ON THE TWO-POINT BOUNDARY VALUE PROBLEM FOR QUADRATIC SECOND-ORDER DIFFERENTIAL EQUATIONS AND INCLUSIONS ON MANIFOLDS

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The two-point boundary value problem for second-order differential inclusions of the form $(D/dt)\dot{m}(t) \in F(t,m(t),\dot{m}(t))$ on complete Riemannian manifolds is investigated for a couple of points, nonconjugate along at least one geodesic of Levi-Civitá connection, where D/dt is the covariant derivative of Levi-Civitá connection and $F(t,m,X)$ is a set-valued vector with quadratic or less than quadratic growth in the third argument. Some interrelations between certain geometric characteristics, the distance between points, and the norm of right-hand side are found that guarantee solvability of the above problem for F with quadratic growth in X. It is shown that this interrelation holds for all inclusions with F having less than quadratic growth in X, and so for them the problem is solvable.

1. Introduction and discussion of the problem

Let M be a finite-dimensional manifold and TM be its tangent bundle with the natural projection $\pi : TM \to M$. Consider a set-valued map $F : R \times TM \to TM$ such that for any point $(m,X) \in TM$ (this means that $X \in T_mM$, i.e., X is a tangent vector to M at the point $m \in M$) the relation $\pi F(t,m,X) = \pi(m,X) = m$ holds.

The main aim of this paper is investigation of two-point boundary value problem for second-order differential inclusions of the form

$$\frac{D}{dt}\dot{m}(t) \in F(t,m(t),\dot{m}(t)) \qquad (1.1)$$

with F having quadratic or less than quadratic growth in the third argument where D/dt is the covariant derivative of a certain connection.

Such inclusions arise in description of complicated mechanical systems on nonlinear configuration spaces where the set-valued right-hand side F is generated by an essentially discontinuous force field or by a force with control (see, e.g., [8, 10]). That is why everywhere below we call F a set-valued force field.

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Besides its mechanical meaning this problem with F quadratic in X is important since it is a generalization of the well-known classical problem on the possibility to join two given points in a manifold by a geodesic curve of a certain connection (see, e.g., [17]). Recall that if ∇ and $\bar{\nabla}$ are covariant derivatives of two different connections on a manifold M, there exists a $(1,2)$-tensor field $S(\cdot,\cdot)$ on M such that for any two vector fields X and Y on M the equality $\bar{\nabla}_X Y = \nabla_X Y + S(X,Y)$ holds (see, e.g., [17, Statement 7.10]). From this it follows that in terms of covariant derivative ∇ the geodesics of another connection $\bar{\nabla}$ are always described by an equation of the form

$$\frac{D}{dt}\dot{m}(t) = \alpha\big(m(t),\dot{m}(t)\big), \tag{1.2}$$

where $\alpha(m,X) = S_m(X,X)$ is a vector filed on M that is quadratic in $X \in T_mM$ at any point $m \in M$.

For the Levi-Civitá connection on a complete Riemannian manifold the solvability of two-point boundary value problem for (1.2) for any points m_0, m_1 follows from Hopf-Rinow theorem (see, e.g., [2, 17]). But it is not the case even for a Riemannian connection with nonzero torsion: in [1, 6, 14] examples of Riemannian connections (in particular, on a compact manifold, two-dimensional torus) are presented for which this problem may not be solvable.

Consider two elementary and nevertheless characteristic examples where the two-point boundary value problem for (1.2) (and so for (1.1)) may not be solvable in spite of the fact that (1.1) is given in terms of Levi-Civitá connection of a complete Riemannian metric.

