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*Research Article*

# Positive Solutions for Singular Semipositone Fractional Differential Equation Subject to Multipoint Boundary Conditions

**Rui Pu,[1] Xingqiu Zhang,[1] Yujun Cui,[2] Peilong Li,[1] and Weiwei Wang[1]**

[1] *School of Medical Information Engineering, Jining Medical College, Rizhao, Shandong 276826, China*
[2] *Department of Applied Mathematics, Shandong University of Science and Technology, Qingdao 266590, China*

Correspondence should be addressed to Xingqiu Zhang; zhxq197508@163.com

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Existence result together with multiplicity result of positive solutions of higher-order fractional multipoint boundary value problems is given by considering the integrations of height functions on some special bounded sets. The nonlinearity may change its sign and may possess singularities on the time and the space variables at the same time.

## 1. Introduction

Very recently, Henderson and Luca [1] obtained the existence result of positive solution for the following singular differential equation with fractional derivative:

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

subject to multipoint BCs

$$\begin{gathered} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0+}^{p} u(1) = \sum_{i=1}^{m} a_i D_{0+}^{q} u(\xi_i). \end{gathered} \tag{2}$$

Here $D_{0+}^{\alpha}$ is the traditional Riemann-Liouville derivative with order $\alpha$, $n-1 < \alpha \le n$ $(n \ge 3)$, $a_i \ge 0$, $i = 1, 2, \ldots, m$ $(m \in \mathbb{N}^+)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $p, q \in \mathbb{R}$, $1 \le p \le n-2$, and $0 \le q \le p$ with $\Delta = \Gamma(\alpha)/\Gamma(\alpha - p) - (\Gamma(\alpha)/\Gamma(\alpha - q)) \sum_{i=1}^{m} a_i \xi_i^{\alpha-q-1} > 0$. The nonlinear term $f$ permits sign-changing and singularities at $t = 0, 1$.

Fractional derivative gives a more precise exhibition on long-memory behavior owing to its nonlocal characteristic. Nowadays, researchers have reached a consensus that fractional calculus is one of the effective tools to describe phenomena in almost every field of science and technology. For example, people have realized that electrical conductance of cell membranes of biological organism took on fractional-order style [2]. As a consequence, fractional-order ordinary differential equations can give a more accurate description on spread process of some infectious diseases such as HIV, hand-foot-mouth disease [3], malaria, tuberculosis, and measles. Arafa et al. [4] formulated a model with orders $\alpha_1, \alpha_2, \alpha_3 > 0$ to describe the infection of $CD4^+$ T cells.

In 2015, Wang et al. [5] concentrated on the positive solutions for a class of modified HIV-1 population dynamics model presented in [6]

$$\begin{gathered} D_t^{\alpha} u(t) + \lambda f\left(t, u(t), D_t^{\beta} u(t), v(t)\right) = 0, \\ D_t^{\gamma} v(t) + \lambda g(t, u(t)) = 0, \quad 0 < t < 1, \\ D_t^{\beta} u(0) = D_t^{\beta+1} u(0) = 0, \\ D_t^{\beta} u(1) = \int_0^1 D_t^{\beta} u(s)\, \mathrm{d}A(s), \\ v(0) = v'(0) = 0, \\ v(1) = \int_0^1 v(s)\, \mathrm{d}B(s), \end{gathered} \tag{3}$$

where $2 < \alpha, \gamma \le 3$, $0 < \beta < 1$, $\lambda > 0$ is a parameter, $\alpha - \beta > 2$, and $D_t^\alpha$, $D_t^\beta$, and $D_t^\gamma$ are traditional R-L derivatives. $\int_0^1 D_t^\beta u(s)\mathrm{d}A(s)$ and $\int_0^1 v(s)\mathrm{d}B(s)$ represent R-S integrals and $A$, $B$ are bounded variations, and the nonlinearities permit sign-changing and singularities at $t = 0, 1$. Other relative papers on nonlocal boundary value problems can be found in [7–20].

