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PSEUDO-SASAKIAN MANIFOLDS ENDOWED WITH A CONTACT CONFORMAL CONNECTION

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ABSTRACT. Pseudo-Sasakian manifolds $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ endowed with a contact conformal connection are defined. It is proved that such manifolds are space forms $\tilde{M}(K), K < 0$, and some remarkable properties of the Lie algebra of infinitesimal transformations of the principal vector field $\tilde{U}$ on $\tilde{M}$ are discussed. Properties of the leaves of a co-isotropic foliation on $\tilde{M}$ and properties of the tangent bundle manifold $T\tilde{M}$ having $\tilde{M}$ as a basis are studied.

KEY WORDS AND PHRASES. *Witt frame, CICR submanifold, relative contact infinitesimal transformation, U-contact concircular pairing, differential form of Godbillon-Vey, form of E. Cartan, Finslerian form, mechanical system, dynamical system, spray, CR product.*
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1. INTRODUCTION.

In the last years many papers have been concerned with Sasakian manifold $\tilde{M}(\phi,\xi,\tilde{\eta},\tilde{g})$ and related structures. Recently Rosca [1] has defined *pseudo-Sasakian* manifolds $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ and Goldberg and Rosca [2] have studied *CICR submanifolds* (i.e. co-isotropic CR submanifolds) of $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$.

In the present paper we study $(2m+1)$-dimensional pseudo-Sasakian manifolds of index $m+1$, $m > 4$, structured by a *contact conformal* (abr. c.c.) connection. It is proved that such manifolds are hyperbolic space forms $\tilde{M}(K)$, $K < 0$, and with the c.c. connection (which in fact is a natural generalization of the connection defined by Rosca [3]) is associated (compare with Rosca [3]) a so denominated *principal vector field* $\tilde{U}$.

The paper is organized as follows. In Section 3 we develop some basic results induced by the c.c. connection and some remarkable properties of the Lie algebra of *infinitesimal transformations* defined by $\tilde{U}$. It is shown that

(i) $\tilde{U}$ (resp. $U\tilde{U}$) is *divergence free* (resp. defines an *infinitesimal homothety*) on $\tilde{M}$ and all connection forms on $\tilde{M}$ are integral relations of *invariance* for $U\tilde{U}$ (see Lichnerowicz [4]);

(ii) $\tilde{U}$ and $U\tilde{U}$ define an *U-contact concircular pairing* (in the sense of Rosca [5]) and any contact extension of $\tilde{U}$ is a *relative contact* *infinitesimal transformation* (in the sense of Rosca [3]) of the canonical 1-form $\tilde{\eta}$;

(iii) $\tilde{U}$ and $U\tilde{U}$ define both infinitesimal automorphisms of $(2q+1)$-forms $\tilde{\beta}_q = L^q\tilde{u}$ $(q<m)$ where $\tilde{u}$ (resp. L) is the dual form of $\tilde{U}$ (resp. the (1,1)-operator taken with respect to the 2-form $\tilde{\Omega} = d\tilde{\eta}/2$). Accordingly, if Σ_β is the exterior differential system defined by $\{\tilde{\beta}_q\}$, $\tilde{U}$ and $U\tilde{U}$ may be considered as *isovectors* of Σ_β.

Section 4 is concerned with a *co-isotropic foliation* F_c on $\tilde{M}$. The leaves M_c of F_c are CICR submanifolds of $\tilde{M}$ and if $\operatorname{codim} M_c = \ell$, then the *form of Godbillon-Vey* on M_c (see Lichnerowicz [6]) is a $(2\ell+1)$-form w_G which is a *relative integral invariant* of $U = \tilde{U}|_{M_c}$.

Further the necessary and sufficient conditions for M_c to be *foliate* is that the isotropic component $U^\perp$ of U vanishes. In this case M_c is a *CR product* (see Yano and Kon [7] and Rosca [8]).

Finally using some notions introduced by Yano and Ishihara [9] and also by Klein [10], we consider in Section 5 certain properties of the tangent bundle manifold $T\tilde{M}$ having the manifold $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ as a basis.

It is proved that the *complete lifts* $\tilde{\Omega}^C$ and $\tilde{u}^C$ of $\tilde{\Omega}$ and $\tilde{u}$ respectively are *homogeneous* of degree one and that the *form of E. Cartan* $\tilde{\Pi}$ on $T\tilde{M}$ is a *Finslerian form*. Furthermore, we may associate with $\tilde{\Pi}$ a *regular mechanical system* whose *dynamical system* is a *spray* on $\tilde{M}$.