Example 1.1. Consider a mechanical system on the unit sphere S^2, embedded into R^3, with the force field $\alpha(\bar{r},\dot{\bar{r}}) = [\bar{r},\dot{\bar{r}}]\|\dot{\bar{r}}\|$ where the square brackets denote vector product. Taking into account the fact that S^2 is embedded into R^3, we can apply d'Alembert principle and reduce (1.2) to the equation of motion with a constraint in the form: $\ddot{\bar{r}} = [\bar{r},\dot{\bar{r}}]\|\dot{\bar{r}}\| - 2T\bar{r}$ where the kinetic energy $T = (1/2)\dot{\bar{r}}^2$. Since the acceleration is everywhere orthogonal to the velocity, it is obvious that $\dot{T} = 0$. Consider the vector $\bar{b} = [\dot{\bar{r}},\ddot{\bar{r}}]$. Direct calculations yield $\dot{\bar{b}} = 0$. This means that any trajectory satisfies the relation $(\bar{b},\bar{r}) = \text{const}$ (the parentheses denote scalar product in R^3), that is, it is a circle on the sphere that also lies in a plane orthogonal to the constant vector $\bar{b}$. Antipodal points are joint by a great circle, that is, $(\bar{b},\bar{r}) = 0$. From this we get the equality for mixed product $(\bar{r},\dot{\bar{r}},\ddot{\bar{r}}) = 0$ that is impossible. Thus the antipodal points on the sphere cannot be connected with a trajectory.

Example 1.2. Let $X = (x,y)$ be a vector from R^2 and let $a > 0$ be a real number; by $\|\cdot\|$ denote the norm in R^2. In R^2 consider the following system of (1.2) type:

$$\ddot{x}(t) = -a\|\dot{X}\|\dot{y}, \qquad \ddot{y}(t) = a\|\dot{X}\|\dot{x} \tag{1.3}$$

with initial condition $X(0) = 0, X(0) = X_0$. Since here the vectors $\dot{X}$ and $\ddot{X}$ are orthogonal to each other along the solution, $\|\dot{X}\|$ is constant. Let $\|X_0\| = C$, represent the vector X_0 in the form $X_0 = C(-\sin\varphi_0, \cos\varphi_0)$. Then the solution of above-mentioned Cauchy problem takes the form $x(t) = (1/a)\cos(\mathrm{Cat}+\varphi_0) - (1/a)\cos\varphi_0$, $y(t) = (1/a)\sin(\mathrm{Cat}+\varphi_0) - (1/a)\sin\varphi_0$. Hence any solution is a circle with the radius $1/a$ and it does not leave the disc of radius $2/a$ with the center at the initial point. We would like to emphasize that the radius is being reduced as a is increasing.

If the points are conjugate along all geodesics of Levi-Civitá connection joining them (like antipodal points in Example 1.1), the problem may not be solvable even for uniformly bounded $\alpha(m,X)$ and for $\alpha(m,X)$ having linear growth in velocities (see [8, 10]). Example 1.2 is representative specially for quadratic right-hand sides.

The two-point boundary value problem for (1.1) and (1.2) with nonconjugate points has been investigated under various conditions, more restrictive than ours in this paper. For (1.2) (i.e., for single-valued force fields) its solvability was shown by Gliklikh for continuous force fields in [7] (bounded case) and in [9] (linear growth in X), by Yakovlev, for example, in [18] for smooth force fields under some complicated conditions and by Ginzburg in [6] for smooth force fields with less than quadratic growth in X. The solvability of this problem for inclusion (1.1) was shown for set-valued force fields of several types (Gel'man and Gliklikh [5], Gliklikh and Obukhovskiĭ [12, 13], Kisielewicz [16], etc.) but only in uniformly bounded case.

In this paper, we consider the above-mentioned problem for (1.1) with force fields having quadratic or less than quadratic growth in X. We deal with $F(t,m,X)$ either almost lower semicontinuous or satisfying upper Carathéodory condition (in the latter case $F(t,m,X)$ has convex images). We suppose that m_0 and m_1 are not conjugate along at least one Levi-Civitá geodesic and show that if $F(t,m,X)$ has less than quadratic growth in X (see Definition 3.1 below), there exists a solution of (1.1) that joins those points. For the case of F having quadratic bound in X (see Definition 3.2 below, it is a natural generalization of quadratic growth property for a right-hand side of (1.2)) we find a certain condition on geometric properties of M, Riemannian distance between m_0 and m_1 and the norm of operator F that guarantees the solvability of the problem (see Remark 3.9 below). The former result is a generalization of that from [6] for second-order differential equations with smooth force fields having less that quadratic growth in velocities. Notice that in [6] the arguments based on uniqueness of solution to Cauchy problem for (1.2) are used that are not applicable to the case of inclusion (1.1).

Preliminary material from set-valued analysis can be found in [3, 4, 15], from geometry of manifolds, in [2, 14, 17].