Motivated by the above papers, we devote ourselves to the existence result as well as multiplicity result of positive solutions for BVP (1)-(2). This article admits some new features. First, the nonlinear term $f$ may take negative infinity and change its sign. Compared with [1], $f$ permits singularities on the time and the space variables at the same time. Second, the method exploited in this paper is different from that in [1] in essence. More precisely, height functions with their integrations on some special bounded set are utilized to get the existence result for positive solutions of BVP (1)-(2). Thirdly, a result of multiple positive solutions is also given in this paper. Conditions employed in this paper are easy to be verified.

## 2. Preliminaries and Several Lemmas

Two traditional Banach spaces $E = C[0,1]$ and $L^1(0,1)$ are involved in this article, where $E = C[0,1]$ and $L^1(0,1)$ represent the spaces of the continuous functions and Lebesgue integrable functions equipped with the norms $\|u\| = \max_{0\le t\le 1}|u(t)|$ and $\|u\|_1 = \int_0^1 |u(t)|\mathrm{d}t$, respectively.

**Lemma 1** (see [1]). *If $\Delta \ne 0$. Given $y \in C(0,1) \cap L^1(0,1)$, the solution of the following differential equation:*

$$\begin{gathered} D_{0+}^\alpha u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0+}^p u(1) = \sum_{i=1}^m a_i D_{0+}^q u(\xi_i) \end{gathered} \tag{4}$$

*satisfies*

$$u(t) = \int_0^1 G(t,s)\, y(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{5}$$

*and here*

$$G(t,s) = G_1(t,s) + \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^m a_i G_2(\xi_i, s), \tag{6}$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \cdot \begin{cases} t^{\alpha-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-p-1}, & 0 \le t \le s \le 1, \end{cases} \tag{7}$$

$$G_2(t,s) = \frac{1}{\Gamma(\alpha-q)} \cdot \begin{cases} t^{\alpha-q-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-q-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-q-1}(1-s)^{\alpha-p-1}, & 0 \le t \le s \le 1, \end{cases} \tag{8}$$

$\forall (t,s) \in [0,1]\times[0,1]$.

**Lemma 2** (see [1]). *Assume that $a_i > 0$ $(i = 1, 2, \ldots, m)$ and $\Delta > 0$. Then the Green function $G$ of (4) given by (6) is continuous on $[0,1]\times[0,1]$ and meets*

(a) *$G(t,s) \le J(s)$, $\forall t, s \in [0,1]$, and here $J(s) = h_1(s) + (1/\Delta)\sum_{i=1}^m a_i G_2(\xi_i, s)$, $h_1(s) = (1-s)^{\alpha-p-1}(1-(1-s)^p)/\Gamma(\alpha)$, and $s \in [0,1]$;*

(b) *$G(t,s) \ge t^{\alpha-1}J(s)$, $\forall t, s \in [0,1]$;*

(c) *$G(t,s) \le \sigma t^{\alpha-1}$ $\forall t, s \in [0,1]$, and here $\sigma = 1/\Gamma(\alpha) + \sum_{i=1}^m a_i \xi_i^{\alpha-q-1}/(\Delta\Gamma(\alpha-q))$.*

**Lemma 3.** *Let $u \in C[0,1]$ satisfy (4), where $y \in L^1(0,1)$, $y(t) \ge 0$, and $0 \le t \le 1$. Then, $u(t) \ge t^{\alpha-1}\|u\|$, and $0 \le t \le 1$.*

*Proof.* It can be easily seen from Lemma 5 in [1]; we omit the details. □

**Lemma 4.** *Suppose that $w(t) \in C[0,1]$ be the solution of*

$$\begin{gathered} D_{0+}^\alpha u(t) + k(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0+}^p u(1) = \sum_{i=1}^m a_i D_{0+}^q u(\xi_i), \end{gathered} \tag{9}$$

*where $k \in L^1(0,1)$ and $k(t) > 0$. Then, $w(t) \le \sigma\|k\|_1 t^{\alpha-1}$ and $0 \le t \le 1$.*

