we prove that for a transcendental entire function $f(z)$ of finite order such that $\lambda(f - a(z)) < \sigma(f)$, where $a(z)$ is an entire function and satisfies $\sigma(a(z)) < 1$, n is a positive integer and if $\Delta_\eta^n f(z)$ and $f(z)$ share the function $a(z)$ CM, where η $(\in \mathbb{C})$ satisfies $\Delta_\eta^n f(z) \not\equiv 0$, then

$$a(z) \equiv 0 \quad \text{and} \quad f(z) = ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

MSC: 39A10; 30D35

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1 Introduction and results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory of meromorphic functions (see [1–3]). In addition, we use the notation $\lambda(f)$ for the exponent of convergence of the sequence of zeros of a meromorphic function f, and $\sigma(f)$ to denote the order growth of f. For a nonzero constant η, the forward differences $\Delta_\eta^n f(z)$ are defined (see [4, 5]) by

$$\Delta_\eta f(z) = \Delta_\eta^1 f(z) = f(z+\eta) - f(z) \quad \text{and}$$
$$\Delta_\eta^{n+1} f(z) = \Delta_\eta^n f(z+\eta) - \Delta_\eta^n f(z), \quad n = 1, 2, \ldots.$$

Throughout this paper, we denote by $S(r,f)$ any function satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of r of finite logarithmic measure. A meromorphic function $\alpha(z)$ is said to be a small function of $f(z)$ if $T(r,\alpha(z)) = S(r,f)$, and we denote by $S(f)$ the set of functions which are small compared to $f(z)$.

Let f and g be two nonconstant meromorphic functions, and let $a \in \mathbb{C}$. We say that f and g share the value a CM (IM) provided that $f - a$ and $g - a$ have the same zeros counting multiplicities (ignoring multiplicities), that f and g share the value ∞ CM (IM) provided that f and g have the same poles counting multiplicities (ignoring multiplicities). Using the same method, we can define that f and g share the function $a(z)$ CM (IM), where $a(z) \in S(f) \cap S(g)$. Nevanlinna's four values theorem [6] says that if two nonconstant meromorphic functions f and g share four values CM, then $f \equiv g$ or f is a Möbius transformation of g. The condition 'f and g share four values CM' has been weakened to 'f and

g share two values CM and two values IM' by Gundersen [7, 8], as well as by Mues [9]. But whether the condition can be weakened to 'f and g share three values IM and another value CM' is still an open question.

In the special case, we recall a well-known conjecture by Brück [10].

Conjecture *Let f be a nonconstant entire function such that hyper order $\sigma_2(f) < \infty$ and $\sigma_2(f)$ is not a positive integer. If f and f' share the finite value a CM, then*

$$f' - a = c(f - a),$$

where c is a nonzero constant.

The notation $\sigma_2(f)$ denotes hyper-order (see [11]) of $f(z)$ which is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}.$$

The conjecture has been verified in the special cases when $a = 0$ [10], or when f is of finite order [12], or when $\sigma_2(f) < \frac{1}{2}$ [13].

Recently, many authors [14–17] started to consider sharing values of meromorphic functions with their shifts. Heittokangas *et al.* proved the following theorems.

Theorem A (See [15]) *Let f be a meromorphic function with $\sigma(f) < 2$, and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share the values a $(\in \mathbb{C})$ and ∞ CM, then*

$$f(z+c) - a = \tau\bigl(f(z) - a\bigr)$$

for some constant τ.

In [15], Heittokangas *et al.* give the example $f(z) = e^{z^2} + 1$ which shows that $\sigma(f) < 2$ cannot be relaxed to $\sigma(f) \le 2$.

Theorem B (See [16]) *Let f be a meromorphic function of finite order, let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions $a_1, a_2, a_3 \in \hat{S}(f)$ with period c CM (where $\hat{S}(f) = S(f) \cup \{\infty\}$), then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Recently, many results of complex difference equations have been rapidly obtained (see [18–25]). In the present paper, we utilize a complex difference equation to consider uniqueness problems.

The main purpose of this paper is to utilize a complex difference equation to study problems concerning sharing values of meromorphic functions and their differences. It is well known that $\Delta_\eta f(z) = f(z+\eta) - f(z)$ (where η $(\in \mathbb{C})$ is a constant satisfying $f(z+\eta) - f(z) \not\equiv 0$) is regarded as the difference counterpart of f'. So, Chen and Yi [20] considered the problem that $\Delta_\eta f(z)$ and $f(z)$ share one value a CM and proved the following theorem.

Theorem C (See [20]) *Let f be a finite order transcendental entire function which has a finite Borel exceptional value a, and let η $(\in \mathbb{C})$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If*

$\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share the value a CM, then

$$a = 0 \quad \textit{and} \quad \frac{f(z+\eta) - f(z)}{f(z)} = A,$$

where A is a nonzero constant.

Question 1 What can be said if we consider the forward difference $\Delta_\eta^n f(z)$ and $f(z)$ share one value or one small function?

In this paper, we answer Question 1 and prove the following theorem.

Theorem 1.1 *Let $f(z)$ be a finite order transcendental entire function such that $\lambda(f - a(z)) < \sigma(f)$, where $a(z)$ is an entire function and satisfies $\sigma(a) < 1$. Let n be a positive integer. If $\Delta_\eta^n f(z)$ and $f(z)$ share $a(z)$ CM, where η $(\in \mathbb{C})$ satisfies $\Delta_\eta^n f(z) \not\equiv 0$, then*

$$a(z) \equiv 0 \quad \textit{and} \quad f(z) = ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

In the special case, if we take $a(z) \equiv a$ in Theorem 1.1, we can get the following corollary.

Corollary 1.1 *Let $f(z)$ be a finite order transcendental entire function which has a finite Borel exceptional value a. Let n be a positive integer. If $\Delta_\eta^n f(z)$ and $f(z)$ share value a CM, where η $(\in \mathbb{C})$ satisfies $\Delta_\eta^n f(z) \not\equiv 0$, then*

$$a = 0 \quad \textit{and} \quad f(z) = ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

Remark 1.1 From Corollary 1.1, we know that $\frac{\Delta_\eta^n f(z)}{f(z)} = (e^{c_1\eta} - 1)^n$ and it shows that the quotient of $\Delta_\eta^n f(z)$ and $f(z)$ is related to η, n and c_1, but not related to c. On the other hand, Corollary 1.1 shows that if f has a nonzero finite Borel exceptional value b^*, then, for any constant η satisfying $\Delta_\eta^n f(z) \not\equiv 0$, the value b^* is not shared CM by $\Delta_\eta^n f(z)$ and $f(z)$. See the following example.