2. Mathematical machinery

In this section, we modify some constructions from [8, 10] for the problem under consideration.

Let M be a complete Riemannian manifold. Consider $m_0 \in M$, $[0,1] \subset R$ and let $v\colon [0,1] \to T_{m_0}M$ be a continuous curve. It is shown that there exists unique C^1-curve $m : [0,1] \to M$ such that $m(0) = m_0$ and the vector $\dot m(t)$ is parallel along $m(\cdot)$ to the vector $v(t) \in T_{m_0}M$ at any $t \in [0,1]$.

Denote the curve $m(t)$, constructed above from the curve $v(t)$, by the symbol $\mathcal{S}v(t)$. Thus, we have defined a continuous operator $\mathcal{S} : C^0([0,1], T_{m_0}M) \to C^1([0,1], M)$ that sends the Banach space $C^0([0,1], T_{m_0}M)$ of continuous maps (curves) from $[0,1]$ to $T_{m_0}M$ into the Banach manifold $C^1([0,1], M)$ of C^1-maps from $[0,1]$ to M.

By $U_k \subset C^0([0,1], T_{m_0}M)$ we denote the ball of radius k centered at the origin in $C^0([0,1], T_{m_0}M)$.

Let a point $m_1 \in M$ be nonconjugate to the point $m_0 \in M$ along a geodesic $g(t)$ of the Levi-Civitá connection. Without loss of generality we postulate that the parameter t on $g(t)$ is taken so that $g(0) = m_0$ and $g(1) = m_1$.

Lemma 2.1. *There exists a ball $U_\varepsilon \subset C^0([0,1], T_{m_0}M)$ with a radius $\varepsilon > 0$ such that for any curve $\hat{u}(t) \in U_\varepsilon \subset C^0([0,1], T_{m_0}M)$ there exists a unique vector $\mathbf{C}_{\hat{u}}$, belonging to a certain bounded neighbourhood V of the vector $\dot{\gamma}(0)$ in $T_{m_0}M$, that is continuous in $\hat{u}$ and such that $\mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(1) = m_1$.*

Proof. By the construction of operator $\mathcal{S}$ its value $\mathcal{S}v_\gamma(1)$ on the constant curve $v_\gamma(t) = \dot{\gamma}(0)$ coincides with $\exp_{m_0} \dot{\gamma}(0) = m_1$. Since m_0 and m_1 are not conjugate along γ, $\exp_{m_0}$ is a diffeomorphisms of a certain neighbourhood $\dot{\gamma}(0) \in T_{m_0}M$ onto a neighbourhood of the point m_1 in M. Applying the implicit function theorem, one can easily show that the perturbation of exponential map, that sends $X \in T_{m_0}M$ to $\mathcal{S}(X + \hat{u})(1)$, is also a diffeomorphism of a certain neighbourhood V of $\dot{\gamma}(0)$ onto a neighbourhood of m_1 in M for any curve $\hat{u}(t)$ from a small enough ε-neighbourhood of the origin in $C^0([0,1], T_{m_0}M)$. □

Introduce the notation $\sup_{\mathbf{C} \in V} \|\mathbf{C}\| = C$ where V is from Lemma 2.1.

Lemma 2.2. *In conditions and notations of Lemma 2.1 let $K > 0$ and $t_1 > 0$ be such that $t_1^{-1}\varepsilon > K$. Then for any curve $u(t) \in U_K \subset C^0([0,t_1], T_{m_0}M)$ there exists a unique vector C_u in a neighbourhood $t_1^{-1}V$ of the vector $t_1^{-1}\dot{\gamma}(0)$ in $T_{m_0}M$, continuously depending on u and such that $S(u + C_u)(t_1) = m_1$.*

