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Research Article

Poisson Summation Formulae Associated with the Special Affine Fourier Transform and Offset Hilbert Transform

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This paper investigates the generalized pattern of Poisson summation formulae from the special affine Fourier transform (SAFT) and offset Hilbert transform (OHT) points of view. Several novel summation formulae are derived accordingly. Firstly, the relationship between SAFT (or OHT) and Fourier transform (FT) is obtained. Then, the generalized Poisson sum formulae are obtained based on above relationships. The novel results can be regarded as the generalizations of the classical results in several transform domains such as FT, fractional Fourier transform, and the linear canonical transform.

1. Introduction

The classical Poisson summation formula, which demonstrates that the sum of infinite samples in the time domain of a signal $f(t)$ is equivalent to the sum of infinite samples of $F(\cdot)$ in the Fourier domain, is of importance in theories and applications of signal processing [1]. The traditional Poisson sum formula can be represented as follows [2]:

$$\sum_{k=-\infty}^{+\infty} f(t+k\tau) = \frac{1}{\tau}\sum_{n=-\infty}^{+\infty} F\left(\frac{n}{\tau}\right) e^{j(nt/\tau)} \quad (1)$$

or

$$\sum_{k=-\infty}^{+\infty} f(k\tau) = \frac{1}{\tau}\sum_{n=-\infty}^{+\infty} F\left(\frac{n}{\tau}\right), \quad t=0, \quad (2)$$

where $F(\cdot)$ denotes the Fourier transform (FT) of a signal $f(t)$ and τ stands for the sampling interval. Not only does Poisson summation formula play a key role in various branches of the mathematics, but also it finds numerous applications in lots of fields, for example, mechanics, signal processing community, and many other scientific fields. The Poisson summation formula is related to the Fourier transform, and, with the development of modern signal processing technologies, there are many other kinds of transforms that have been proposed, it is therefore worthwhile and interesting to investigate the Poisson sum formula in deep associated with these kinds of new integral transforms.

The special affine Fourier transform (SAFT) [3, 4], also known as the offset linear canonical transform [5, 6] or the inhomogeneous canonical transform [5], is a six-parameter (a,b,c,d,u_0,w_0) class of linear integral transform. Many well-known transforms in signal processing and optic systems are its special cases such as Fourier transform (FT), fractional Fourier transform (FRFT), the linear canonical transform (LCT), time shifting and scaling, frequency modulation, pulse chirping, and others [7, 8]. SAFT can be interpreted as a time shifting and frequency modulated version of LCT [9–11], that is much more flexible because of its extra parameters (u_0,w_0). Recently, it has been widely noticed in many practical applications along with the rapid development of LCT [12–14]. Thus, developing relevant theorems for SAFT is of importance and necessary in optical systems and many signal processing applications as well.

In addition, the generalized Hilbert transform closely related to SAFT, called offset Hilbert transform (OHT), is another powerful tool in the fields of optics and signal processing community [15]. It has been presented recently and widely used for image processing, especially for edge detection and enhancement, because it can emphasize the derivatives of the image [16, 17]. In recent decades, many essential theories and useful applications of SAFT and OHT have been derived from in-depth researching on it [8, 15, 18, 19].

To the best of our knowledge, Poisson sum formula has been generalized with many transforms such as FRFT, LCT, fractional Laplace transform and fractional Hilbert transform [1, 2, 20, 21]. However, none of the research papers throw light on the study of the traditional Poisson sum formula associated with the SAFT and OHT yet. Based on the existing results, the motivation of this paper is to generalize the above-mentioned Poisson sum formula into SAFT and OHT domains.

The rest of this paper is organized as follows. Section 2 gives some fundamental knowledge of SAFT and OHT. In Section 3, we give the relationships between SAFT/OHT and FT in detail. Some novel Poisson summation formulae associated with SAFT are presented in Section 4. Section 5 concludes the paper.