Example 1.1 Suppose that $f(z) = e^z + b^*$, where b^* is a nonzero finite value. Then f has a nonzero finite Borel exceptional value b^*. For any $\eta \neq 2k\pi i$, $k \in \mathbb{Z}$, the value b^* is not shared CM by $\Delta_\eta^n f(z)$ and $f(z)$. Observe that $\Delta_\eta^n f(z) = \sum_{j=0}^n (-1)^j C_n^j f(z + (n-j)\eta)$, where C_n^j are the binomial coefficients. Thus, for any $\eta \neq 2k\pi i$, $k \in \mathbb{Z}$, we have $\Delta_\eta^n f(z) = (e^\eta - 1)^n \cdot e^z$. Thus, we can see that $f(z) - b^* = e^z$ has no zero, but $\Delta_\eta^n f(z) - b^* = (e^\eta - 1)^n e^z - b^*$ has infinitely many zeros. Hence, the value b^* is not shared CM by $\Delta_\eta^n f(z)$ and $f(z)$.

In the special case, if we take $n = 1$ in Theorem 1.1 and $n = 1$ in Corollary 1.1, we can obtain the following corollaries.

Corollary 1.2 *Let $f(z)$ be a finite order transcendental entire function such that $\lambda(f - a(z)) < \sigma(f)$, where $a(z)$ is an entire function and satisfies $\sigma(a) < 1$. If $\Delta_\eta f(z) = f(z+\eta) - f(z)$*

and $f(z)$ share $a(z)$ CM, where η ($\in \mathbb{C}$) satisfies $f(z+\eta) \not\equiv f(z)$, then

$$a(z) \equiv 0 \quad \textit{and} \quad f(z) = ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

Corollary 1.3 *Let $f(z)$ be a finite order transcendental entire function which has a finite Borel exceptional value a. If $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share value a CM, where η ($\in \mathbb{C}$) satisfies $f(z+\eta) \not\equiv f(z)$, then*

$$a = 0 \quad \textit{and} \quad f(z) = ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

Remark 1.2 The Corollary 1.2 shows that if a nonzero polynomial $a(z)$ satisfies $\lambda(f-a) < \sigma(f)$, then $a(z)$ is not shared CM by $\Delta f(z)$ and $f(z)$. For example, if we take $a(z) \equiv z$, and $\lambda(f-z) < \sigma(f)$ holds, then $\Delta f(z)$ and $f(z)$ do not have any common fixed point (counting multiplicities). See the following example.

Example 1.2 Suppose that $f(z) = e^z + z$. Then $f(z)$ satisfies $\lambda(f(z) - z) = 0 < 1 = \sigma(f)$ and has no fixed point. But for any $\eta \neq 2k\pi i$, $k \in \mathbb{Z}$, the function $\Delta_\eta f(z) = f(z+\eta) - f(z) = (e^\eta - 1)e^z + \eta$ has infinitely many fixed points by Milloux's theorem (see [1, 3]). Hence, the nonzero polynomial $a(z) \equiv z$ is not shared CM by $\Delta_\eta f(z)$ and $f(z)$.

Remark 1.3 From Corollary 1.3, we can see that under the hypothesis of Theorem C, we can get the expression of $f(z)$, that is, $f(z) = ce^{c_1 z}$. Thus, we know that the constant A in Theorem C is related to η and c_1, but not related to c. Actually, from the proof of Lemma 2.9, we have $A = e^{c_1\eta} - 1$ (obviously, we can obtain $A \neq -1$). Hence, Corollary 1.3 contains and improves Theorem C. Obviously, Theorem 1.1 generalizes Theorem C.

2 Lemmas for the proof of theorems

Lemma 2.1 (See [21]) *Let f be a meromorphic function with a finite order σ, η be a nonzero constant. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset (1,\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0,1]$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \le \left|\frac{f(z+\eta)}{f(z)}\right| \le \exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 2.2 (See [11, 26]) *Suppose that $n \ge 2$ and let $f_1(z), \dots, f_n(z)$ be meromorphic functions and $g_1(z), \dots, g_n(z)$ be entire functions such that*

(i) $\sum_{j=1}^{n} f_j(z) \exp\{g_j(z)\} = 0$;

(ii) *when $1 \le j < k \le n$, $g_j(z) - g_k(z)$ is not constant*;

(iii) *when $1 \le j \le n$, $1 \le h < k \le n$,*

$$T(r,f_j) = o\{T(r, \exp\{g_h - g_k\})\} \quad (r \to \infty, r \notin E),$$

where $E \subset (1,\infty)$ has finite linear measure or logarithmic measure.

Then $f_j(z) \equiv 0$, $j = 1, \dots, n$.

ε**-set** Following Hayman [1], we define an ε-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set, then the set of $r \geq 1$, for which the circle $S(0,r)$ meets E, has finite logarithmic measure, and for almost all real θ, the intersection of E with the ray $\arg z = \theta$ is bounded.

Lemma 2.3 (See [4]) *Let f be a function transcendental and meromorphic in the plane of order < 1. Let $h > 0$. Then there exists an ε-set E such that*

$$f(z+c) - f(z) = cf'(z)\bigl(1 + o(1)\bigr) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leq h$.

Lemma 2.4 (See [25]) *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f),$$

where $U(z,f)$, $P(z,f)$, $Q(z,f)$ are difference polynomials such that the total degree $\deg U(z,f) = n$ in $f(z)$ and its shifts, and $\deg Q(z,f) \leq n$. Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then, for each $\varepsilon > 0$,

$$m\bigl(r, P(z,f)\bigr) = O\bigl(r^{\rho-1+\varepsilon}\bigr) + S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Remark 2.1 From the proof of Lemma 2.4 in [25], we can see that if the coefficients of $U(z,f)$, $P(z,f)$, $Q(z,f)$, namely $\alpha_\lambda(z)$, satisfy $m(r,\alpha_\lambda) = S(r,f)$, then the same conclusion still holds.

Lemma 2.5 (See [27]) *Let $P_n(z), \ldots, P_0(z)$ be polynomials such that $P_n P_0 \not\equiv 0$ and satisfy*

$$P_n(z) + \cdots + P_0(z) \not\equiv 0. \tag{2.1}$$

Then every finite order transcendental meromorphic solution $f(z)$ ($\not\equiv 0$) of the equation

$$P_n(z)f(z+n) + P_{n-1}(z)f(z+n-1) + \cdots + P_0(z)f(z) = 0 \tag{2.2}$$

satisfies $\sigma(f) \geq 1$, and $f(z)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often and $\lambda(f-a) = \sigma(f)$.

Remark 2.2 If equation (2.2) satisfies condition (2.1) and all $P_j(z)$ are constants, we can easily see that equation (2.2) does not possess any nonzero polynomial solution.

Lemma 2.6 (See [27]) *Let $F(z), P_n(z), \ldots, P_0(z)$ be polynomials such that $FP_nP_0 \not\equiv 0$. Then every finite order transcendental meromorphic solution $f(z)$ ($\not\equiv 0$) of the equation*

$$P_n(z)f(z+n) + P_{n-1}(z)f(z+n-1) + \cdots + P_0(z)f(z) = F \tag{2.3}$$

satisfies $\lambda(f) = \sigma(f) \geq 1$.

Remark 2.3 From the proof of Lemma 2.5 in [27], we can see that if we replace $f(z+j)$ by $f(z+j\eta)$ $(j=1,\dots,n)$ in equation (2.2) or (2.3), then the corresponding conclusion still holds.