2. PRELIMINARIES.

Let $(\tilde{M},\tilde{g})$ be a $(2m+1)$-dimensional connected pseudo-Riemannian manifold of signature $(m+1,m)$ and suppose that $m > 4$.

At each point $\tilde{p} \in \tilde{M}$ one has the standard decomposition (see Rosca [1]):

$$T_{\tilde{p}}(\tilde{M}) = H_{\tilde{p}} \oplus T_{\tilde{p}} \tag{2.1}$$

where $T_{\tilde{p}}$, $H_{\tilde{p}}$, and $T_{\tilde{p}}$ are the tangent space, a $(2m)$-dimensional *neutral* vector space, and a *time-like* line orthogonal to $H_{\tilde{p}}$, respectively.

Let $S_{\tilde{p}}, \overset{*}{S}_{\tilde{p}} \subset H_{\tilde{p}}$ be two *self-orthogonal* (abbreviation s.o.) m-distributions which define an *involutive* automorphism U of square $+1$ (U is the *para complex* operator defined by Libermann [11]). Let $\xi \in T_{\tilde{p}}$ and $\tilde{\eta} \in \Lambda^1(\tilde{M})$ be the pairing which defines a contact structure σ_c on $\tilde{M}$, and $\tilde{\nabla}$ be the covariant differentiation operator defined by the metric tensor $\tilde{g}$. Then if for any vector fields $\tilde{Z}$, $\tilde{Z}'$ on $\tilde{M}$ the structure tensors $(U,\xi,\tilde{\eta},\tilde{g})$ satisfy

$$\begin{cases} U^2(\tilde{Z}) = \tilde{Z} - \tilde{\eta}(\tilde{Z})\xi, & \tilde{g}(U\tilde{Z},U\tilde{Z}') = -\tilde{g}(\tilde{Z},\tilde{Z}') + \tilde{\eta}(\tilde{Z})\tilde{\eta}(\tilde{Z}'), \\ \tilde{g}(\tilde{Z},\xi) = \tilde{\eta}(\tilde{Z}), & \tilde{\nabla}_{\tilde{Z}}\xi = U\tilde{Z}, \\ d\tilde{\eta}(\tilde{Z},\tilde{Z}') = -2\tilde{g}(U\tilde{Z},\tilde{Z}'), & \tilde{\eta}(\xi) = 1, \end{cases} \tag{2.2}$$

the manifold $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ has been called a *pseudo-Sasakian manifold* (see Rosca [1]).

In order to study real *co-isotropic* and *isotropic foliations* on $\tilde{M}$ (that is *improper* immersions in $\tilde{M}$), we consider an adapted field of *Witt frames:* $\tilde{W} = \{h_A\colon A,B,C = 0,1,\ldots,2m\}$. The vectors h_a and h_{a^*} $(a=1,\ldots,m; a^*=a+m)$ are *null* and $h_0 = \xi$ is the *anisotropic* vector field of the W-basis $\{h_A\}$. We set

$$\tilde{S}_{\tilde{p}} = \{h_a\}, \qquad \tilde{S}^*_{\tilde{p}} = \{h_{a^*}\} \tag{2.3}$$

and as is known, one has

$$\begin{cases} \tilde{g}(h_a,h_b{*}) = \delta_{ab}, & \tilde{g}(\xi,h_a) = 0, \\ \tilde{g}(\xi,h_a{*}) = 0, & \tilde{g}(\xi,\xi) = 1 \end{cases} \tag{2.4}$$

and

$$Uh_a = h_a, \quad Uh_a{*} = -h_a{*}, \quad U\xi = 0. \tag{2.5}$$

If $\tilde{W}^* = \{\tilde{\omega}^\Lambda\}$ is the cobasis associated with $\tilde{W}$, we set $\tilde{\omega}^0 = \tilde{\eta}$ and the line element $d\tilde{p}$ ($d\tilde{p}$ is a canonical vector 1-form and is independent on any connection on $\tilde{M}$) is given by

$$d\tilde{p} = \tilde{\omega}^\Lambda \otimes h_\Lambda. \tag{2.6}$$

It follows from (2.4) that the metric tensor $\tilde{g}$ is:

$$\tilde{g} = 2\sum_a \tilde{\omega}^a \otimes \tilde{\omega}^{a^*} + \tilde{\eta}\otimes\tilde{\eta}. \tag{2.7}$$