*Proof.* For any $0 \le t \le 1$, we get by Lemma 2

$$\begin{aligned} w(t) &= \int_0^1 G(t,s)\,k(s)\,\mathrm{d}s \le \sigma t^{\alpha-1}\int_0^1 k(s)\,\mathrm{d}s \\ &= \sigma\|k\|_1 t^{\alpha-1}. \end{aligned} \tag{10}$$

□

**Lemma 5** (see [21]). *Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in Banach space $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ and $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ a completely continuous operator, where $\theta$ denotes the zero element of $E$ and $P$ a cone of $E$. Suppose that one of the two conditions holds:*

(i) $\|Au\| \le \|u\|$, $\forall u \in P \cap \partial\Omega_1$; $\|Au\| \ge \|u\|$, $\forall u \in P \cap \partial\Omega_2$.

(ii) $\|Au\| \ge \|u\|$, $\forall u \in P \cap \partial\Omega_1$; $\|Au\| \le \|u\|$, $\forall u \in P \cap \partial\Omega_2$.

*Then $A$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

## 3. Main Results

Let

$$K = \{u : u(t) \geq t^{\alpha-1} \|u\|\}. \quad (11)$$

Obviously, $K$ is a cone in $E$ and $(E, K)$ is a partial ordering Banach space.

($A_0$) $f \in C((0,1) \times (0,+\infty), (-\infty,+\infty))$, and there exists a function $k \in L^1(0,1)$ and $k(t) > 0$, such that $f(t,u) \geq -k(t)$, $\forall t \in (0,1)$, $u > 0$.

($A_1$) For any positive numbers $r_1 < r_2$, there exists a nonnegative function $\gamma_{r_1,r_2} \in L^1(0,1)$ such that

$$|f(t,u)| \leq \gamma_{r_1,r_2}(t), \quad 0 < t < 1, \; r_1 t^{\alpha-1} \leq u \leq r_2 \quad (12)$$

with

$$\int_0^1 (1-s)^{\alpha-p-1} \gamma_{r_1,r_2}(s)\,ds < +\infty. \quad (13)$$

($A_2$) There exists a $\tilde{r}_1 > \sigma\|k\|_1$ such that

$$\int_0^1 J(s)\varphi(s,\tilde{r}_1)\,ds < \tilde{r}_1, \quad (14)$$

where $\varphi(t,\tilde{r}_1) = \max\{f(t,u) : (\tilde{r}_1 - \sigma\|k\|_1)t^{\alpha-1} \leq u \leq \tilde{r}_1\} + k(t)$.

($A_3$) There exists a $\tilde{r}_2 > \tilde{r}_1$ such that

$$\int_0^1 J(s)\psi(s,\tilde{r}_2)\,ds > \tilde{r}_2, \quad (15)$$

where $\psi(t,\tilde{r}_2) = \min\{f(t,u) : (\tilde{r}_2 - \sigma\|k\|_1)t^{\alpha-1} \leq u \leq \tilde{r}_2\} + k(t)$.

**Theorem 6.** *Suppose that conditions* ($A_0$), ($A_1$), ($A_2$), *and* ($A_3$) *hold. Then BVP (1)-(2) admits at least one positive solution.*

*Proof.* First of all, we concentrate on the following modified approximating BVP (MABVP for short) to overcome difficulties caused by singularities:

$$\begin{gathered} D_{0+}^{\alpha} u(t) + f(t, \chi_n(u-w)(t)) + k(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0+}^{\beta}(1) = \sum_{i=1}^{m} a_i D_{0+}^{q} u(\xi_i), \end{gathered} \quad (16)$$

and, here,

$$\chi_n(u) = \begin{cases} u, & u \geq \dfrac{1}{n}, \\ \dfrac{1}{n}, & u < \dfrac{1}{n}. \end{cases} \quad (17)$$

The operator $T_n$ is given by

$$(T_n u)(t) = \int_0^1 G(t,s)[f(s,\chi_n(u-w)(s)) + k(s)]\,ds, \quad 0 \leq t \leq 1. \quad (18)$$

Next, we will give the proof from three steps.