2. Preliminaries

2.1. The Special Affine Fourier Transform. The special affine Fourier transform (SAFT) with real parameters $\mathbf{A} = (a, b, c, d, u_0, w_0)$ of a signal $f(t)$ is defined by the following [5, 22]:

$$F_{\mathbf{A}}(u) = O_L^{\mathbf{A}}[f(t)](u) = \begin{cases} \int_{-\infty}^{+\infty} f(t) h_{\mathbf{A}}(t,u)\,dt & b \neq 0 \\ \sqrt{d} e^{j(cd/2)(u-u_0)^2 + jw_0 u} f[d(u-u_0)] & b = 0, \end{cases} \quad (3)$$

where

$$h_{\mathbf{A}}(t,u) = K_{\mathbf{A}} e^{(j/2b)[at^2 + 2t(u_0 - u) - 2u(du_0 - bw_0) + du^2]}, \quad K_{\mathbf{A}} = \sqrt{\frac{1}{j2\pi b}} e^{j(d/2b)u_0^2} \quad (4)$$

and $ad - bc = 1$. Note that, for $b = 0$, the SAFT of a signal is essentially a chirp multiplication and it is of no particular interest for our objective in this work. Hence, without loss of generality, we set $b \neq 0$ in the following section unless stated otherwise. The inverse of an SAFT with parameter $\mathbf{A} = (a, b, c, d, u_0, w_0)$ is given by an SAFT with parameter $\mathbf{A}^{-1} = (d, -b, -c, a, bw_0 - du_0, cu_0 - aw_0)$, which is

$$f(t) = O_L^{\mathbf{A}^{-1}}[F_{\mathbf{A}}(u)](t) = C \int_{-\infty}^{+\infty} F_{\mathbf{A}}(u) h_{\mathbf{A}^{-1}}(u,t)\,du, \quad (5)$$

where $C = e^{j(1/2)(cdu_0^2 - 2adu_0 w_0 + abw_0^2)}$. This can be verified by the definition of SAFT. Most of important transforms can be its special cases when parameter $\mathbf{A}$ is replaced with specific parameters. For example, when $\mathbf{A} = (0, 1, -1, 0, 0, 0)$, SAFT coincides with FT; when $\mathbf{A} = (\cos\alpha, \sin\alpha, -\cos\alpha, \sin\alpha, 0, 0)$, SAFT is FRFT; when $\mathbf{A} = (a, b, c, d, 0, 0)$, SAFT equals LCT. Furthermore, many important theories on SAFT have been investigated [8, 15, 23, 24].

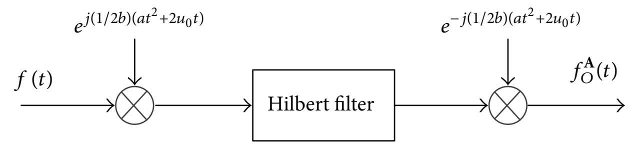

Figure 1: Offset Hilbert filter.

2.2. Offset Hilbert Transform. The offset Hilbert transform (OHT) of a signal $f(t)$ is defined as follows [15]:

$$f_O^{\mathbf{A}}(t) = H_O^{\mathbf{A}}[f](t) = \text{p.v.}\frac{e^{-j((at^2+2u_0t)/2b)}}{\pi} \int_{-\infty}^{+\infty} \frac{f(x)}{t-x} e^{j((ax^2+2u_0x)/2b)}\,dx. \quad (6)$$

It should be noted that the above definition uses the Cauchy principal value of the integral (denoted here by p.v.). To obtain the relationship between the stand and HT and OHT, we can rewrite (6) as

$$f_O^{\mathbf{A}}(t) = H_O^{\mathbf{A}}[f](t) = e^{-j((at^2+2u_0t)/2b)} \left(f(t) e^{j((at^2+2u_0t)/2b)} \right) * h(t). \quad (7)$$

Notice that computing the OHT of a signal f is equivalent to multiplying it by a chirp, $e^{j((at^2+2u_0t)/2b)}$, then passing the product through a standard Hilbert filter and finally multiplying the output by the chirp, $e^{-j((at^2+2u_0t)/2b)}$. This relationship between OHT and the classical HT can be shown in Figure 1.