If $\tilde{\theta}^\Lambda_B = \tilde{\gamma}^\Lambda_{BC}\tilde{\omega}^C$ ($\tilde{\gamma}^\Lambda_{BC} \in C^\infty(\tilde{M})$) and $\tilde{\Theta}^\Lambda_B$ are the connection forms and the curvature 2-forms on the bundle $\tilde{W}(\tilde{M})$ respectively, then the structure equations (E. Cartan) may be written in the indexless form as follows:

$$\tilde{\nabla}h = \tilde{\theta}\otimes h, \tag{2.8}$$

$$d\tilde{\omega} = -\tilde{\theta}\wedge\tilde{\omega}, \tag{2.9}$$

$$d\tilde{\theta} = -\tilde{\theta}\wedge\tilde{\theta} + \tilde{\Theta}. \tag{2.10}$$

Referring to (2.4) and (2.8), one has

$$\begin{cases} \tilde{\theta}^a_b + \tilde{\theta}^{b^*}_{a^*} = 0, & \tilde{\theta}^{a^*}_b = 0, \quad \tilde{\theta}^a_{b^*} = 0, \\ \tilde{\theta}^o_a + \tilde{\theta}^{a^*}_0 = 0, & \tilde{\theta}^a_0 + \tilde{\theta}^0_{a^*} = 0 \end{cases} \tag{2.11}$$

and

$$\tilde{\theta}^0_a = \tilde{\omega}^{a^*}, \quad \tilde{\theta}^0_{a^*} = -\tilde{\omega}^a. \tag{2.12}$$

By virtue of (2.8), (2.9), and (2.11) one has

$$d\tilde{\eta} = 2\sum_a \tilde{\omega}^a \wedge \tilde{\omega}^{a^*} \tag{2.13}$$

and

$$\tilde{\nabla}\xi = Ud\tilde{p} \Longrightarrow \langle\tilde{\nabla}_{\tilde{Z}}\xi,\tilde{Z}'\rangle + \langle\tilde{\nabla}_{\tilde{Z}'}\xi,\tilde{Z}\rangle = 0 \tag{2.14}$$

where $\tilde{Z}$ and $\tilde{Z}'$ are any vector fields on $\tilde{M}$.

In the following we agree to call the 2-form

$$\tilde{\Omega} = \sum_a \tilde{\omega}^a \wedge \tilde{\omega}^{a^*} \tag{2.15}$$

the *fundamental 2-form* on $\tilde{M}$

Since by (2.11) one has

$$\tilde{\theta}^a_a + \tilde{\theta}^{a^*}_{a^*} = 0, \quad \tilde{\Theta}^a_a + \tilde{\Theta}^{a^*}_{a^*} = 0, \tag{2.16}$$

we shall call

$$\tilde{\theta}_R = \sum_a \tilde{\theta}^a_a \tag{2.17}$$

and

$$\tilde{\Theta}_R = \sum_a \tilde{\Theta}^a_a \tag{2.18}$$

the *Ricci 1-form* and the *Ricci 2-form* respectively (see Rosca [12]). As is known, the form $\tilde{\Theta}_R$ defines the *first class of Chern* of $\tilde{M}$.

Using (2.10) and referring to (2.12) and (2.15), one quickly obtains

$$d\tilde{\theta}_R = \tilde{\Theta}_R - \tilde{\Omega} . \tag{2.19}$$

The above equation proves that the 2-forms $\tilde{\Theta}_R$ and $\tilde{\Omega}$ are homologous. Hence the two *cocycles* $\tilde{\Theta}_R$ and $\tilde{\Omega}$ belong to the *2-cohomology class* $H^2(\tilde{M})$ of $\tilde{M}$.

Let now F_c be a coisotropic foliation on $\tilde{M}$ and denote by M_c a maximal integral manifold (leave) of F_c. It has been shown by Goldberg and Rosca [2] that M_c is a *contact CR submanifold* of $\tilde{M}$, that is there exists a differentiable distribution $D: p \to D_p \subset T_p(M_c)$, $p \in M_c$ (one denotes the induced elements on M_c by suppressing $\sim$) satisfying:

(i) D is *invariant* i.e. $UD_p \subseteq D_p$, and

(ii) the complementary orthogonal distribution $D^\perp: p \to D_p^\perp \subset T_p(M_c)$ is *anti-invariant* i.e. $UD_p^\perp \subseteq T_p^\perp(M_c)$.