Lemma 2.7 (See [4]) *Let f be a function transcendental and meromorphic in the plane which satisfies $\underline{\lim}_{r\to\infty}\frac{T(r,f)}{r}=0$. Then $g(z)=f(z+1)-f(z)$ and $G(z)=\frac{f(z+1)-f(z)}{f(z)}$ are both transcendental.*

Remark 2.4 From the proof of Lemma 2.7 in [4], we can see that, under the same hypotheses of Lemma 2.7, we can obtain the following conclusion: $\Delta_\eta f(z)=f(z+\eta)-f(z)$ and $G(z)=\frac{\Delta_\eta f(z)}{f(z)}=\frac{f(z+\eta)-f(z)}{f(z)}$ are both transcendental.

Lemma 2.8 *Let $f(z)=H(z)e^{c_1z}$, where $H(z)$ $(\not\equiv 0)$ is an entire function such that $\sigma(H)<1$ and c_1 is a nonzero constant. If $\Delta_\eta^n f(z)\not\equiv 0$ for some constant η, and*

$$\frac{\Delta_\eta^n f(z)}{f(z)}=A \tag{2.4}$$

holds, where A is a constant, then $H(z)$ is a constant.

Proof From $\Delta_\eta^n f(z)\not\equiv 0$, we can see that $A\neq 0$. In order to prove that $H(z)$ is a constant, we only need to prove $H'(z)\equiv 0$. Substituting $f(z)=H(z)e^{c_1z}$ into (2.4), we can obtain

$$\sum_{j=0}^{n-1}(-1)^jC_n^je^{(n-j)c_1\eta}H\bigl(z+(n-j)\eta\bigr)+\bigl((-1)^n-A\bigr)H(z)=0. \tag{2.5}$$

First, we assert that the sum of all coefficients of equation (2.5) is equal to zero, that is,

$$e^{nc_1\eta}-C_n^1e^{(n-1)c_1\eta}+\cdots+(-1)^{n-1}C_n^{n-1}e^{c_1\eta}+\bigl((-1)^n-A\bigr)=0. \tag{2.6}$$

On the contrary, we suppose that

$$e^{nc_1\eta}-C_n^1e^{(n-1)c_1\eta}+\cdots+(-1)^{n-1}C_n^{n-1}e^{c_1\eta}+\bigl((-1)^n-A\bigr)\neq 0.$$

Thus, applying Lemma 2.5 and Remarks 2.2-2.3 to (2.5), we have $\sigma(H)\geq 1$, a contradiction. Hence, (2.6) holds. Thus, by (2.6) and (2.5), we have

$$\sum_{j=0}^{n-1}(-1)^jC_n^je^{(n-j)c_1\eta}\bigl(H\bigl(z+(n-j)\eta\bigr)-H(z)\bigr)=0. \tag{2.7}$$

By Lemma 2.3, we see that there exists an ε-set E such that for $j=1,2,\dots,n$,

$$H(z+j\eta)-H(z)=j\eta H'(z)\bigl(1+o(1)\bigr)\quad \text{as } z\to\infty \text{ in } \mathbb{C}\backslash E. \tag{2.8}$$

Substituting (2.8) into (2.7), we can get

$$\eta H'(z)\cdot K+\eta H'(z)\cdot K\cdot o(1)=0\quad \text{as } z\to\infty \text{ in } \mathbb{C}\backslash E, \tag{2.9}$$

where K is a constant and satisfies

$$K = ne^{nc_1\eta} - C_n^1(n-1)e^{(n-1)c_1\eta} + \cdots + (-1)^{n-2}C_n^{n-2}2e^{2c_1\eta} + (-1)^{n-1}C_n^{n-1}e^{c_1\eta}.$$

Secondly, we assert that $K \not\equiv 0$. If $n = 1$, then $K = e^{c_1\eta} \not\equiv 0$; if $n \geq 2$, on the contrary, we suppose that $K = 0$. Then, for $j = 0, 1, \ldots, n-1$, noting that

$$C_n^j \cdot (n-j) = \frac{n! \cdot (n-j)}{(n-j)!j!} = \frac{(n-1)! \cdot n}{(n-1-j)!j!} = nC_{n-1}^j,$$

we have

$$\sum_{j=0}^{n-1}(-1)^j C_n^j(n-j)e^{(n-j)c_1\eta} = ne^{c_1\eta}\left(e^{c_1\eta} - 1\right)^{n-1} = 0.$$

Thus, we can obtain from the equality above that $e^{c_1\eta} = 1$ since $n - 1 \geq 1$. By (2.6) we have $A = (e^{c_1\eta} - 1)^n = 0$, which contradicts $A \not\equiv 0$. Hence $K \not\equiv 0$ and (2.9) implies $H'(z) \not\equiv 0$. Thus, we can know that $H(z)$ is a nonzero constant.

Thus, Lemma 2.8 is proved. □

Lemma 2.9 *Suppose that $f(z)$ is a finite order transcendental entire function such that $\lambda(f - a(z)) < \sigma(f)$, where $a(z)$ is an entire function and satisfies $\sigma(a) < 1$. Let n be a positive integer. If $\Delta_\eta^n f(z) \not\equiv 0$ for some constant η $(\in \mathbb{C})$, and*

$$\frac{\Delta_\eta^n f(z) - a(z)}{f(z) - a(z)} = A \tag{2.10}$$

holds, where A is a constant, then

$$a(z) \equiv 0, \qquad A \not\equiv 0 \quad \textit{and} \quad f(z) = ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

Proof Since $f(z)$ is a transcendental entire function of finite order and satisfies $\lambda(f - a(z)) < \sigma(f)$, we can write $f(z)$ in the form

$$f(z) = a(z) + H(z)e^{h(z)}, \tag{2.11}$$

where H $(\not\equiv 0)$ is an entire function, h is a polynomial with $\deg h = k$ $(k \geq 1)$, H and h satisfy

$$\lambda(H) = \sigma(H) = \lambda\left(f - a(z)\right) < \sigma(f) = \deg h. \tag{2.12}$$

First, we assert that $a(z) \equiv 0$. Substituting (2.11) into (2.10), we can get that

$$\frac{\Delta_\eta^n f(z) - a(z)}{f(z) - a(z)} = \frac{\sum_{j=0}^{n}(-1)^j C_n^j H(z + (n-j)\eta)e^{h(z+(n-j)\eta)} + b(z)}{H(z)e^{h(z)}} = A, \tag{2.13}$$

where $b(z) = \Delta_\eta^n a(z) - a(z)$. Rewrite (2.13) in the form

$$\sum_{j=0}^{n-1}(-1)^j C_n^j H\bigl(z+(n-j)\eta\bigr)e^{h(z+(n-j)\eta)-h(z)} + \bigl((-1)^n - A\bigr)H(z) = -b(z)e^{-h(z)}. \tag{2.14}$$