*(I) For Any $\sigma\|k\|_1 < r_1 < r_2$, $T_n : K \cap (\Omega_{r_2} \backslash \Omega_{r_1}) \to K$, $n > 1/r_1$, Is Completely Continuous.* For any $0 < t < 1$, considering that $u(t) - w(t) \geq (r_1 - \sigma\|k\|_1)t^{\alpha-1} > 0$, one has

$$(r_1 - \sigma\|k\|_1)t^{\alpha-1} \leq \max\left\{u(t) - w(t), \frac{1}{n}\right\} \leq r_2, \quad n > \frac{1}{r_1}, \quad (19)$$

which means

$$(r_1 - \sigma\|k\|_1)t^{\alpha-1} \leq \chi_n(u-w)(t) \leq r_2, \quad 0 < t < 1. \quad (20)$$

Thus, one gets from ($A_0$), ($A_1$), and Lemma 2 that

$$\begin{aligned} (T_n u)(t) &= \int_0^1 G(t,s)[f(s,\chi_n(u-w)(s)) + k(s)]\,ds \\ &\leq \int_0^1 J(s)[f(s,\chi_n(u-w)(s)) + k(s)]\,ds \\ &\leq \int_0^1 \left[\frac{(1-s)^{\alpha-p-1}(1-(1-s)^p)}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^{m} a_i \xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1}\right] \\ &\quad \cdot [f(s,\chi_n(u-w)(s)) + k(s)]\,ds \\ &\leq \int_0^1 \left[\frac{(1-s)^{\alpha-p-1}(1-(1-s)^p)}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^{m} a_i \xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1}\right] \\ &\quad \cdot \left(\gamma_{r_1-\sigma\|k\|_1,r_2}(s) + k(s)\right)ds < +\infty. \end{aligned} \quad (21)$$

So, $T_n : K \cap (\Omega_{r_2} \setminus \Omega_{r_1}) \to E$ is well defined.

At the same time, for given $u \in K \cap (\Omega_{r_2} \setminus \Omega_{r_1})$, $t \in [0,1]$, one gets from (21) and Lemma 2 that

$$\begin{aligned} (T_n u)(t) &= \int_0^1 G(t,s)[f(s,\chi_n(u-w)(s)) + k(s)]\,ds \\ &\leq \int_0^1 J(s)[f(s,\chi_n(u-w)(s)) + k(s)]\,ds. \end{aligned} \quad (22)$$

Hence,

$$\|T_n u\| \leq \int_0^1 J(s)[f(s,\chi_n(u-w)(s)) + k(s)]\,ds. \quad (23)$$

Given $u \in K$, it follows from Lemma 2 that

$$\begin{aligned}(T_n u)(t) &= \int_0^1 G(t,s)\left[f(s,\chi_n(u-w)(s)) + k(s)\right] ds \\ &\geq t^{\alpha-1}\int_0^1 J(s)\left[f(s,\chi_n(u-w)(s)) + k(s)\right] ds \\ &\geq t^{\alpha-1}\|T_n u\|,\end{aligned} \tag{24}$$

which means that $T_n : K \cap (\Omega_{r_2} \setminus \Omega_{r_1}) \to K$.

