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A new Alon-Babai-Suzuki-type inequality on set systems

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Abstract

Let $K = \{k_1, k_2, \ldots, k_r\}$ and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be two sets of nonnegative integers. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that $|F| \in K$ for every $F \in \mathcal{F}$ and $|E \cap F| \in \mathcal{L}$ for every pair of distinct subsets $E, F \in \mathcal{F}$. We prove that

$$|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$$

when we have the conditions that $K \cap \mathcal{L} = \emptyset$ and $n \ge s + \max k_i$. This result gives an improvement for Alon, Babai, and Suzuki's conjecture under the nonmodular. This paper gets an improvement of a theorem of Hwang *et al.* with $K \cap \mathcal{L} = \emptyset$.

MSC: 05D05

Keywords: Alon-Babai-Suzuki inequalities; Frankl-Ray-Chaudhuri-Wilson theorems; multilinear polynomials

1 Introduction

In this paper, let X denote the set $[n] = \{1, 2, \ldots, n\}$ and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. A family $\mathcal{F}$ of subsets of X is called $\mathcal{L}$-*intersecting* if $|E \cap F| \in \mathcal{L}$ for every pair of distinct subsets $E, F \in \mathcal{F}$. A family $\mathcal{F}$ is k-*uniform* if it is a collection of k-subsets of X.

In 1975, Ray-Chaudhuri and Wilson proved the following fundamental result.

Theorem 1.1 (Ray-Chaudhuri and Wilson [1]) *If $\mathcal{F}$ is a k-uniform $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \le \binom{n}{s}.$$

In terms of the parameters n and s, this inequality is best possible, as shown by the set of all s-subsets of X with $\mathcal{L} = \{0, 1, \ldots, s-1\}$. A nonuniform version of Theorem 1.1 was proved by Frankl and Wilson in 1981.

Theorem 1.2 (Frankl and Wilson [2]) *If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

This result is again best possible in terms of the parameters n and s, as shown by the family of all subsets of size at most s of X with $\mathcal{L} = \{0, 1, \dots, s-1\}$.

In 1991, Alon, Babai, and Suzuki proved the following inequality, which is a generalization of the well-known Frankl-Ray-Chaudhuri-Wilson theorems (Theorem 1.1 and Theorem 1.2).

Theorem 1.3 (Alon, Babai, and Suzuki [3]) *Let $K = \{k_1, k_2, \dots, k_r\}$, $\mathcal{L} = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of X such that $|F| \in K$ for every $F \in \mathcal{F}$. If $\min k_i > s - r$, then*

$$|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

Note that it is best possible in terms of the parameters n, r, and s, as shown by the set of all subsets of X with size at least $s - r + 1$ and at most s, and $\mathcal{L} = \{0, 1, \dots, s-1\}$.

Since then, many Alon-Babai-Suzuki-type inequalities have been proved. Below is a list of related results in this field obtained by others.

Theorem 1.4 *Let $K = \{k_1, k_2, \dots, k_r\}$, $\mathcal{L} = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of X such that $|F| \in K$ for every $F \in \mathcal{F}$. If $K \cap \mathcal{L} = \emptyset$, then*

$$|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}.$$

Theorem 1.5 (Snevily [4]) *Let $K = \{k_1, k_2, \dots, k_r\}$, $\mathcal{L} = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of X such that $|F| \in K$ for every $F \in \mathcal{F}$. If $\min k_i > \max l_j$, then*

$$|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

Theorem 1.6 (Hwang and Sheikh [5]) *Let $K = \{k, k+1, \dots, k+r-1\}$, $\mathcal{L} = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of X such that $|F| \in K$ for every $F \in \mathcal{F}$. If $K \cap \mathcal{L} = \emptyset$ and $k > s - r$, then*

$$|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}.$$

Theorem 1.7 (Hwang and Sheikh [5]) *Let $K = \{k_1, k_2, \dots, k_r\}$, $\mathcal{L} = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of X such that $|F| \in K$ for every $F \in \mathcal{F}$. If $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$ and $\min k_i > s - r$, then*

$$|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}.$$

Conjecture 1.8 (Alon, Babai, and Suzuki [3]) *Let K and $\mathcal{L}$ be subsets of $\{0, 1, \dots, p-1\}$ such that $K \cap \mathcal{L} = \emptyset$, where p is a prime and $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ be a family of subsets of $[n]$ such that $|F_i| \pmod{p} \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \pmod{p} \in \mathcal{L}$ for $i \neq j$. If $n \ge s + \max k_i$ for every i, then $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

In the following paper, they prove the above conjecture in the nonmodular.