The distribution D (resp. $D^\perp$) is called the *horizontal* (resp. *vertical*) distribution. Such type of CR submanifolds is called *CICR submanifolds* (see Goldberg and Rosca [2]).

3. PSEUDO-SASAKIAN MANIFOLDS ENDOWED WITH A CONTACT CONFORMAL CONNECTION.

As a natural generalization of the definition given by Rosca [3], we assume that the structure equations (2.9) are written in the form

$$\begin{cases} d\tilde{\omega}^a = (\tilde{u}+\tilde{\eta}) \wedge \tilde{\omega}^a + \tilde{t}_a\tilde{\Omega} , \\ d\tilde{\omega}^{a^*} = (\tilde{u}-\tilde{\eta}) \wedge \tilde{\omega}^{a^*} + \tilde{t}_{a^*}\tilde{\Omega} \end{cases} \tag{3.1}$$

where $\tilde{\Omega} = d\tilde{\eta}/2$, $\tilde{t}_a, \tilde{t}_{a^*} \in C^\infty(M)$, and $u \in \Lambda^1(\tilde{M})$ is a *closed* 1-form. Note that $\tilde{t}_a$ and $\tilde{t}_{a^*}$ are the components of a vector field

$$\tilde{U} = \sum_a (\tilde{t}_a h_a + \tilde{t}_{a^*} h_{a^*}) \tag{3.2}$$

of *constant* length.

We shall say (see Rosca [3]) that in this case the pseudo-Sasakian manifold $\tilde{M}$ is endowed with a *contact conformal* (abr. c.c.) *connection*. We also agree to call $\tilde{U}$ the *principal vector field* associated with this connection.

Since $\tilde{g}(\tilde{U},\tilde{U})$ = const, we may write by (3.2) that

$$\sum_a \tilde{t}_{a^*}\tilde{t}_{a^*} = c, \quad c = \text{const.} \tag{3.3}$$

Taking exterior differentials of (3.1), we get

$$\begin{cases} d\tilde{t}_a = (\tilde{u}+\tilde{\eta})\tilde{t}_a - 2\tilde{\omega}^a, \\ d\tilde{t}_{a^*} = (\tilde{u}-\tilde{\eta})\tilde{t}_{a^*} - 2\tilde{\omega}^{a^*} . \end{cases} \tag{3.4}$$

Denote by Σ the exterior differential system defined by equations (3.1) and (3.4) and by I the *ideal* corresponding to Σ. The exterior differentiation of (3.4) where $\tilde{\omega}^a$ and $\tilde{\omega}^{a^*}$ satisfy (3.1), $\tilde{\Omega} = d\tilde{\eta}/2$, $d\tilde{u} = 0$, leads to the identity. Because of this, $dI \subset I$, that is Σ is a *closed* system. It follows from this that the system Σ defining the pseudo-Sasakian manifold $\tilde{M}$ endowed with a c.c. connection is *completely integrable* and its solution depends on $2m$ constants (the number of equations in (3.4)).

From (3.4) and (3.3) we also obtain

$$c\tilde{u} = \sum_a (t_{a^*}\tilde{\omega}^a - t_a\omega^{a^*}) \tag{3.5}$$

and $\tilde{u}(\tilde{U}) = 0$ which shows that $\tilde{u}$ is an *integral relation of invariance* for $\tilde{U}$ (see Lichnerowicz [4]). In the following we agree to call $\tilde{u}$ the *principal Pfaffian* associated with the c.c. connection.

Consider now the 1-form

$$\tilde{v} = \sum_a (\tilde{t}_a\tilde{\omega}^{a^*} + \tilde{t}_{a^*}\tilde{\omega}^a). \tag{3.6}$$

Taking the exterior differential of $\tilde{v}$, one finds with the help of (3.1) and (3.4) that $c = 2$. In this case we deduce

$$d\tilde{v} = 2\tilde{u} \wedge \tilde{v}, \tag{3.7}$$

and this equation asserts that $\tilde{v}$ is *exterior recurrent* (see Datta [13] with $2\tilde{u}$ as *the recurrence 1-form*.