3. The Relationships between SAFT/OHT and Fourier Transform

In order to derive novel Poisson summation formulae based on SAFT and OHT, some relationships between SAFT/OHT and FT are obtained in this section firstly.

Lemma 1. *Suppose the SAFT of a signal $f(t)$ with parameters $\mathbf{A} = (a, b, c, d, u_0, w_0)$ is $F_{\mathbf{A}}(\cdot)$, and set $g(t) = f(t)e^{j(a/2b)t^2}$, and then the following relations hold:*

$$F_{\mathbf{A}}(u) = \frac{1}{\sqrt{j2\pi b}} e^{j(1/2b)[d(u_0^2+u^2) - 2u(du_0 - bw_0)]} G\left(\frac{u-u_0}{b}\right), \quad (8)$$
$$G(v) = \sqrt{j2\pi b} e^{-j(1/2)[dbv^2 + 2bvw_0 + 2u_0w_0]} F_{\mathbf{A}}(bv + u_0),$$

where $G(\cdot)$ is the FT of signal $g(t)$.

Proof. It is easy to verify Lemma 1 by the definitions of SAFT and FT. □

Lemma 2. *Suppose the SAFT of a signal $f(t)$ with parameters $\mathbf{A} = (a, b, c, d, u_0, w_0)$ is $F_{\mathbf{A}}(\cdot)$, and set $q(x) = (f(t-x)/x)e^{j(a/2b)x^2}$, and then the following relations hold:*

$$f_O^{\mathbf{A}}(t) = \frac{1}{\pi} Q\left(\frac{at+u_0}{b}\right), \quad Q(v) = \pi f_O^{\mathbf{A}}\left(\frac{bv-u_0}{a}\right), \tag{9}$$

where $Q(\cdot)$ is the FT of signal $q(x)$.

Proof. According to the definition of OHT,

$$\begin{aligned} f_O^{\mathbf{A}}(t) &= \frac{e^{-j((at^2+2u_0t)/2b)}}{\pi}\int_{-\infty}^{+\infty}\frac{f(x)\,e^{j((ax^2+2u_0x)/2b)}}{t-x}dx \\ &= \frac{e^{-j((at^2+2u_0t)/2b)}}{\pi} \cdot \int_{-\infty}^{+\infty}\frac{f(t-x)\,e^{j((a(t-x)^2+2u_0(t-x))/2b)}}{x}dx = \frac{1}{\pi} \\ &\cdot \int_{-\infty}^{+\infty}\frac{f(t-x)}{x}e^{j(a/2b)x^2}e^{-j((at+u_0)/b)x}dx = \frac{1}{\pi} \\ &\cdot \int_{-\infty}^{+\infty} q(x)\,e^{-j((at+u_0)/b)x}dx = \frac{1}{\pi}Q\left(\frac{at+u_0}{b}\right). \end{aligned} \tag{10}$$

By replacing $v = (at+u_0)/b$, (10) can be rewritten as

$$f_O^{\mathbf{A}}\left(\frac{bv-u_0}{a}\right) = \frac{1}{\pi}Q(v). \tag{11}$$

This completes the proof of Lemma 2. □

4. Main Results

Based on the relationships in Lemmas 1 and 2, the generalized Poisson summation formulae associated with SAFT and OHT are derived in following subsections, respectively.

4.1. The Poisson Sum Formula Based on SAFT

Theorem 3. *The Poisson summation formulae of a signal $f(t)$ in the SAFT domain with parameter $\mathbf{A}$ are*

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} f(t+k\tau)\,e^{j(a/2b)(2kt\tau+k^2\tau^2)} &= \frac{\sqrt{j2\pi b}}{\tau} \\ &\cdot e^{-j(u_0w_0+(a/2b)t^2)}\sum_{n=-\infty}^{+\infty}e^{-j(dbn^2/2\tau^2+bnw_0/\tau)}F_{\mathbf{A}}\left(\frac{bn}{\tau}\right. \\ &\left.+u_0\right)e^{j(nt/\tau)}, \end{aligned} \tag{12}$$