Theorem 1.9 (Hwang *et al.* [6]) *Let $K = \{k_1, k_2, \dots, k_r\}$, $\mathcal{L} = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of X such that $|F| \in K$ for every $F \in \mathcal{F}$. If $n \geq s + \max k_i$, then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

In this paper, we will prove the following Alon-Babai-Suzuki-type inequality which gives an improvement for Theorem 1.9 with $K \cap \mathcal{L} = \emptyset$ and for Alon, Babai and Suzuki's conjecture in the nonmodular.

Theorem 1.10 *Let $K = \{k_1, k_2, \dots, k_r\}$, $\mathcal{L} = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of X such that $|F| \in K$ for every $F \in \mathcal{F}$. If $K \cap \mathcal{L} = \emptyset$ and $n \geq s + \max k_i$, then*

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

We note that in some cases the conditions $K \cap \mathcal{L} = \emptyset$ and $n \geq s + \max k_i$ in Theorem 1.10 holds, but Snevily's condition $\min k_i > \max l_j$ in Theorem 1.5 does not. For instance, if $n = 12$, $K = \{3, 6\}$ and $\mathcal{L} = \{0, 1, 2, 4, 5\}$, then it is clear that $K \cap \mathcal{L} = \emptyset$ and $s + \max k_i = 5 + 6 < 12 = n$, but $\min k_i = 3 < 5 = \max l_j$.

If the condition $K \cap \mathcal{L} = \emptyset$ in Theorem 1.10 is replaced by $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$, we have the following result which gives a better bound than Theorem 1.10 and Theorem 1.7.

Theorem 1.11 *Let $K = \{k_1, k_2, \dots, k_r\}$, $\mathcal{L} = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of X such that $|F| \in K$ for every $F \in \mathcal{F}$. If $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$ and $n \geq s + \max k_i$, then*

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r+1}.$$

2 Proofs of theorems

In this section, we will give a proof for Theorem 1.10, which is motivated by the methods used in [3–7].

Proof of Theorem 1.10 Let $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$. We may assume (after relabeling) that for $1 \leq i \leq t$, $n \in F_i$ and that for $i > t$, $n \notin F_i$. With each set $F_i \in \mathcal{F}$, we associate its characteristic vector $v_i = (v_{i_1}, \dots, v_{i_n}) \in \mathbb{R}^n$, where $v_{i_j} = 1$ if $j \in F_i$ and $v_{i_j} = 0$ otherwise. Let $\overline{v_i}$ be the characteristic vector of F_i^c.

Recall that a polynomial in n variables is multilinear if its degree in each variable is at most 1. Let us restrict the domain of the polynomials we will work with to the n-cube $\Omega = \{0, 1\}^n \subseteq \mathbb{R}^n$. Since in this domain $x_i{}^2 = x_i$ for each variable, every polynomial in our proof is multilinear.

For each $F_i \in \mathcal{F}$, define

$$f_i(x) = \prod_{l_j \in \mathcal{L}} \bigl(v_i \cdot x - \bigl(|F_i| - l_j\bigr)\bigr).$$

Then $f_i(\overline{v_i}) \neq 0$ for every $1 \leq i \leq m$ and $f_i(\overline{v_j}) = 0$ for $i \neq j$, since $K \cap \mathcal{L} = \emptyset$. Thus, $\{f_i(x)|1 \leq i \leq m\}$ is a linearly independent family.

Let $\mathcal{G} = \{G_1, \ldots, G_p\}$ be the family of subsets of X with size at most s that contain n, which is ordered by size, that is, $|G_i| \leq |G_j|$ if $i < j$, where $p = \sum_{i=0}^{s-1} \binom{n-1}{i}$. Let u_i denote the characteristic vector of G_i. For $i = 1, \ldots, p$, we define

$$g_i(x) = \prod_{j \in G_i} x_j.$$

Since $g_i(u_i) \neq 0$ for every $1 \leq i \leq p$ and $g_i(u_j) = 0$ for any $j < i$, $\{g_i(x)|1 \leq i \leq p\}$ is a linearly independent family.

Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be the family of subsets of $[n] - \{n\}$ with size at most $s - 2r$, where $q = \sum_{i=0}^{s-2r} \binom{n-1}{i}$. We order the members of $\mathcal{H}$ such that $|H_i| \leq |H_j|$ if $i < j$. Let w_i be the characteristic vector of H_i.

Let $W = \{n - k_i - 1 | k_i \in K\} \cup \{n - k_i | k_i \in K\}$. Then $|W| \leq 2r$. Set

$$f(x) = \prod_{h \in W} \left(\sum_{j=1}^{n-1} x_j - h \right).$$

For $i = 1, \ldots, q$, define

$$h_i(x) = f(x) \prod_{j \in H_i} x_j.$$

Note that $h_i(w_j) = 0$ for any $j > i$ and $h_i(w_i) \neq 0$ for every $1 \leq i \leq q$ since $n \geq s + \max k_i$, and thus $\{h_i(x)|1 \leq i \leq q\}$ is a linearly independent family.