By (2.4) and (2.5) one easily finds

$$\tilde{u}(U\tilde{U}) = \tilde{v}(\tilde{U}) = \tilde{g}(\tilde{U},\tilde{U}) = \tilde{g}(U\tilde{U},U\tilde{U}) = 2\sum \tilde{t}_a\tilde{t}_{a^*}. \tag{3.8}$$

Hence if $\flat : T(\tilde{M}) \to T^*(\tilde{M})$ is the *musical isomorphism* with respect to $\tilde{g}$ (see Poor [14]), we may write: $\tilde{u} = \flat(U\tilde{U})$, $\tilde{v} = \flat(\tilde{U})$. Since $\tilde{u}$ is closed, it follows from (3.7) that the manifold $\tilde{M}$ under consideration is foliated by 2-codimensional submanifolds orthogonal to $\tilde{U}$ and $U\tilde{U}$.

Next if μ: $\tilde{Z} \to i_{\tilde{Z}}\tilde{\Omega}$, $T(\tilde{M}) \to T^*(\tilde{M})$ is the bundle isomorphism defined by $\tilde{\Omega} = d\tilde{\eta}/2$, one readily finds

$$\mu(\tilde{U}) = 2\tilde{u}. \tag{3.9}$$

In the following we agree to call the *presymplectic* form $\tilde{\Omega}$ (dim ker($\tilde{\Omega}$) $\neq 0$) the *fundamental* 2-form on $\tilde{M}$.

Let now $\tilde{U}_f = \tilde{U} + \tilde{f}\xi$ ($\tilde{f} \in C^\infty(\tilde{M})$) be a *contact extension* of $\tilde{U}$ and $\mathcal{L}_{\tilde{U}_f}$ the Lie derivative with respect to $\tilde{U}_f$. Then by (3.9) one quickly finds $d\mathcal{L}_{\tilde{U}_f}\tilde{\eta} = 0$. Therefore according to the definition given by Rosca [3], we may say that $\tilde{U}_f$ is a *relative contact infinitesimal transformation* of $\tilde{\eta}$.

Denote now by $\tilde{\sigma}_S$ (resp. $\tilde{\sigma}_{S^*}$) the simple unit form which corresponds to $\tilde{S}_{\tilde{p}}$ (resp. $\tilde{S}^*_{\tilde{p}}$). One has

$$\begin{cases} \tilde{\sigma}_S = \tilde{\omega}^1 \wedge \ldots \wedge \tilde{\omega}^m, \\ \tilde{\sigma}_{S^*} = \tilde{\omega}^{1^*} \wedge \ldots \wedge \tilde{\omega}^{m^*}, \end{cases} \tag{3.10}$$

and by (3.1) the exterior differentials of (3.10) are

$$\begin{cases} d\tilde{\sigma}_S = [m(\tilde{u}+\tilde{\eta})-\tilde{v}] \wedge \tilde{\sigma}_S, \\ d\tilde{\sigma}_{S^*} = [m(\tilde{u}-\tilde{\eta})+\tilde{v}] \wedge \tilde{\sigma}_{S^*}. \end{cases} \tag{3.11}$$

Since $\tilde{\sigma}_S$ and $\tilde{\sigma}_{S^*}$ are both exterior recurrent, it follows from a well-known property that both co-isotropic distributions $\tilde{S} + \{\xi\}$ and $\tilde{S}^* + \{\xi\}$ are *involutive* (orth. $(\tilde{S}+\{\xi\}) = \tilde{S}$; orth. $(\tilde{S}^*+\{\xi\}) = S^*$). It is worth to emphasize that this property is true for any pseudo-Sasakian manifold.

Now with the help of (3.1), one finds that the connection forms are given by

$$\begin{cases} \tilde{\theta}^a_a = \tilde{t}_a \tilde{\omega}^{a^*} + \tilde{t}_{a^*}\tilde{\omega}^a + \tilde{v}/2 \quad \text{(no summation)}, \\ \tilde{\theta}^a_b = \tilde{t}_{b^*}\tilde{\omega}^a + \tilde{t}_a \tilde{\omega}^{b^*} . \end{cases} \tag{3.12}$$

By (3.12) and (3.6) one finds

$$\tilde{\theta}_R = (m+2)\tilde{v}/2 \tag{3.13}$$

and (3.7) shows that $\tilde{\theta}_R$ is exterior recurrent.