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} f(k\tau)\,e^{j(a/2b)k^2\tau^2} &= \frac{\sqrt{j2\pi b}}{\tau} \\ &\cdot e^{-ju_0w_0}\sum_{n=-\infty}^{+\infty}e^{-j(dbn^2/2\tau^2+bnw_0/\tau)}F_{\mathbf{A}}\left(\frac{bn}{\tau}+u_0\right), \\ &\qquad t = 0. \end{aligned} \tag{13}$$

Proof. If we set $g(t) = f(t)e^{j(a/2b)t^2}$, from the traditional Poisson sum formula for $g(t)$ in the Fourier domain, that is, (1), we obtain

$$\sum_{k=-\infty}^{+\infty} g(t+k\tau) = \frac{1}{\tau}\sum_{n=-\infty}^{+\infty} G\left(\frac{n}{\tau}\right)e^{j(nt/\tau)}. \tag{14}$$

By directly using Lemma 1, we derive that

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} f(t+k\tau)\,e^{j(a/2b)(t+k\tau)^2} &= \frac{1}{\tau} \\ &\cdot \sum_{n=-\infty}^{+\infty}\sqrt{j2\pi b}e^{-j(1/2b)[db^2(n/\tau)^2+2b^2(n/\tau)w_0+2bu_0w_0]}F_{\mathbf{A}}\left(b\right. \\ &\left.\cdot\frac{n}{\tau}+u_0\right)e^{j(nt/\tau)}. \end{aligned} \tag{15}$$

Theorem 3 is proved by simple calculation on (15). □

Equations (13) and (15) can be regarded as the generalization of classical Poisson sum formula based on SAFT. It should be noticed that when the parameters of the SAFT are chosen to be the special cases of the SAFT, the derived results reduce to the classical results of Fourier transform domain, fractional Fourier transform domain, and linear canonical transform domains. It clearly demonstrates that the infinite sum of periodic phase-shifted replica of a signal $f(t)$ in the time domain is equivalent to the infinite sum of periodic phase-shifted replica $F_{\mathbf{A}}(\cdot)$ in the SAFT domain.

In addition, it is of importance to investigate the Poisson sum formula of signals with compact support in SAFT domain. A signal $f(t)$ is said to have compact support in SAFT domain if its SAFT $F_{\mathbf{A}}(u) > \Omega_{\mathbf{A}}$, where $\Omega_{\mathbf{A}} > 0$ is some real number. Without loss of generality, let $a > 0, b > 0, u_0 > 0$ in the following analysis.

Corollary 4. *Suppose a signal $f(t)$ is band-limited in SAFT domain with a compact support $\Omega_{\mathbf{A}}$; then the Poisson sum formula derived in Theorem 3 can be rewritten as the following forms according to the replica period τ:*

(a) *When $b/\tau > \Omega_{\mathbf{A}} + u_0$ and $\Omega_{\mathbf{A}} > u_0$,*

$$\sum_{k=-\infty}^{+\infty} f(t+k\tau)\,e^{j(a/2b)(2kt\tau+k^2\tau^2)} = \frac{\sqrt{j2\pi b}}{\tau}e^{-j(a/2b)t^2}e^{-ju_0w_0}F_{\mathbf{A}}(u_0). \tag{16}$$

(b) *When* $(\Omega_{\mathbf{A}} + u_0)/2 < b/\tau < \Omega_{\mathbf{A}} - u_0$ *and* $\Omega_{\mathbf{A}} > 3u_0$,

$$\sum_{k=-\infty}^{+\infty} f(t+k\tau) e^{j(a/2b)(2kt\tau+k^2\tau^2)} = \frac{\sqrt{j2\pi b}}{\tau} \cdot e^{-j(a/2b)t^2} e^{-ju_0w_0} \times \left\{ F_{\mathbf{A}}(u_0) + e^{-j(1/2)[db/\tau^2+2bw_0/\tau]} F_{\mathbf{A}}\left(\frac{b}{\tau}+u_0\right) e^{j(t/\tau)} + e^{-j(1/2)[db/\tau^2-2bw_0/\tau]} F_{\mathbf{A}}\left(-\frac{b}{\tau}+u_0\right) e^{-j(t/\tau)} \right\}. \quad (17)$$