Proof. For $u(t) \in U_K \subset C^0([0,t_1], T_{m_0}M)$ introduce $\hat{u}(t) = t_1 u(t_1 \cdot t) \in U_\varepsilon \subset C^0([0,1], T_{m_0}M)$ and $C_u = t_1^{-1}\mathbf{C}_{\hat{u}}$. From Lemma 2.1 we get $\mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(1) = m_1$ and $(d/dt)\mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(t)$ is parallel to $\hat{u}(t) + \mathbf{C}_{\hat{u}}$. For the curve $\gamma(t) = \mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(t \cdot t_1)$ we have $(d/dt)\gamma(t) = t_1^{-1}(d/dt)\mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(t \cdot t_1)$ and this vector is parallel along the same curve to the vector $t_1^{-1}(\hat{u}(t) + \mathbf{C}_{\hat{u}}) = u(t) + C_u$. Thus $\gamma(t) = \mathcal{S}(u + C_u)(t) = \mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(t \cdot t_1^{-1})$ for $t \in [0,t_1]$. Hence $\mathcal{S}(u + C_u)(t_1) = \mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(1) = m_1$. □

Lemmas 2.1 and 2.2 form a modification of [10, Theorem 3.3].

Lemma 2.3. *For specified $t_1 > 0$ and $K > 0$ all curves $S(v(t) + C_v)(t)$ with $v \in U_K \subset C^0([0,t_1], T_{m_0}M)$ lie in a compact set $\Xi \subset M$ where Ξ depends on ε and C introduced above.*

Indeed, since the parallel translation preserves the norm of a vector, for any $v(t)$ as above the length of $S(v(t) + C_v)(t)$ is not greater than $\int_0^{t_1}(K + \|C_v\|)dt \le \int_0^{t_1} t_1^{-1}(\varepsilon + C)dt = \int_0^1(\varepsilon + C)dt = \varepsilon + C$. Since M is complete, by Hopf-Rinow theorem any metric ball of finite radius $\varepsilon + C$ is compact.

Lemma 2.4. *Let a real number δ satisfy the inequality $0 < \delta < \varepsilon/(\varepsilon + C)^2$. Then there exists a small enough positive number φ such that $(\varepsilon t_1^{-1} - \varphi) > 0$ and the inequality $\delta((\varepsilon t_1^{-1} - \varphi) + Ct_1^{-1})^2 < \varepsilon t_1^{-2} - \varphi t_1^{-1}$ holds.*

Proof. For δ satisfying the hypothesis of the lemma we get $\delta(\varepsilon t_1^{-1} + Ct_1^{-1})^2 < \varepsilon t_1^{-2}$. From continuity of both sides of this inequality it follows that there exists a small enough number $\varphi > 0$ such that $(\varepsilon t_1^{-1} - \varphi) > 0$ and the inequality $\delta((\varepsilon t_1^{-1} - \varphi) + Ct_1^{-1})^2 < \varepsilon t_1^{-2} - \varphi t_1^{-1}$ holds. □

3. The main statements

Everywhere below M is a complete Riemannian manifold, by $\|\cdot\|$ we denote the norm in a tangent space generated by the Riemannian metric. Introduce the norm of the set $\|F(t,m,X)\| \in T_mM$ by usual formula $\|F(t,m,X)\| = \sup_{y\in F(t,m,X)} \|y\|$.

Definition 3.1. We say that $F(t,m,X)$ has less than quadratic growth in X if for any compact $\Theta \subset M$ and any finite interval $[0,l]$ the relation

$$\lim_{\|X\|\to\infty} \frac{||F(t,m,X)||}{\|X\|^2} = 0 \tag{3.1}$$

holds uniformly in $t \in [0,l]$ and $m \in \Theta$.

Definition 3.2. We say that $F(t,m,X)$ has quadratic bound in X if for any compact $\Theta \subset M$ and any finite interval $[0,l]$ the relation

$$\lim_{\|X\|\to\infty} \frac{||F(t,m,X)||}{\|X\|^2} = a(t,m) \tag{3.2}$$

holds uniformly in $t \in [0,l]$ and $m \in \Theta$ where $a(t,m) \geq 0$ is a real bounded function on $[0,l] \times \Theta$ that is not identical zero.

Definition 3.3. We say that $F(t,m,X)$ satisfies upper Carathéodory conditions if:

(1) for every $(m,X) \in TM$ the map $F(\cdot,m,X) : I \to T_mM$ is measurable,

(2) for almost all $t \in I$ the map $F(t,\cdot,\cdot) : TM \to TM$ is upper semicontinuous.

Definition 3.4. Let $I = [0,l] \subset R$. The set-valued force field $F : I \times TM \to TM$ is called almost lower semicontinuous if there exists a countable sequence of disjoint compact sets $\{I_n\}$, $I_n \subset I$ such that: (i) the measure of $I \setminus \cup_n I_n$ is equal to zero; (ii) the restriction of F on each $I_n \times TM$ is lower semicontinuous.