Suppose that $b(z) \not\equiv 0$. Then, from $\sigma(H(z+(n-j)\eta)) = \sigma(H(z)) < \deg h(z) = k$ ($j = 0,1,\ldots,n-1$), $\deg(h(z+(n-j)\eta) - h(z)) = k-1$ and $\sigma(b(z)) \le \sigma(a(z)) < 1 \le k$, we can see that the order of growth of the left-hand side of (2.14) is less than k, and the order of growth of the right-hand side of (2.14) is equal to k. This is a contradiction. Hence, $b(z) \equiv \Delta_\eta^n a(z) - a(z) \equiv 0$. Namely,

$$a(z+n\eta) - C_n^1 a\bigl(z+(n-1)\eta\bigr) + \cdots + (-1)^{n-1}C_n^{n-1}a(z+\eta) + \bigl((-1)^n - 1\bigr)a(z) = 0. \tag{2.15}$$

Suppose that $a(z) \not\equiv 0$. Note that the sum of all coefficients of (2.15) does not vanish. Then we can apply Lemma 2.5 and Remarks 2.2-2.3 to (2.15) and obtain $\sigma(a(z)) \ge 1$, which contradicts our hypothesis. Hence, $a(z) \equiv 0$. Thus, (2.13) can be rewritten as

$$\frac{\Delta_\eta^n f(z)}{f(z)} = \frac{\sum_{j=0}^{n}(-1)^j C_n^j H(z+(n-j)\eta)e^{h(z+(n-j)\eta)-h(z)}}{H(z)} = A. \tag{2.16}$$

Secondly, we prove that $A \ne 0$. In fact, if $A = 0$, we obtain from (2.16) that $\Delta_\eta^n f(z) \equiv 0$, which contradicts our hypothesis.

Thirdly, we prove that $\sigma(f) = k = 1$. On the contrary, we suppose that $\sigma(f) = k \ge 2$. Thus, we will deduce a contradiction for cases $A = (-1)^n$ and $A \ne (-1)^n$, respectively.

Case 1. Suppose that $A = (-1)^n$. Thus, for a positive integer n, there are three subcases: (1) $n = 1$; (2) $n = 2$; (3) $n \ge 3$.

Subcase 1.1. Suppose that $n = 1$. Then, by $A = -1$, we can obtain from (2.16) that

$$e^{h(z+\eta)-h(z)} = (1+A)\cdot\frac{H(z)}{H(z+\eta)} \equiv 0,$$

a contradiction.

Subcase 1.2. Suppose that $n = 2$. Then, by $A = (-1)^2 = 1$ and (2.16), we have

$$e^{h(z+2\eta)-h(z+\eta)} = \frac{2H(z+\eta)}{H(z+2\eta)}. \tag{2.17}$$

Set $Q_1(z) = \frac{2H(z+2\eta)}{H(z+\eta)}$. Then, from (2.17), we can know that $Q_1(z)$ is a nonconstant entire function. Set $\sigma(H) = \sigma_1$. Then $\sigma_1 < \sigma(f) = k$. By Lemma 2.1, we see that for any given ε_1 ($0 < 3\varepsilon_1 < k - \sigma_1$), there exists a set $E_1 \subset (1,\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\exp\bigl\{-r^{\sigma_1-1+\varepsilon_1}\bigr\} \le \left|\frac{2H(z+\eta)}{H(z+2\eta)}\right| \le \exp\bigl\{r^{\sigma_1-1+\varepsilon_1}\bigr\}. \tag{2.18}$$

Since $Q_1(z)$ is an entire function, by (2.18), we have

$$T\bigl(r, Q_1(z)\bigr) = m\bigl(r, Q_1(z)\bigr) \le m\left(r, \frac{2H(z+\eta)}{H(z+2\eta)}\right) + O(1) \le r^{\sigma_1-1+\varepsilon_1},$$

so that $\sigma(Q_1(z)) \leq \sigma_1 - 1 + \varepsilon_1 < k-1$. Thus, by $\deg(h(z+\eta) - h(z)) = k-1$ and $\sigma(Q_1) < k-1$, we can see that the order of growth of the left-hand side of (2.17) is equal to $k-1$, and the order of growth of the right-hand side of (2.17) is less than $k-1$. This is a contradiction.

Subcase 1.3. Suppose that $n \geq 3$. Then we can obtain from (2.16) that

$$\sum_{j=0}^{n-2} (-1)^j C_n^j \frac{H(z+(n-j)\eta)}{H(z+\eta)} e^{h(z+(n-j)\eta)-h(z+\eta)} + (-1)^{n-1} C_n^{n-1} = 0. \tag{2.19}$$

Set $Q_2(z) = e^{h(z+2\eta)-h(z+\eta)}$. Then $Q_2(z)$ is a transcendental entire function since $\sigma(Q_2(z)) = k-1 \geq 1$. For $j = 3, 4, \dots, n$, we have

$$e^{h(z+j\eta)-h(z+\eta)} = Q_2\bigl(z+(j-2)\eta\bigr) Q_2\bigl(z+(j-3)\eta\bigr) \cdots Q_2(z).$$

Thus, (2.19) can be rewritten as

$$U_2\bigl(z, Q_2(z)\bigr) \cdot Q_2(z) = (-1)^n C_n^{n-1}, \tag{2.20}$$

where

$$\begin{aligned} U_2\bigl(z, Q_2(z)\bigr) = {} & \frac{H(z+n\eta)}{H(z+\eta)} Q_2\bigl(z+(n-2)\eta\bigr) Q_2\bigl(z+(n-3)\eta\bigr) \cdots Q_2(z+\eta) \\ & - C_n^1 \frac{H(z+(n-1)\eta)}{H(z+\eta)} Q_2\bigl(z+(n-3)\eta\bigr) Q_2\bigl(z+(n-4)\eta\bigr) \cdots Q_2(z+\eta) \\ & + \cdots + (-1)^{n-2} C_n^{n-2} \frac{H(z+2\eta)}{H(z+\eta)}. \end{aligned}$$

Noting that $(-1)^n C_n^{n-1} \neq 0$, we can see that $U_2(z, Q_2(z)) \not\equiv 0$. Set $\sigma(H) = \sigma_2$. Then $\sigma_2 < k$. Since $Q_2(z)$ is of regular growth and $\sigma(Q_2(z)) = k-1$, for any given ε_2 $(0 < 3\varepsilon_2 < k - \sigma_2)$ and all $r > r_0$ (> 0), we have

$$T\bigl(r, Q_2(z)\bigr) > r^{k-1-\varepsilon_2}. \tag{2.21}$$

By Lemma 2.1, we see that for ε_2, there exists a set $E_2 \subset (1, \infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$, we have

$$\exp\bigl\{-r^{\sigma_2-1+\varepsilon_2}\bigr\} \leq \left| \frac{H(z+j\eta)}{H(z+\eta)} \right| \leq \exp\bigl\{r^{\sigma_2-1+\varepsilon_2}\bigr\} \quad (j = 2, 3, \dots, n). \tag{2.22}$$

Thus, from (2.21) and (2.22), we can get that for $j = 2, 3, \dots, n$,

$$\frac{m(r, \frac{H(z+j\eta)}{H(z+\eta)})}{T(r, Q_2(z))} \leq \frac{r^{\sigma_2-1+\varepsilon_2}}{r^{k-1-\varepsilon_2}} \to 0 \quad \bigl(r \to \infty \text{ and } r \notin [0,1] \cup E_2\bigr),$$

that is,

$$m\left(r, \frac{H(z+j\eta)}{H(z+\eta)}\right) = S(r, Q_2) \quad (j = 2, 3, \dots, n). \tag{2.23}$$

Noting that $\deg_{Q_2} U(z, Q_2) = n - 2 \geq 1$ and by Lemma 2.4 and Remark 2.1, we have

$$T(r, Q_2) = m(r, Q_2) = S(r, Q_2),$$

a contradiction.