Coming back to relations (3.12), one readily finds

$$\tilde{\theta}^a_a(U\tilde{U}) = 0, \quad \tilde{\theta}^a_b(U\tilde{U}) = 0 . \tag{3.14}$$

Therefore we may say that all connection forms of the pseudo-Sasakian manifold $\tilde{M}$ under consideration are *integral relations of invariance* for the vector field $U\tilde{U}$.

Denote now by $\tilde{\tau}$ the volume element of $\tilde{M}$. One may take a local orientation such that

$$\tilde{\tau} = \tilde{\sigma}_S \wedge \tilde{\sigma}_{S^*} \wedge \tilde{\eta} \tag{3.15}$$

and denote by $*: \Lambda^q T^*\tilde{M} \to \Lambda^{2m+1-q} T^*\tilde{M}$ the *star operator* determined by $\tilde{\tau}$. If, like usually, $\mathfrak{X}\tilde{M}$ means the vector space of sections over $T\tilde{M}$, then, as is known, for any vector field $\tilde{Z} \in \mathfrak{X}\tilde{M}$ one has

$$*\text{div}\, \tilde{Z} = (\text{div}\, \tilde{Z})\tilde{\tau} = di_{\tilde{Z}}\tilde{\tau} = \mathcal{L}_{\tilde{Z}}\tilde{\tau} . \tag{3.16}$$

Making use of (3.4), (3.11), (3.16), and the fact that

$$\tilde{U} = \sum_a (\tilde{t}_a h_a + \tilde{t}_{a^*} h_{a^*}), \tag{3.17}$$

one finds after some calculations:

$$\text{div}\, \tilde{U} = 0, \quad \text{div}(U\tilde{U}) = 2 \sum_a \tilde{t}_a \tilde{t}_{a^*} = 4 . \tag{3.18}$$

Hence $\tilde{U}$ is *divergence free* and $U\tilde{U}$ is an *infinitesimal homothety* on $\tilde{M}$.

Now if $\tilde{Z} = \tilde{Z}^A h_A$, $\tilde{Z}' = (\tilde{Z}')^A h_A \in \tilde{M}$ are any vector fields, then, as is known (see Poor [14]), one has

$$\tilde{\nabla}_{\tilde{Z}'}\tilde{Z} = (\mathcal{L}_{\tilde{Z}'}\tilde{z}^A) h_A + \tilde{z}^A(\tilde{\nabla}_{\tilde{Z}'} h_A) .$$

Therefore, by (2.3), (3.4), and (3.12) we get

$$\begin{cases} \tilde{\nabla}_{\tilde{Z}}\tilde{U} = (\tilde{\eta}(\tilde{Z}) + \tilde{v}(\tilde{Z}))U\tilde{U} - 2\tilde{u}(\tilde{Z})\xi , \\ \tilde{\nabla}_{\tilde{Z}}U\tilde{U} = (\tilde{\eta}(\tilde{Z}) + \tilde{v}(\tilde{Z}))\tilde{U} + \tilde{v}(\tilde{Z})\xi . \end{cases} \tag{3.19}$$

We also note that since $\flat(U\tilde{U}) = \tilde{u}$ is a closed form, we may say (see Poor [14]) that $\tilde{\nabla}U\tilde{U}$ is *self-adjoint*.

According to the definition given by Rosca [5] and Rosca and Verstraelen [15], the formulae (3.19) show that the vector field $\tilde{U}$ defines a *U-contact concircular pairing*.

Denote by D_U the 3-distribution defined by $\{\tilde{U}, U\tilde{U}, \xi\}$. By (2.2), (3.5), and (3.6) one readily finds from (3.19) that

$$[\tilde{U}, \xi] = 0, \quad [U\tilde{U}, \xi] = 0 . \tag{3.20}$$

Hence both vector fields $\tilde{U}$ and $U\tilde{U}$ commute with ξ and by (3.19) and (3.20) we see that D_U defines a *3-foliation* on $\tilde{M}$.

It is worth now to make the following considerations.