Theorem 3.5. *Let $F(t,m,X)$ satisfy the upper Carathéodory condition, has convex closed bounded images and has less than quadratic growth in X. Let the points m_1 and m_0 be nonconjugate along a certain geodesic g of the Levi-Civitá connection. Then there exists a positive number $L(m_0,m_1,g)$ such that if $0 < t_1 < L(m_0,m_1,g)$ there exists a solution $m(t)$ of (1.1), for which $m(0) = m_0$ and $m(t_1) = m_1$.*

Proof. For a C^1-curve $\gamma(t) = \mathcal{S}v(t)$, $v(\cdot) \in C^0(I,T_{m_0}M)$, consider the set-valued vector field $F(t,\gamma(t),\dot\gamma(t))$. Denote by Γ the operator of parallel translation of vectors along $\gamma(\cdot)$ at the point $\gamma(0) = m_0$. Apply operator Γ to all sets $F(t,\gamma(t),\dot\gamma(t))$ along $\gamma(\cdot)$. As a result for any $v \in C^0(I,T_{m_0}M)$ we obtain a set-valued map $\Gamma F\mathcal{S}v : [0,l] \to T_{m_0}M$ that has convex images. It is shown in [13] that the map $\Gamma F\mathcal{S} : C^0([0,l],T_{m_0}M) \times [0,l] \to T_{m_0}M$ satisfies upper Carathéodory conditions. Denote by $\mathcal{P}\Gamma F\mathcal{S}v$ the set of all measurable selections of $\Gamma F\mathcal{S}v : [0,l] \to T_{m_0}M$ (such selections exist by [3]). Define on $C^0([0,t_1],T_{m_0}M)$ the set-valued operator $\int\mathcal{P}\Gamma F\mathcal{S}$ by the formula

$$\int\mathcal{P}\Gamma F\mathcal{S}v = \left\{\int_0^t f(\tau)d\tau \mid f(\cdot) \in \mathcal{P}\Gamma F\mathcal{S}v\right\}. \tag{3.3}$$

It is shown in [13] that $\int\mathcal{P}\Gamma F\mathcal{S}$ is upper semicontinuous, has convex images and sends bounded sets from $C^0([0,t_1],T_{m_0}M)$ into compacts.

Consider the numbers ε and C constructed for the points m_0 and m_1 and geodesic g. Let Ξ be a compact from Lemma 2.3, and let $[0,l]$ be a certain interval. Choose a positive number $\delta < \varepsilon/(\varepsilon+C)^2$. Since F satisfies Definition 3.1, one can easily see that there exists a number $Q > 0$ such that for $\|X\| \geq Q$ the inequality

$$\max_{(t,m)\in I\times\Xi} ||F(t,m,Y)|| < \delta\|X\|^2 \tag{3.4}$$

holds for all $\|Y\| < \|X\|$. For $t_1 > 0$ small enough we get $t_1 \in [0,l]$ and $t_1^{-1}\varepsilon - \varphi > Q$ where φ is from Lemma 2.4. Determine $L(m_0,m_1,g)$ as the upper bound of t_1 such that the above relations hold. Let $0 < t_1 < L(m_0,m_1,g)$. For this t_1 denote by K the corresponding number $t_1^{-1}\varepsilon - \varphi$.

By the construction $t_1^{-1}\varepsilon > K$ and so by Lemma 2.2 the operator $\mathcal{Z} : U_K \to C^0([0,t_1], T_{m_0}M)$:

$$\mathcal{Z}(v) = \int\mathcal{P}\Gamma F\mathcal{S}(v+C_v) \tag{3.5}$$

is well posed. As well as $\int\mathcal{P}\Gamma F\mathcal{S}$ this operator is upper semicontinuous, has convex images and sends bounded sets from $C^0([0,t_1],T_{m_0}M)$ into compacts.