Case 2. Suppose that $A \neq (-1)^n$. Thus, for a positive integer n, there are two subcases: (1) $n = 1$; (2) $n \geq 2$.

Subcase 2.1. Suppose that $n = 1$. Thus, (2.16) can be rewritten as

$$\frac{H(z+\eta)}{H(z)} = \bigl(A - (-1)^n\bigr)e^{h(z)-h(z+\eta)} = (A+1)e^{h(z)-h(z+\eta)}.$$

Noting the $A + 1 \neq 0$, we can use the same method as in the proof of Subcase 1.2 and deduce a contradiction.

Subcase 2.2. Suppose that $n \geq 2$. Then we can obtain from (2.16) that

$$\sum_{j=0}^{n-1}(-1)^j C_n^j \frac{H(z+(n-j)\eta)}{H(z)} e^{h(z+(n-j)\eta)-h(z)} + (-1)^n - A = 0. \tag{2.24}$$

Set $Q_3(z) = e^{h(z+\eta)-h(z)}$. Then $Q_3(z)$ is a transcendental entire function since $\sigma(Q_3(z)) = k - 1 \geq 1$. For $j = 1, 2, \ldots, n$, we have

$$e^{h(z+j\eta)-h(z)} = Q_3\bigl(z + (j-1)\eta\bigr)Q_3\bigl(z + (j-2)\eta\bigr)\cdots Q_3(z).$$

Thus, (2.24) can be rewritten as

$$U_3\bigl(z, Q_3(z)\bigr) \cdot Q_3(z) = A - (-1)^n, \tag{2.25}$$

where

$$\begin{aligned} U_3\bigl(z, Q_3(z)\bigr) = {}& \frac{H(z+n\eta)}{H(z)} Q_3\bigl(z+(n-1)\eta\bigr)Q_3\bigl(z+(n-2)\eta\bigr)\cdots Q_3(z+\eta) \\ & - C_n^1 \frac{H(z+(n-1)\eta)}{H(z)} Q_3\bigl(z+(n-2)\eta\bigr)Q_3\bigl(z+(n-3)\eta\bigr)\cdots Q_3(z+\eta) \\ & + \cdots + (-1)^{n-1}C_n^{n-1}\frac{H(z+\eta)}{H(z)}. \end{aligned}$$

We can see that $U_3(z, Q_3(z)) \not\equiv 0$ since $A - (-1)^n \neq 0$. Noting that $\deg_{Q_3} U_3(z, Q_3(z)) = n - 1 \geq 1$, we can use the same method as in the proof of Subcase 1.3 and deduce a contradiction.

Thus, we have proved that $\sigma(f) = k = 1$. And $f(z)$ can be written as

$$f(z) = H(z)e^{c_1 z + c_0} = H^*(z)e^{c_1 z}, \tag{2.26}$$

where c_0, c_1 ($\neq 0$) are two constants and $H^*(z) = e^{c_0}H(z)$ ($\not\equiv 0$) is an entire function and satisfies

$$\sigma\bigl(H^*(z)\bigr) = \lambda\bigl(H^*(z)\bigr) = \lambda(f) < \sigma(f) = 1. \tag{2.27}$$

Thus, by (2.26), (2.27), (2.16) and Lemma 2.8, we can get that $H^*(z)$ is a nonzero constant, and so, $f(z)$ can be written as

$$f(z) = ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

Thus, Lemma 2.9 is proved. □

Remark 2.5 From the proof of Lemma 2.9 or Remark 1.3, we can see that $A \neq -1$ in Lemma 2.9 when $n = 1$ and Theorem C. Unfortunately, we cannot obtain $A \neq (-1)^n$ when $n \geq 2$ in Lemma 2.9. This is because we can get a contradiction from the equality $e^{c_1\eta} - 1 = -1$, but we cannot obtain a contradiction from the equality $(e^{c_1\eta} - 1)^n = (-1)^n$ when $n \geq 2$.

3 Proof of Theorem 1.1

By the hypotheses of Theorem 1.1, we can write $f(z)$ in the form (2.11), and (2.12) holds. Since $\Delta_\eta^n f(z)$ and $f(z)$ share an entire function $a(z)$ CM, then

$$\frac{\Delta_\eta^n f(z) - a(z)}{f(z) - a(z)} = \frac{\sum_{j=0}^n (-1)^{n-j} C_n^j H(z + j\eta)e^{h(z+j\eta)} + b(z)}{H(z)e^{h(z)}} = e^{P(z)}, \tag{3.1}$$

where $P(z)$ is a polynomial and $b(z) = \Delta_\eta^n a(z) - a(z)$. Obviously, $\sigma(b(z)) \leq \sigma(a(z)) < 1$.

First step. We prove

$$\frac{\Delta_\eta^n f(z) - a(z)}{f(z) - a(z)} = A, \tag{3.2}$$

where A ($\neq 0$) is a constant. If $P(z) \equiv 0$, then, by (3.1), we see that (3.2) holds and $A = 1$.

Now suppose that $P(z) \not\equiv 0$ and $\deg P(z) = s$. Set

$$h(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0, \qquad P(z) = b_s z^s + b_{s-1} z^{s-1} + \cdots + b_0, \tag{3.3}$$

where $k = \sigma(f) \geq 1$, a_k ($\neq 0$), $a_{k-1}, \ldots, a_0$, b_s ($\neq 0$), $b_{s-1}, \ldots, b_0$ are constants. By (3.1), we can see that $0 \leq \deg P = s \leq \deg h = k$.

In this case, we prove that $P(z)$ is a constant, that is, $s = 0$. To this end, we will deduce a contradiction for the cases $s = k$ and $1 \leq s < k$, respectively.

Case 1. Suppose that $1 \leq s = k$. Thus, there are two subcases: (1) $a(z) \not\equiv 0$; (2) $a(z) \equiv 0$.