Given bounded set $D \subset K \cap (\Omega_{r_2} \setminus \Omega_{r_1})$, we can see from (21) that $T_m(D)$ is uniformly bounded. By Ascoli-Arzela theorem, in order to show the complete continuity, we need only to prove that $T_n(D)$ is equicontinuous. It follows from (6), (7), (8), and $(A_1)$ that

$$\begin{aligned}\left|(T_n u)'(t)\right| &= \left|\int_0^1 \frac{\partial}{\partial t}G(t,s)\left[f(s,\chi_n(u-w)(s))\right.\right. \\ &\quad \left.\left. + k(s)\right] ds\right| \leq \left|\int_0^1 \frac{\partial}{\partial t}G_1(t,s)\right. \\ &\quad \left.\cdot\left[f(s,\chi_n(u-w)(s)) + k(s)\right] ds\right| \\ &\quad + \left|\int_0^1 \frac{(\alpha-1)t^{\alpha-2}}{\Delta}\sum_{i=1}^m a_i G_2(\xi_i,s)\right. \\ &\quad \left.\cdot\left[f(s,\chi_n(u-w)(s)) + k(s)\right] ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\left|\int_0^t \left[(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-p-1} - (\alpha-1)\right.\right. \\ &\quad \left.\cdot (t-s)^{\alpha-2}\right]\left[f(s,\chi_n(u-w)(s)) + k(s)\right] ds \\ &\quad + \int_t^1 (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-p-1}\left[f(s,\chi_n(u-w)(s))\right. \\ &\quad \left.\left. + k(s)\right] ds\right| \\ &\quad + \left|\int_0^1 \frac{(\alpha-1)}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^m a_i \xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1}\right. \\ &\quad \left.\cdot\left[f(s,\chi_n(u-w)(s)) + k(s)\right] ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\left[\int_0^1 \left[(\alpha-1)(1-s)^{\alpha-p-1} + (\alpha-1)\right.\right. \\ &\quad \left.\left.\cdot(1-s)^{\alpha-2}\right]\left(\gamma_{r_1-\sigma\|k\|_1,r_2}(s) + k(s)\right) ds\right] \\ &\quad + \left|\int_0^1 \frac{(\alpha-1)}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^m a_i \xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1}\right. \\ &\quad \left.\cdot\left(\gamma_{r_1-\sigma\|k\|_1,r_2}(s) + k(s)\right) ds\right| < +\infty, \quad t \in [0,1].\end{aligned} \tag{25}$$

Equicontinuity of $T_n(D)$ can be derived from the absolute continuity of Lebesgue integral and (25). Thus, the fact that $T_n : K \cap (\Omega_{r_2} \setminus \Omega_{r_1}) \to K$, $n > 1/r_1$, is completely continuous has been proved.

*(II) For Sufficiently Large n, MABVP (16) Admits at Least One Positive Solution.* Given $u \in K \cap \partial\Omega(\widetilde{r}_1)$, one has $u(t) \geq \widetilde{r}_1 t^{\alpha-1}$, $0 \leq t \leq 1$. Thus, for $n > 1/\widetilde{r}_1$, similar to (20), we get

$$\left(\widetilde{r}_1 - \sigma\|k\|_1\right)t^{\alpha-1} \leq \chi_n(u-w)(t) \leq \widetilde{r}_1, \quad 0 < t < 1. \tag{26}$$

From the definition of $\varphi(t,\widetilde{r}_1)$, we have

$$f(t,\chi_n(u-w)(t)) + k(t) \leq \varphi(t,\widetilde{r}_1). \tag{27}$$

By $(A_2)$ and Lemma 2, one gets

$$\begin{aligned}&\|T_n u\| \\ &\quad = \max_{0\leq t\leq 1}\int_0^1 G(t,s)\left[f(s,\chi_n(u-w)(s)) + k(s)\right] ds \\ &\quad \leq \int_0^1 J(s)\varphi(s,\widetilde{r}_1) ds < \widetilde{r}_1;\end{aligned} \tag{28}$$

that is,

$$\|T_n u\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega(\widetilde{r}_1). \tag{29}$$

Given $u \in K \cap \partial\Omega(\widetilde{r}_2)$, one has $u(t) \geq \widetilde{r}_2 t^{\alpha-1}$, $0 \leq t \leq 1$. Thus, for $n > 1/\widetilde{r}_2$, we get

$$\widetilde{r}_2 \geq \chi_n(u-w)(t) \geq \left(\widetilde{r}_2 - \sigma\|k\|_1\right)t^{\alpha-1}. \tag{30}$$