We will show that the polynomials in

$$\{f_i(x)|1 \leq i \leq m\} \cup \{g_i(x)|1 \leq i \leq p\} \cup \{h_i(x)|1 \leq i \leq q\}$$

are linearly independent. Suppose that we have a linear combination of these polynomials that equals zero:

$$\sum_{i=1}^{m} \alpha_i f_i(x) + \sum_{i=1}^{p} \beta_i g_i(x) + \sum_{i=1}^{q} \gamma_i h_i(x) = 0. \tag{2.1}$$

We will prove that the coefficients must be zero. First substitute the characteristic vector $\overline{v_i}$ of F_i^c with $n \in F_i$ into equation (2.1). Since $n \in G_j$, $g_j(\overline{v_i}) = 0$ for every $1 \leq j \leq p$. Note that we have $h_j(\overline{v_i}) = 0$ for every $1 \leq j \leq q$. Recall that $f_j(\overline{v_i}) = 0$ for $j \neq i$, we obtain $\alpha_i f_i(\overline{v_i}) = 0$. Thus, $\alpha_i = 0$ for every $1 \leq i \leq t$, since $f_i(\overline{v_i}) \neq 0$. It follows that

$$\sum_{i=t+1}^{m} \alpha_i f_i(x) + \sum_{i=1}^{p} \beta_i g_i(x) + \sum_{i=1}^{q} \gamma_i h_i(x) = 0. \tag{2.2}$$

Then we substitute the characteristic vector $\overline{v_i}^\star$ of $F_i^c - \{n\}$ with $n \notin F_i$ into equation (2.2). For every $1 \leq j \leq q$ and $1 \leq k \leq p$, $h_j(\overline{v_i}^\star) = 0$, $g_k(\overline{v_i}^\star) = 0$. Note that $f_j(\overline{v_i}^\star) = 0$ for $j \neq i$, we obtain $\alpha_i f_i(\overline{v_i}^\star) = 0$. Since $f_i(\overline{v_i}^\star) \neq 0$, $\alpha_i = 0$ for every $t+1 \le i \le m$. Therefore, equation (2.2) reduces to

$$\sum_{i=1}^{p} \beta_i g_i(x) + \sum_{i=1}^{q} \gamma_i h_i(x) = 0. \tag{2.3}$$

First, we substitute the characteristic vector w_i of H_i with the smallest size into equation (2.3). We follow the same process to substitute the characteristic vector w_i of H_i with the smallest size after deleting first H_i. Note that $h_i(w_j) = 0$ for any $j > i$ and $h_i(w_i) \neq 0$ for every $1 \le i \le q$, since H_i does not contain n and G_i contains n, $g_k(w_i) = 0$ for $1 \le k \le p$. Thus, we reduce (2.3) to

$$\sum_{i=1}^{p} \beta_i g_i(x) = 0. \tag{2.4}$$

We prove that $\{g_i(x) | 1 \le i \le p\}$ is already a linearly independent family. To complete the proof, simply note that each polynomial in $\{f_i(x) | 1 \le i \le m\} \cup \{g_i(x) | 1 \le i \le p\} \cup \{h_i(x) | 1 \le i \le q\}$ can be written as a linear combination of the multilinear polynomials of degree at most s. The space of such multilinear polynomials has dimension $\sum_{i=0}^{s} \binom{n}{i}$. It follows that

$$m + p + q = |\mathcal{F}| + \sum_{i=0}^{s-1} \binom{n-1}{i} + \sum_{i=0}^{s-2r} \binom{n-1}{i} \le \sum_{i=0}^{s} \binom{n}{i}$$

which implies

$$|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

This completes the proof of the theorem. □

Proof of Theorem 1.11 Let $n \in \bigcap_{F \in \mathcal{F}} F$. Then consider $\mathcal{F}^* = \{F_1^*, F_2^*, \ldots, F_m^*\}$ where $F_i^* = F_i \setminus \{n\}$ for $1 \le i \le m$. Now, $|F_i^*| \in K^* = \{k_1^*, k_2^*, \ldots, k_r^*\}$, where $k_i^* = k_i - 1$. Similarly, $|F_i^* \cap F_j^*| \in \mathcal{L}^* = \{l_1^*, l_2^*, \ldots, l_s^*\}$, where $l_i^* = l_i - 1$. Since $n \ge s + \max k_i$, then $n - 1 \ge s + \max k_i^*$. Thus, it follows from Theorem 1.9 that

$$m \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r+1}.$$ □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
KWH made the first polynomial and proved the first polynomials are linear independent. RXJL made the other polynomials and proved that the other polynomials are linearly independent. All authors read and approved the final manuscript.

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