(c) *When* $(\Omega_{\mathbf{A}} + u_0)/(m+1) < b/\tau < (\Omega_{\mathbf{A}} - u_0)/m$ *and* $\Omega_{\mathbf{A}} > (2m+1)u_0$,

$$\sum_{k=-\infty}^{+\infty} f(t+k\tau) e^{j(a/2b)(2kt\tau+k^2\tau^2)} = \frac{\sqrt{j2\pi b}}{\tau} \cdot e^{-j(a/2b)t^2} e^{-ju_0w_0} \sum_{n=-m}^{m} e^{-j(dbn^2/2\tau^2+bnw_0/\tau)} F_{\mathbf{A}}\left(\frac{bn}{\tau}+u_0\right) e^{j(nt/\tau)}. \quad (18)$$

Proof. (a) Since $f(t)$ is a band-limited signal in SAFT domain with a compact support $\Omega_{\mathbf{A}}$, it is easy to derive the right hand $F_{\mathbf{A}}(bn/\tau+u_0)$ of (12) that is equal to zeros when $n \neq 0$. That is, from $b/\tau > \Omega_{\mathbf{A}} + u_0$, we derive that $b/\tau + u_0 > b/\tau - u_0 > \Omega_{\mathbf{A}}$ and $-2b/\tau + u_0 < -b/\tau + u_0 < -\Omega_{\mathbf{A}}$. Thus, it is easy to derive that

$$\frac{\sqrt{j2\pi b}}{\tau} e^{-j(u_0w_0+(a/2b)t^2)} \sum_{n=-\infty}^{+\infty} e^{-j(dbn^2/2\tau^2+bnw_0/\tau)} F_{\mathbf{A}}\left(\frac{bn}{\tau}+u_0\right) e^{j(nt/\tau)} = \frac{\sqrt{j2\pi b}}{\tau} e^{-j(a/2b)t^2} e^{-ju_0w_0} F_{\mathbf{A}}(u_0). \quad (19)$$

Substituting (19) into (12) yields the final results.

(b) It is easy to prove that only when $n = -1, 0, +1$, $F_{\mathbf{A}}(bn/\tau + u_0)$ is nonzero. The right hand of (12) is

$$\frac{\sqrt{j2\pi b}}{\tau} \cdot e^{-j(u_0w_0+(a/2b)t^2)} \sum_{n=-\infty}^{+\infty} e^{-j(dbn^2/2\tau^2+bnw_0/\tau)} F_{\mathbf{A}}\left(\frac{bn}{\tau}+u_0\right) e^{j(nt/\tau)} = \frac{\sqrt{j2\pi b}}{\tau} e^{-j(a/2b)t^2} e^{-ju_0w_0} \left\{ F_{\mathbf{A}}(u_0) + e^{-j(1/2)[db/\tau^2+2bw_0/\tau]} F_{\mathbf{A}}\left(\frac{b}{\tau}+u_0\right) e^{j(t/\tau)} + e^{-j(1/2)[db/\tau^2-2bw_0/\tau]} F_{\mathbf{A}}\left(-\frac{b}{\tau}+u_0\right) e^{-j(t/\tau)} \right\}. \quad (20)$$

Substituting (20) into (12) yields the final results.

(c) The proof of this situation is similar to the proof of (a) and (b), and we omit it here. □

4.2. The Poisson Sum Formula Based on OHT

Theorem 5. *The Poisson sum formula of a signal* $f(t)$ *in the OHT domain with parameter* $\mathbf{A}$ *is as follows:*

$$\sum_{k=-\infty}^{+\infty} \frac{f(t-y-k\tau)}{y+k\tau} e^{j(a/2b)(2ky\tau+k^2\tau^2)} = \frac{\pi}{\tau} e^{-j(a/2b)y^2} \sum_{n=-\infty}^{+\infty} f_O^{\mathbf{A}}\left(\frac{bn-u_0\tau}{a\tau}\right) e^{j(ny/\tau)}. \quad (21)$$