According to the definition of $\psi(t,\widetilde{r}_2)$, one gets

$$f(t,\chi_n(u-w)(t)) + k(t) \geq \psi(t,\widetilde{r}_2). \tag{31}$$

By Lemma 2, $(A_3)$, and (31), we have

$$\begin{aligned}&\|T_n u\| \\ &\quad = \max_{0\leq t\leq 1}\int_0^1 G(t,s)\left[f(s,\chi_n(u-w)(s)) + k(s)\right] ds \\ &\quad \geq \max_{0\leq t\leq 1} t^{\alpha-1}\int_0^1 J(s)\left[f(s,\chi_n(u-w)(s)) + k(s)\right] ds \\ &\quad \geq \int_0^1 J(s)\psi(s,\widetilde{r}_2) ds \geq \widetilde{r}_2;\end{aligned} \tag{32}$$

that is,

$$\|T_n u\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega(\widetilde{r}_2). \tag{33}$$

Take $m_0 = \max\{1/\widetilde{r}_1, 1/\widetilde{r}_2\}$. Let $N = \{m_0, m_0+1, \ldots\}$. Then, for $n \in N$, both (29) and (33) hold. This together with Lemma 5 shows that $T_n$ $(n \in N)$ has at least one fixed point $\widetilde{u}_n \in K \cap \overline{\Omega}_{\widetilde{r}_2} \setminus \Omega_{\widetilde{r}_1}$.

*(III) BVP (1)-(2) Admits at Least One Positive Solution.* By Lemma 4, one has

$$\widetilde{u}_n(t) \geq \|\widetilde{u}_n\| t^{\alpha-1} \geq \widetilde{r}_1 t^{\alpha-1} > \sigma \|k\|_1 t^{\alpha-1} \geq w(t), \quad m > N, \ t \in [0,1], \tag{34}$$

$$\widetilde{u}_n(t) = \int_0^1 G(t,s)\left(f\left(s, \chi_n(\widetilde{u}_n - w)(s)\right) + k(s)\right) \mathrm{d}s, \quad 0 < t < 1. \tag{35}$$

By $(\mathrm{A}_1)$, we know $\{\widetilde{u}_n : n > N\}$ are bounded and equicontinuous sets $[0,1]$. It follows from Arzela-Ascoli theorem that there exist a subsequence $N_0$ of $N$ and a function $\widetilde{u} \in C[0,1]$ such that $\widetilde{u}_n$ converges to $\widetilde{u}$ uniformly on $[0,1]$ as $n \to \infty$ through $N_0$. Taking limit as $n \to \infty$ on both sides of (35) together with the fact $\widetilde{u}_n(t) \geq t^{\alpha-1}\|\widetilde{u}_n\| \geq \widetilde{r}_1 t^{\alpha-1}$, we get

$$\widetilde{u}(t) = \int_0^1 G(t,s)\left(f\left(s, \widetilde{u}(s) - w(s)\right) + k(s)\right) \mathrm{d}s, \quad 0 < t < 1. \tag{36}$$

Let $u(t) = \widetilde{u}(t) - w(t)$, and then from (36) one has that $u(t)$ is a positive solution for BVP (1)-(2).

A multiplicity result follows if we replace $(\mathrm{A}_2)$ and $(\mathrm{A}_3)$ by the following:

$(\mathrm{A}_2')$ There exists $\widetilde{r}_i > \sigma\|k\|_1$ $(i = 1, 2, \ldots, m)$ such that

$$\int_0^1 J(s)\varphi(s, \widetilde{r}_i)\,\mathrm{d}s < \widetilde{r}_i, \tag{37}$$

where $\varphi(t, \widetilde{r}_i) = \max\{f(t,u) : (\widetilde{r}_i - \sigma\|k\|_1)t^{\alpha-1} \leq u \leq \widetilde{r}_i\} + k(t)$.