Subcase 1.1. Suppose that $a(z) \not\equiv 0$. First we suppose that $b_k \neq -a_k$. Then (3.1) is rewritten as

$$g_{11}(z)e^{P(z)} + g_{12}e^{-h(z)} + g_{13}e^{h_0(z)} = 0, \tag{3.4}$$

where $h_0(z) \equiv 0$ and

$$g_{11}(z) = -H(z); \qquad g_{12}(z) = b(z); \qquad g_{13}(z) = \sum_{j=0}^n (-1)^{n-j} C_n^j H(z + j\eta)e^{h(z+j\eta)-h(z)}.$$

Since $\sigma(H) < k$, $\sigma(b) < 1 \leq k$ and $\deg(h(z+j\eta) - h(z)) = k-1$ ($j = 1, 2, \ldots, n$), we can see that $\sigma(g_{1m}(z)) < k$ ($m = 1, 2, 3$). On the other hand, by $b_k \neq -a_k$, we can see that $\deg(P - (-h)) = \deg(P - h_0) = \deg(-h - h_0) = k$. Since $e^{P-(-h)}$, e^{P-h_0} and e^{-h-h_0} are of regular growth, and $\sigma(g_{1m}) < k$ ($m = 1, 2, 3$), we can see that for $m = 1, 2, 3$,

$$T(r, g_{1m}) = o\bigl(T\bigl(r, e^{P-(-h)}\bigr)\bigr) = o\bigl(T\bigl(r, e^{P-h_0}\bigr)\bigr) = o\bigl(T\bigl(r, e^{-h-h_0}\bigr)\bigr). \tag{3.5}$$

Thus, applying Lemma 2.2 to (3.4), by (3.5), we can obtain $g_{1m}(z) \equiv 0$ ($m = 1, 2, 3$). Clearly, this is a contradiction.

Now we suppose that $b_k = -a_k$. Then (3.1) is rewritten as

$$\bigl[H(z)e^{P(z)+h(z)} - b(z)\bigr]e^{-h(z)} = \sum_{j=0}^{n}(-1)^{n-j}C_n^j H(z+j\eta)e^{h(z+j\eta)-h(z)}. \tag{3.6}$$

We affirm that $H(z)e^{P(z)+h(z)} - b(z) \not\equiv 0$. In fact, if $H(z)e^{P(z)+h(z)} - b(z) \equiv 0$, then, by (3.6), we can obtain

$$\sum_{j=1}^{n}(-1)^{n-j}C_n^j H(z+j\eta)e^{h(z+j\eta)-h(z)} + (-1)^n H(z) \equiv 0, \tag{3.7}$$

this is the special case of (2.14) when $b(z) \equiv 0$ and $A = 0$. Hence, using the same method as in the proof of Case 2 in the proof of Lemma 2.9, we can get that $\sigma(f) = k = 1$. Hence, substituting $h(z) = c_1 z + c_0$ into (3.7), we have

$$\sum_{j=0}^{n}(-1)^j C_n^j e^{(n-j)c_1\eta}H\bigl(z + (n-j)\eta\bigr) = 0. \tag{3.8}$$

On this occasion, we assert that $(e^{c_1\eta} - 1)^n = 0$. On the contrary, we suppose that $(e^{c_1\eta} - 1)^n \neq 0$. Then the sum of all coefficients of (3.8) is $(e^{\eta} - 1)^n$, which does not vanish. By Lemma 2.5 and Remarks 2.2-2.3, we have $\sigma(H) \geq 1$, a contradiction. Hence, $(e^{c_1\eta} - 1)^n = 0$. Thus, $e^{c_1\eta} = 1$. Substituting it into (3.8), we have

$$\sum_{j=0}^{n}(-1)^j C_n^j H\bigl(z + (n-j)\eta\bigr) = 0. \tag{3.9}$$

First, we suppose that $H(z)$ is transcendental. Then, noting that $\sigma(H) < 1$ implies $\underline{\lim}_{r\to\infty}\frac{T(r,H)}{r} = 0$, by Lemma 2.7 and Remark 2.4, we know that $\Delta_\eta H(z) = H(z+\eta) - H(z)$ is transcendental. Moreover, $\sigma(\Delta_\eta H(z)) \leq \sigma(H(z)) < 1$ implies $\underline{\lim}_{r\to\infty}\frac{T(r,\Delta_\eta H)}{r} = 0$. Repeating the process above $n-1$ times, we can see that $\Delta_\eta^n H(z)$ is transcendental. That is, the left-hand side of (3.9) is a transcendental function. Hence (3.9) is impossible.

Secondly, we suppose that $H(z)$ is a nonzero polynomial. Then, noting that $b_k = -a_k$, we can see that $e^{p(z)+h(z)}$ is a nonzero constant. Thus, from $b(z) = H(z)e^{p(z)+h(z)}$, we can know that $b(z)$ is a nonzero polynomial. Thus, applying Lemma 2.6 to the equation $\Delta_\eta^n a(z) - a(z) = b(z)$ and by Remark 2.3, we have $\sigma(a) \geq 1$, a contradiction. Hence, $H(z)e^{P(z)+h(z)} - b(z) \not\equiv 0$. Thus, since $\deg(P+h) \leq k-1$, $\deg(-h) = k$, $\deg(h(z+j\eta) - h(z)) = k-1$ ($j = 1, 2, \ldots, n$) and $\sigma(H) < k$, we see that the order of growth of the left-hand side of

(3.6) is equal to k, and the order of growth of the right-hand side of (3.6) is less than k. This is a contradiction.

Subcase 1.2. Suppose that $a(z) \equiv 0$. Then (3.1) is rewritten as

$$H(z)e^{P(z)} = \sum_{j=0}^{n}(-1)^{n-j}C_n^j H(z+j\eta)e^{h(z+j\eta)-h(z)}. \tag{3.10}$$

Since $H(z) \not\equiv 0$, $\sigma(H) < k$, $\deg P = s = k$ and $\deg(h(z+j\eta)-h(z)) = k-1$ ($j = 1,2,\dots,n$), we can see that the order of growth of the left-hand side of (3.10) is equal to k, and the order of growth of the right-hand side of (3.10) is less than k. This is a contradiction.

Case 2. Suppose that $1 \le s < k$. Thus, there are two subcases: (1) $a(z) \not\equiv 0$; (2) $a(z) \equiv 0$.

Subcase 2.1. Suppose that $a(z) \not\equiv 0$. Then, by (3.1), we can obtain

$$\sum_{j=0}^{n}(-1)^{n-j}C_n^j H(z+j\eta)e^{h(z+j\eta)-h(z)} - H(z)e^{P(z)} = b(z)e^{-h(z)}. \tag{3.11}$$

We assert that $b(z) \not\equiv 0$. In fact, if $b(z) \equiv 0$, then (2.15) obviously holds. Hence, using the same method as in the proof of Lemma 2.9, by Lemma 2.5 and Remarks 2.2-2.3, we can get that $\sigma(a) \ge 1$, a contradiction. Hence, $b(z) \not\equiv 0$. Since $\deg h = k$, $\deg(h(z+j\eta)-h(z)) = k-1$ ($j = 1,2,\dots,n$), $\deg P = s < k$ and $\sigma(H) < k$, we see that the order of growth of the left-hand side of (3.11) is less than k, and the order of growth of the right-hand side of (3.11) is equal to k. This is a contradiction.