For $v \in U_K \subset C^0([0,t_1],T_{m_0}M)$, since the parallel translation preserves the norm of a vector, from the construction of operator $\mathcal{S}$, from (3.4) and from Lemma 2.4 it follows that

$$\left\|F\left(t,\mathcal{S}(v(t)+C_v),\frac{d}{dt}\mathcal{S}(v(t)+C_v)\right)\right\| < \delta\left(t_1^{-1}\varepsilon-\varphi+Ct_1^{-1}\right)^2 < \left(t_1^{-2}\varepsilon-t_1^{-1}\varphi\right). \tag{3.6}$$

Since parallel translation preserves the norm of a vector, from the last inequality it follows that

$$||\mathcal{Z}(v+C_v)|| = \left\|\int\mathcal{P}\Gamma F\mathcal{S}(v(\tau)+C_v)\right\|_{C^0([0,t_1],T_{m_0}M)} < \left(t_1^{-1}\varepsilon-\varphi\right) = K. \tag{3.7}$$

Thus $\mathcal{Z}$ sends the ball U_K into itself and from Schauder's principle for upper semicontinuous set-valued maps (see, e.g., [3]) it follows that it has a fixed point $u^* \in U_K$, that is, $u^* \in \mathcal{Z}u^*$. Let us show that $m(t) = \mathcal{S}(u^*(t)+C_{u^*})$ is the desired solution. By the construction we have $m(0) = m_0$ and $m(t_1) = m_1$, $m(t)$ is a C^1-curve and $\dot m(t)$ is absolutely continuous. Note that $\dot u^*$ is a selection of $\Gamma F(t,\mathcal{S}(u^*+C_{u^*}),(d/dt)\mathcal{S}(u^*+C_{u^*}))$ because u^* is a fixed point of $\mathcal{Z}$. In other words, the inclusion $\dot u^*(t) \in \Gamma F(t,\mathcal{S}(u^*+C_{u^*}),(d/dt)\mathcal{S}(u^*+C_{u^*}))$ holds for all points t at which the derivative exists. Using the properties of the covariant derivative and the definition of u^*, one can show that $\dot u^*(t)$ is parallel to $(D/dt)\dot{m}(t)$ along $m(\cdot)$ and $\Gamma F(t,\mathcal{S}(u^* + C_{u^*}),(d/dt)\mathcal{S}(u^* + C_{u^*}))$ is parallel to $F(t,m(t),\dot{m}(t))$. Hence, $(D/dt)\dot{m}(t) \in F(t,m(t),\dot{m}(t))$. □

Theorem 3.6. *Let $F(t,m,X)$ satisfy the upper Carathéodory condition, has convex closed bounded images and has quadratic bound in X. Let the points m_1 and m_0 be nonconjugate along a certain geodesic g of the Levi-civitá connection. Let in addition for $t \in [0,l]$ and $m \in \Xi$, where $[0,l]$ is a certain interval and Ξ is the compact from Lemma 2.3, for the function $a(t,m)$ from Definition 3.2 there exists a real number δ such that the estimate $a(t,m) < \delta < \varepsilon/(\varepsilon + C)^2$ holds. Then there exists a positive number $L(m_0,m_1,g)$ such that if $0 < t_1 < L(m_0,m_1,g)$ there exists a solution $m(t)$ of (1.1), for which $m(0) = m_0$ and $m(t_1) = m_1$.*

The proof of Theorem 3.6 follows the same scheme of arguments as that for Theorem 3.5. The only modification is that here for F with quadratic bound in X we assume the existence of δ such that $a(t,m) < \delta < \varepsilon/(\varepsilon + C)^2$ while in the proof of Theorem 3.5 analogous δ is shown to exist for any F with less than quadratic growth in X.

Theorem 3.7. *Let $F(t,m,X)$ be almost lower semicontinuous, has closed bounded images and has less than quadratic growth in X. Let the points m_1 and m_0 be nonconjugate along a certain geodesic g of the Levi-civitá connection. Then there exists a positive number $L(m_0, m_1,g)$ such that if $0 < t_1 < L(m_0,m_1,g)$ there exists a solution $m(t)$ of (1.1), for which $m(0) = m_0$ and $m(t_1) = m_1$.*