$(\mathrm{A}_3')$ There exists $\widetilde{R}_i$ $(i = 1, 2, \ldots, m)$ with

$$0 < \widetilde{r}_1 < \widetilde{R}_1 < \widetilde{r}_2 < \widetilde{R}_2 < \cdots < \widetilde{r}_m < \widetilde{R}_m \tag{38}$$

such that

$$\int_0^1 J(s)\psi(s, \widetilde{R}_i)\,\mathrm{d}s > \widetilde{R}_i, \tag{39}$$

where $\psi(t, \widetilde{R}_i) = \min\{f(t,u) : (\widetilde{R}_i - \sigma\|k\|_1)t^{\alpha-1} \leq u \leq \widetilde{R}_i\} + k(t)$. □

**Theorem 7.** *Suppose that conditions* $(\mathrm{A}_0)$, $(\mathrm{A}_1)$, $(\mathrm{A}_2')$, *and* $(\mathrm{A}_3')$ *hold. Then BVP (1)-(2) has at least m positive solutions.*

## 4. An Example

Consider the fractional differential equation

$$D_{0+}^{8/3}u(t) + \frac{1}{16\sqrt[3]{t^2(1-t)}}\left(u^6 + \frac{1}{\sqrt[8]{u}}\right) - \frac{1}{12\sqrt[3]{t}} = 0, \quad 0 < t < 1, \tag{40}$$

with multipoint BCs

$$u(0) = u'(0) = 0,$$
$$D_{0+}^{1/2}u(1) = \frac{1}{2}D_{0+}^{1/3}u\left(\frac{1}{4}\right) + \frac{2}{3}D_{0+}^{1/3}u\left(\frac{1}{2}\right) + \frac{1}{4}D_{0+}^{1/4}u\left(\frac{4}{5}\right). \tag{41}$$

It is clear that (40)-(41) has the form of (1)-(2), where $\alpha = 8/3$, $n = 3$, $p = 1/2$, $q = 1/3$, $m = 3$, $\xi_1 = 1/2$, $\xi_2 = 1/4$, $\xi_3 = 4/5$, $a_1 = 1/2$, $a_2 = 2/3$, and $a_3 = 1/4$. Obviously, $(\mathrm{A}_0)$ holds for $k(t) = 1/12\sqrt[3]{t}$. After direct calculation, we have $\alpha - p - 1 = 7/6$, $\alpha - q - 1 = 4/3$, $\Gamma(\alpha) = 1.5046$, $\Gamma(\alpha - p) = 1.0823$, $\Gamma(\alpha - q) = 1.1906$, $\sum_{i=1}^{3} a_i\xi_i^{\alpha-q-1} = 0.5290$, $\Delta = 0.7217$, $\sigma = 1.2803$, $\|k\|_1 = 0.1250$, and $\sigma\|k\|_1 = 0.1600$. It is clear that $(\mathrm{A}_1)$ is valid for $\gamma_{r_1,r_2} = (1/16\sqrt[3]{t^2(1-t)})[{r_2}^6 + (r_1 t^{5/3})^{-1/8}] + 1/12\sqrt[3]{t}$. Next, we check $(\mathrm{A}_2)$ and $(\mathrm{A}_3)$.