Subcase 2.2. Suppose that $a(z) \equiv 0$. Then, by (3.1), we obtain

$$\sum_{j=1}^{n}(-1)^{n-j}C_n^j \frac{H(z+j\eta)}{H(z)}e^{h(z+j\eta)-h(z)} + (-1)^n = e^{P(z)}. \tag{3.12}$$

Thus, there are two subcases: (1) $n = 1$; (2) $n \ge 2$.

Subcase 2.2.1. Suppose that $n = 1$. Then (3.12) can be rewritten as

$$\frac{H(z+\eta)}{H(z)}e^{h(z+\eta)-h(z)} - 1 = e^{P(z)}. \tag{3.13}$$

By (3.13), we see that $\frac{H(z+\eta)}{H(z)}$ is a nonzero entire function. Set $\sigma(H) = \sigma_4$. Then $\sigma_4 < \sigma(f) = k$. By Lemma 2.1, we see that for any given ε_4 ($0 < 3\varepsilon_4 < k - \sigma_4$), there exists a set $E_4 \subset (1,\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_4$, we have

$$\exp\{-r^{\sigma_4-1+\varepsilon_4}\} \le \left|\frac{H(z+\eta)}{H(z)}\right| \le \exp\{r^{\sigma_4-1+\varepsilon_4}\}. \tag{3.14}$$

Since $\frac{H(z+\eta)}{H(z)}$ is an entire function, by (3.13), we have

$$T\left(r, \frac{H(z+\eta)}{H(z)}\right) = m\left(r, \frac{H(z+\eta)}{H(z)}\right) \le r^{\sigma_4-1+\varepsilon_4},$$

so that

$$\sigma\left(\frac{H(z+\eta)}{H(z)}\right) \le \sigma_4 - 1 + \varepsilon_4 < k - 1. \tag{3.15}$$

Since $s < k$, we can see that $\deg P \leq k-1$. If $\deg P < k-1$, then, by (3.15) and $\deg(h(z+\eta) - h(z)) = k-1$, we can see that the order of growth of the left-hand side of (3.13) is equal to $k-1$, and the order of growth of the right-hand side of (3.13) is equal to $\deg P$ which is less than $k-1$. This is a contradiction.

If $\deg P = k-1$, then since $\frac{H(z+\eta)}{H(z)}$ is an entire function and $\deg(h(z+\eta)-h(z)) = k-1 \geq 1$, by (3.15), we can see that the entire function $\frac{H(z+\eta)}{H(z)}e^{h(z+\eta)-h(z)}$ has a Borel exceptional value 0, thus the value 1 must be not its Borel exceptional value. Hence, the left-hand side of (3.13), $\frac{H(z+\eta)}{H(z)}e^{h(z+\eta)-h(z)} - 1$, has infinitely many zeros, but the right-hand side of (3.13), $e^{P(z)}$, has no zero. This is a contradiction.

Subcase 2.2.2. Now we suppose that $n \geq 2$. Thus, for s (= $\deg P$), there are two subcases: (1) $s < k-1$; (2) $s = k-1$.

Subcase 2.2.2.1. Now we suppose that $s < k-1$. Set $Q_5(z) = e^{h(z+\eta)-h(z)}$. Since $\sigma(Q_5) = k-1 \geq 1$, $Q_5(z)$ is a transcendental entire function. Thus, (3.12) can be rewritten as

$$U_5\bigl(z, Q_5(z)\bigr) \cdot Q_5(z) = e^{P(z)} - (-1)^n, \tag{3.16}$$

where

$$\begin{aligned} U_5\bigl(z, Q_5(z)\bigr) = {} & \frac{H(z+n\eta)}{H(z)} Q_5\bigl(z+(n-1)\eta\bigr) Q_5\bigl(z+(n-2)\eta\bigr) \cdots Q_5(z+\eta) \\ & - C_n^1 \frac{H(z+(n-1)\eta)}{H(z)} Q_5\bigl(z+(n-2)\eta\bigr) Q_5\bigl(z+(n-3)\eta\bigr) \cdots Q_5(z+\eta) \\ & + \cdots + (-1)^{n-1} C_n^{n-1} \frac{H(z+\eta)}{H(z)}. \end{aligned} \tag{3.17}$$

Thus, using the same method as in the proof of Subcase 1.3 in the proof of Lemma 2.9 and noting that $\sigma(e^{P(z)} - (-1)^n) = \deg P < k-1$, we have

$$m\bigl(r, e^{P(z)} - (-1)^n\bigr) = S(r, Q_5).$$

Noting that $n \geq 2$ and so $\deg U_5(z, Q_5) = n-1 \geq 1$. Using the same method as in the proof of Subcase 1.3 in the proof of Lemma 2.9, we can obtain

$$T(r, Q_5) = m(r, Q_5) = S(r, Q_5).$$

Clearly, this is a contradiction.

Subcase 2.2.2.2. Now we suppose that $s = k-1$. Thus, (3.12) is written as

$$\sum_{j=1}^{n} (-1)^{n-j} C_n^j \frac{H(z+j\eta)}{H(z)} e^{T_j(z)} + (-1)^n - e^{P(z)} = 0, \tag{3.18}$$

where $T_j(z) = h(z+j\eta) - h(z)$ ($j = 1, 2, \ldots, n$). Thus, by (3.3), we have

$$T_j(z) = jk\eta a_k z^{k-1} + P_{k-2,j}(z), \tag{3.19}$$

where $P_{k-2,j}(z)$ is a polynomial with degree at most $k-2$. Thus, we have

$$T_j(z) - T_t(z) = (j-t)k\eta a_k z^{k-1} + P_{j,t}(z) \quad (1 \leq j \neq t \leq n),$$

where $P_{j,t}(z)$ is a polynomial with degree at most $k-2$.

First, we suppose that there is some j_0 $(1 \le j_0 \le n)$ such that $j_0 k\eta a_k = b_{k-1}$, that is, $\deg(T_{j_0}(z) - P(z)) \le k-2$. Thus, (3.18) can be written as

$$\sum_{1\le j\le n, j\ne j_0} (-1)^{n-j} C_n^j \frac{H(z+j\eta)}{H(z)} e^{h(z+j\eta)-h(z)} + B_{j_0}(z) e^{h(z+j_0\eta)-h(z)} = (-1)^{n+1}, \tag{3.20}$$

where

$$B_{j_0}(z) = (-1)^{n-j_0} C_n^{n-j_0} \frac{H(z+j_0\eta)}{H(z)} - e^{P(z)+h(z)-h(z+j_0\eta)}.$$