Proof. Here we use the same notations as in the proof of Theorem 3.5. Notice that from the condition of less than quadratic growth for F it follows that for all $v \in C^0([0,l],T_{m_0}M)$ the curves from $\mathcal{P}\Gamma F\mathcal{S}v$ are integrable. Hence the set-valued map $\mathcal{P}\Gamma F\mathcal{S}$ sends $C^0([0,l], T_{m_0}M)$ into $L^1(([0,l],\mathcal{A},\mu),T_{m_0}M)$, where $\mathcal{A}$ is the Borel σ-algebra and μ is the normalized Lebesgue's measure. Since F is almost lower semicontinuous, in complete analogy with [15] one can easily show that $\mathcal{P}\Gamma F\mathcal{S} : C^0([0,l],T_{m_0}M) \to L^1(([0,l],\mathcal{A},\mu),T_{m_0}M)$ is lower semicontinuous and has decomposable images (see the definition of decomposable image, e.g., in [4]). Then by Bressan-Kolombo theorem (see, e.g., [4]) it has a continuous selection that we denote by $p\Gamma F\mathcal{S}$.

Choose the numbers Q, $L(m_0,m_1,g)$, $0 < t_1 < L(m_0,m_1,g)$ and K as in the proof of Theorem 3.5. Then on the ball $U_K \subset C^0([0,t_1],T_{m_0}M)$ the operator

$$\mathcal{G}v = \int_0^t p\Gamma F\mathcal{S}\Big((v(s)+C_v),\frac{d}{dt}\mathcal{S}(v(s)+C_v)\Big)ds : U_K \longrightarrow C^0([0,t_1],T_{m_0}M) \tag{3.8}$$

is well posed. As a corollary to [11, Lemma 19], we get that $\mathcal{G}$ is completely continuous. Since parallel translation preserves the norm of a vector, from the construction of $\mathcal{S}$ for any $u \in U_K$ with given F we get

$$\begin{aligned}||\mathcal{G}v|| &= \Big|\Big|\int_0^t p\Gamma F\Big(s,\mathcal{S}(v(s)+C_v),\frac{d}{dt}\mathcal{S}(v(s)+C_v)\Big)ds\Big|\Big|_{C^0([0,t_1],T_{m_0}M)} \\ &\le (t_1^{-1}\varepsilon - \varphi) = K.\end{aligned} \tag{3.9}$$

Hence $\mathcal{G}$ sends U_K into itself and by classical Schauder's principle it has a fixed point $u^* \in U_K$. Using the same arguments, as in the proof of Theorem 3.5, one can easily prove that $m(t) = \mathcal{S}(u^* + C_u^*)(t)$ is a solution of (1.1) such that $m(0) = m_0$ and $m(t_1) = m_1$. □

Theorem 3.8. *Let $F(t,m,X)$ be almost lower semicontinuous, has closed bounded images and quadratic bound in X. Let the points m_1 and m_0 be nonconjugate along a certain geodesic g of the Levi-civitá connection. Let in addition for $t \in [0,l]$ and $m \in \Xi$, where $[0,l]$ is a certain interval and Ξ is the compact from Lemma 2.3, for the function $a(t,m)$ from Definition 3.2 there exists a real number δ such that the estimate $a(t,m) < \delta < \varepsilon/(\varepsilon + C)^2$ holds. Then there exists a positive number $L(m_0,m_1,g)$ such that if $0 < t_1 < L(m_0,m_1,g)$ there exists a solution $m(t)$ of (1.1), for which $m(0) = m_0$ and $m(t_1) = m_1$.*

As well as in the case of Theorems 3.5 and 3.6, Theorem 3.8 is proved in complete analogy with Theorem 3.7 with the following minor modification: in Theorem 3.8 for F with quadratic bound in X we assume the existence of δ such that $a(t,m) < \delta < \varepsilon/(\varepsilon+C)^2$ while in the proof of Theorem 3.7 we use the fact that analogous δ does exist for any F with less than quadratic growth in X (see the proof of Theorem 3.5).

Remark 3.9. Notice that if a geodesic, along which m_0 and m_1 are not conjugate, is a length minimizing one, the number C characterizes the Riemannian distance between these points. The numbers C and ε together provide a certain characteristics of the Riemannian geometry on M in a neighbourhood of m_0. Theorems 3.6 and 3.8 establishes an interrelation between C, ε and the quadratic bounds of (1.1), under which the two-point boundary value problem for nonconjugate points m_0 and m_1 is solvable for sure.

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