Proof. If we set $q(x) = (f(t-x)/x)e^{j(a/2b)x^2}$, it is easy to verify Theorem 5 via (1) and Lemma 2:

$$\sum_{k=-\infty}^{+\infty} q(y+k\tau) = \frac{1}{\tau} \sum_{n=-\infty}^{+\infty} Q\left(\frac{n}{\tau}\right) e^{j(ny/\tau)}. \quad (22)$$

That is

$$\sum_{k=-\infty}^{+\infty} \frac{f(t-y-k\tau)}{y+k\tau} e^{j(a/2b)(y+k\tau)^2} = \frac{\pi}{\tau} \sum_{n=-\infty}^{+\infty} f_O^{\mathbf{A}}\left(\frac{bn-u_0\tau}{a\tau}\right) e^{j(ny/\tau)} \quad (23)$$

By simple calculation, Theorem 5 is completed. □

Equation (21) can be seen as the Poisson sum formula associated with offset Hilbert transform. Furthermore, it is worthwhile and interesting to study the signals with compact support in offset Hilbert transform domain. Let $f_O^{\mathbf{A}}(y)$ be the OHT of a signal $f(t)$. Then $f(t)$ is said to have compact support in OHT domain, if $f_O^{\mathbf{A}}(y) = 0$ for $|y| > \Omega_{\mathbf{A}}$, where $\Omega_{\mathbf{A}} > 0$ is some real number.

Corollary 6. *Suppose signal* $f(t)$ *is band-limited in OHT domain with a compact support* $\Omega_{\mathbf{A}}$*; then the Poisson sum formula derived in Theorem 5 can be rewritten as the following forms according to the replica period* τ*:*

(a) *When* $b/a\tau > \Omega_{\mathbf{A}} + u_0/a$ *and* $\Omega_{\mathbf{A}} > u_0/a$,

$$\sum_{k=-\infty}^{+\infty} \frac{f(t-y-k\tau)}{y+k\tau} e^{j(a/2b)(y+k\tau)^2} = \frac{\pi}{\tau} f_O^{\mathbf{A}}\left(-\frac{u_0}{a}\right). \quad (24)$$

(b) *When* $(\Omega_{\mathbf{A}} + u_0/a)/2 < b/\tau < \Omega_{\mathbf{A}} - u_0/a$ *and* $\Omega_{\mathbf{A}} > 3(u_0/a)$,

$$\sum_{k=-\infty}^{+\infty} \frac{f(t-y-k\tau)}{y+k\tau} e^{j(a/2b)(y+k\tau)^2} = \frac{\pi}{\tau} \left[f_O^{\mathbf{A}}\left(\frac{-b-u_0\tau}{a\tau}\right) e^{-j(y/\tau)} + f_O^{\mathbf{A}}\left(-\frac{u_0}{a}\right) + f_O^{\mathbf{A}}\left(\frac{b-u_0\tau}{a\tau}\right) e^{j(y/\tau)} \right]. \quad (25)$$

(c) *When* $(\Omega_{\mathbf{A}} + u_0/a)/(m+1) < b/\tau < (\Omega_{\mathbf{A}} - u_0/a)/m$ *and* $\Omega_{\mathbf{A}} > (2m+1)(u_0/a)$,

$$\sum_{k=-\infty}^{+\infty} \frac{f(t-y-k\tau)}{y+k\tau} e^{j(a/2b)(y+k\tau)^2} = \frac{\pi}{\tau} \sum_{n=-m}^{m} f_{\mathrm{O}}^{\mathbf{A}}\left(\frac{bn-u_0\tau}{a\tau}\right) e^{j(ny/\tau)}. \quad (26)$$

Proof. It is easy to verify this corollary using Theorem 5 and the similar method in Corollary 4. □

5. Conclusion

In this paper, the traditional Poisson summation formula has been generalized into SAFT and OHT domain. Theorems 3 and 5 are the generalizations of Poisson summation formulae based on SAFT and OHT, respectively. In addition, signals with compact support are mostly used in signal processing and considered in this paper as well. Some novel results associated with Poisson summation formula have been derived in the form of Corollaries 4 and 6.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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