Set $Q_6(z) = e^{h(z+\eta)-h(z)}$ and $\sigma(H) = \sigma_6$. Then (3.20) can be rewritten as

$$U_6\bigl(z, Q_6(z)\bigr) \cdot Q_6(z) = (-1)^{n+1}, \tag{3.21}$$

where

$$\begin{aligned} &U_6\bigl(z, Q_6(z)\bigr) \\ &\quad = \sum_{1\le j\le n, j\ne j_0} (-1)^{n-j} C_n^{n-j} \frac{H(z+j\eta)}{H(z)} Q_6\bigl(z+(j-1)\eta\bigr) Q_6\bigl(z+(j-2)\eta\bigr) \cdots Q_6(z+\eta) \\ &\qquad + B_{j_0}(z) Q_6\bigl(z+(j_0-1)\eta\bigr) Q_6\bigl(z+(j_0-2)\eta\bigr) \cdots Q_6(z+\eta) \quad (j_0 \ge 2), \end{aligned} \tag{3.22}$$

or

$$\begin{aligned} &U_6\bigl(z, Q_6(z)\bigr) \\ &\quad = \sum_{2\le j\le n} (-1)^{n-j} C_n^{n-j} \frac{H(z+j\eta)}{H(z)} Q_6\bigl(z+(j-1)\eta\bigr) Q_6\bigl(z+(j-2)\eta\bigr) \cdots Q_6(z+\eta) \\ &\qquad + B_{j_0}(z) \quad (j_0 = 1). \end{aligned} \tag{3.23}$$

Noting that $(-1)^{n+1} \ne 0$, we can see that $U_6(z, Q_6(z)) \not\equiv 0$. Since $\sigma(H) < k$ and $\sigma(e^{P(z)+h(z)-h(z+j_0\eta)}) \le k-2 < k-1$, using the same method as in the proof of Subcase 1.3 in the proof of Lemma 2.9, we have

$$m\bigl(r, B_{j_0}(z)\bigr) = S(r, Q_6). \tag{3.24}$$

Noting that $n \ge 2$ and so $\deg U_6(z, Q_6) = n-1 \ge 1$. Combining (3.21)-(3.24), using the same method as in the proof of Subcase 1.3 in the proof of Lemma 2.9, we can obtain

$$T(r, Q_6) = m(r, Q_6) = S(r, Q_6).$$

Clearly, this is a contradiction.

Secondly, we suppose that $jk\eta a_k \ne b_{k-1}$ for any $1 \le j \le n$. Thus, equation (3.18) can be rewritten as

$$e^{P(z)} = e^{b_{k-1}z^{k-1}} \cdot e^{P_{k-2}(z)} = \sum_{j=0}^{n} (-1)^{n-j} C_n^j \frac{H(z+j\eta)}{H(z)} e^{h(z+j\eta)-h(z)}, \tag{3.25}$$

where $P_{k-2}(z) = P(z) - b_{k-1}z^{k-1} = b_{k-2}z^{k-2} + b_{k-3}z^{k-3} + \cdots + b_0$. For dealing with equation (3.25), we just compare $|b_{k-1}|$ with $nk|\eta a_k|$ since $nk|\eta a_k| > (n-1)k|\eta a_k| > \cdots > k|\eta a_k|$. Without loss of generality, we suppose that $nk|\eta a_k| \le |b_{k-1}|$. Let $\arg b_{k-1} = \theta_1$, $\arg(\eta a_k) = \theta_2$ and $\sigma(H) = \sigma_7 < k$. Take θ_0 such that $\cos((k-1)\theta_0 + \theta_1) = 1$. By Lemma 2.1, we see that for any given ε_7 $(0 < 3\varepsilon_7 < k - \sigma_7)$, there exists a set $E_7 \subset (1,\infty)$ of finite logarithmic measure such that for all $z = re^{i\theta_0}$ satisfying $|z| = r \notin [0,1] \cup E_7$, we have

$$\exp\{-r^{\sigma_7-1+\varepsilon_7}\} \le \left|\frac{H(z+j\eta)}{H(z)}\right| \le \exp\{r^{\sigma_7-1+\varepsilon_7}\} \quad (j = 1,\ldots,n). \tag{3.26}$$

Thus, noting that $e^{P_{k-2}(z)}$ is of regular growth, we can deduce from (3.25) and (3.26) that

$$\begin{aligned} \left|e^{b_{k-1}z^{k-1}}\right| &= \left|\frac{e^{P(z)}}{e^{P_{k-2}(z)}}\right| \\ &\le \frac{|\sum_{j=0}^{n}(-1)^j C_n^j \frac{H(z+(n-j)\eta)}{H(z)} e^{h(z+(n-j)\eta)-h(z)}|}{|e^{b_{k-2}z^{k-2}+b_{k-3}z^{k-3}+\cdots+b_0}|} \\ &\le \frac{(n+1)n!\exp\{r^{\sigma_7-1+\varepsilon_7}\}\exp\{nk|\eta a_k|\cos((k-1)\theta_0+\theta_2)r^{k-1} + O(r^{k-2})\}}{\exp\{\frac{|b_{k-2}|}{2}r^{k-2}\}}, \end{aligned}$$

that is,

$$\begin{aligned} &\exp\{|b_{k-1}|r^{k-1}\} \\ &\quad \le \exp\left\{nk|\eta a_k|\cos\bigl((k-1)\theta_0+\theta_2\bigr)r^{k-1} + r^{\sigma_7-1+\varepsilon_7} + O\bigl(r^{k-2}\bigr) - \frac{|b_{k-2}|}{2}r^{k-2}\right\} \\ &\quad \le \exp\{nk|\eta a_k|\cos\bigl((k-1)\theta_0+\theta_2\bigr)r^{k-1} + o\bigl(r^{k-1}\bigr)\}. \end{aligned} \tag{3.27}$$

We assert that

$$nk|\eta a_k|\cos\bigl((k-1)\theta_0+\theta_2\bigr) < |b_{k-1}|.$$

In fact, if $nk|\eta a_k| = |b_{k-1}|$, then, by $b_{k-1} \not= nk\eta a_k$, we know that $\cos((k-1)\theta_0+\theta_2) \not= 1$, that is, $\cos((k-1)\theta_0+\theta_2) < 1$, and hence $nk|\eta a_k|\cos((k-1)\theta_0+\theta_2) < nk|\eta a_k| = |b_{k-1}|$. If $nk|\eta a_k| < |b_{k-1}|$, then we have $nk|\eta a_k|\cos((k-1)\theta_0+\theta_2) \le nk|\eta a_k| < |b_{k-1}|$.

Thus, taking a positive constant ε_8 $(0 < \varepsilon_8 < \frac{|b_{k-1}|-nk|\eta a_k|\cos((k-1)\theta_0+\theta_2)}{3})$, we can deduce from (3.27) that

$$\begin{aligned} \exp\{|b_{k-1}|r^{k-1}\} &\le \exp\{nk|\eta a_k|\cos\bigl((k-1)\theta_0+\theta_2\bigr)r^{k-1} + o\bigl(r^{k-1}\bigr)\} \\ &\le \exp\{\bigl(|b_{k-1}| - \varepsilon_8\bigr)r^{k-1}\}, \end{aligned}$$

a contradiction. Thus, we have proved that P is only a constant and (3.2) holds.

Second step. Applying Lemma 2.9 to (3.2), we can obtain the conclusion.

Thus, Theorem 1.1 is proved.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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