Take $\widetilde{r}_1 = 1 > 0.1600 = \sigma\|k\|_1$, and then by Lemma 2 we have

$$\begin{aligned}
&\int_0^1 J(s)\varphi(s,1)\,\mathrm{d}s \leq \int_0^1 \left[\frac{(1-s)^{\alpha-p-1}\left(1-(1-s)^p\right)}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^{m} a_i\xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1}\right] \\
&\cdot \frac{1}{16\sqrt[3]{s^2(1-s)}} \max\left\{\left(u^6 + \frac{1}{\sqrt[8]{u}}\right) : 0.8400s^{5/3} \leq u \leq 1\right\} \mathrm{d}s = \int_0^1 \left[\frac{(1-s)^{7/6}\left(1-(1-s)^{1/2}\right)}{\Gamma(\alpha)} + \frac{\sum_{i=1}^{3} a_i\xi_i^{4/3}}{\Delta\Gamma(\alpha-q)}(1-s)^{7/6}\right] \\
&\cdot \frac{1}{16\sqrt[3]{s^2(1-s)}} \max\left\{\left(u^6 + \frac{1}{\sqrt[8]{u}}\right) : 0.8400s^{5/3} \leq u \leq 1\right\} \mathrm{d}s < \frac{1}{16}\int_0^1 \left[0.6646(1-s)^{7/6} - 0.6646(1-s)^{5/3} + 0.6156 \times (1-s)^{7/6}\right] s^{-2/3}(1-s)^{-1/3}\left(1 + \frac{1}{\sqrt[8]{0.8400s^{5/3}}}\right) \mathrm{d}s \\
&< \frac{1}{16}\int_0^1 \left[1.2802(1-s)^{7/6} - 0.6646(1-s)^{5/3}\right] s^{-2/3}(1-s)^{-1/3}\left(1 + 1.0220s^{-5/24}\right) \mathrm{d}s = \frac{1}{16}(1.2802 \times 2.3283 \\
&- 0.6646 \times 2.1200 + 1.2802 \times 1.0220 \times 7.2087 - 0.6646 \times 1.0220 \times 6.9455) = \frac{1}{16} \times 6.2858 = 0.3929 < 1.
\end{aligned} \tag{42}$$

Thus, $(\mathrm{A}_2)$ is verified.

Take $\widetilde{r}_2 = 20$, and, then, we have

$$\int_0^1 J(s)\psi(s,20)\,\mathrm{d}s \geq \int_0^1 \left[\frac{(1-s)^{\alpha-p-1}\left(1-(1-s)^p\right)}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-q)}\left(\sum_{i=1}^{m} a_i\xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1} - (1-s)^{\alpha-q-1}\right)\right]$$

$$\cdot \frac{1}{16\sqrt[3]{s^2(1-s)}} \min\left\{\left(u^6 + \frac{1}{\sqrt[8]{u}}\right) : 9.8400 s^{5/3} \le u \le 20\right\} \mathrm{d}s \ge \int_0^1 \left[\frac{(1-s)^{7/6}\left(1-(1-s)^{1/2}\right)}{\Gamma(\alpha)} + \frac{\sum_{i=1}^{3} a_i \xi_i^{4/3}}{\Delta\Gamma(\alpha-q)}(1-s)^{7/6} - \frac{1}{\Delta\Gamma(\alpha-q)}(1-s)^{4/3}\right] \cdot \frac{1}{16\sqrt[3]{s^2(1-s)}}\left(9.8400^6 s^{10} + \frac{1}{\sqrt[8]{20}}\right)\mathrm{d}s \ge \frac{9.8400^6}{16}\int_0^1 \left[\left(\frac{1}{\Gamma(\alpha)} + \frac{\sum_{i=1}^{3} a_i \xi_i^{4/3}}{\Delta\Gamma(\alpha-q)}\right) s^{28/3}(1-s)^{5/6} - \frac{1}{\Gamma(\alpha)} s^{28/3}(1-s)^{4/3} - \frac{1}{\Delta\Gamma(\alpha-q)} s^{28/3}(1-s)\right]\mathrm{d}s = 56735\int_0^1 (0.6646 + 0.6156) s^{28/3}(1-s)^{5/6} - 0.6646 s^{28/3}(1-s)^{4/3} - 1.1637 s^{28/3}(1-s)\,\mathrm{d}s = 56735 \times (1.2802 \times 0.0121 - 0.6646 \times 0.0044 - 1.1637 \times 0.0085) = 56735 \times 0.0027 = 153.1845 > 20. \tag{43}$$

Thus, $(A_3)$ is valid. According to Theorem 6, BVP (40)-(41) admits at least one positive solution.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

The authors contributed to each part of this work equally and read and approved the final version of the manuscript.

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