Let $\tilde{Z} \in \mathfrak{X}\tilde{M}$ be any vector field on $\tilde{M}$. Then one has the general *Bochner formula* (see Poor [14]) on $\tilde{M}$:

$$2\langle \mathrm{tr}\, \tilde{\nabla}^2\tilde{Z},\tilde{Z}\rangle + 2\|\tilde{\nabla}\tilde{Z}\|^2 + \tilde{\Delta}\|\tilde{Z}\|^2 = 0 \tag{3.21}$$

where $\tilde{\Delta} = d\circ\delta + \delta\circ d$ is the *Laplace-Beltrami operator* (or *Laplacian*) on $\Lambda T^*\tilde{M}$, and the trace (abr. tr) is calculated with respect to the metric tensor $\tilde{g}$ of $\tilde{M}$.

Applying formula (3.21) to the principal vector field $\tilde{U}$ and taking into account (2.7), one has

$$\mathrm{tr}\, \tilde{\nabla}^2\tilde{U} = \sum_a \tilde{\nabla}_{h_a}(\tilde{\nabla}_{h_{a^*}}\tilde{U}) + \sum_a \tilde{\nabla}_{h_{a^*}}(\tilde{\nabla}_{h_a}\tilde{U}) + \tilde{\nabla}_\xi(\tilde{\nabla}_\xi\tilde{U}) \tag{3.22}$$

and

$$\|\tilde{\nabla}\tilde{U}\|^2 = 2\sum_a \langle\tilde{\nabla}_{h_a}\tilde{U},\tilde{\nabla}_{h_{a^*}}\tilde{U}\rangle + \langle\tilde{\nabla}_\xi\tilde{U},\tilde{\nabla}_\xi\tilde{U}\rangle . \tag{3.23}$$

Now by (2.14), (3.4), (3.5), (3.16), and (3.19) one finds

$$\begin{cases} \tilde{\nabla}_{h_{a^*}}\tilde{\nabla}_{h_a}\tilde{U} = \tilde{t}_a\tilde{t}_{a^*}\tilde{U} + (2-\tilde{t}_a\tilde{t}_{a^*}/2)U\tilde{U} + (3\tilde{t}_a\tilde{t}_{a^*}/2-2)\xi + \tilde{t}_{a^*}h_{a^*}, \\ \tilde{\nabla}_{h_a}\tilde{\nabla}_{h_{a^*}}\tilde{U} = \tilde{t}_a\tilde{t}_{a^*}\tilde{U} - (2-\tilde{t}_a\tilde{t}_{a^*}/2)U\tilde{U}+(3\tilde{t}_a\tilde{t}_{a^*}/2-2)\xi + \tilde{t}_a h_a, \\ \tilde{\nabla}_\xi\tilde{\nabla}_\xi U = U. \end{cases} \tag{3.24}$$

Since we have found $\sum_a \tilde{t}_a\tilde{t}_{a^*} = 2$, we derive from (3.22), (3.23), (3.24), and (3.21) that $\tilde{U}$ satisfies (3.21) and this equation is consistent with $\|\tilde{U}\|^2 = 4$.

Let L be the operator of type (1,1) defined by the fundamental 2-form $\tilde{\Omega}$. Denote then by $\tilde{\beta}_q = L^q\tilde{u} = \tilde{u}\wedge(\Lambda\tilde{\Omega})^q \in \Lambda^{2q+1}\tilde{M}$. Since $\tilde{u}$ and $\tilde{\Omega}$ are both closed, one finds by (3.9) and making use of the properties of the Lie derivative $\mathcal{L} = i\circ d + d\circ i$ that

$$\mathcal{L}_{\tilde{U}}\tilde{\beta}_q = 0 . \tag{3.25}$$

Hence $\tilde{U}$ is an *infinitesimal automorphism* of all (2q+1)-forms $\tilde{\beta}_q$ $(q < m)$.

On the other hand, since $\tilde{g}(\tilde{U},\tilde{U}) = \mathrm{const}$, we may say in similar manner as in the case of a Sasakian manifold that $\tilde{U}$ defines with $U\tilde{U}$ an *U-section*.

Like usually denote by

$$R(\tilde{Z},\tilde{Z}') = [\tilde{\nabla}_{\tilde{Z}},\tilde{\nabla}_{\tilde{Z}'}] - \tilde{\nabla}_{[\tilde{Z},\tilde{Z}']}, \quad \tilde{Z},\tilde{Z}' \in \mathfrak{X}\tilde{M} \tag{3.26}$$

the *curvature operator*. Then, as is known, the *sectional curvature* $K(\tilde{U},U\tilde{U})$ defined by $\tilde{U}$ and $U\tilde{U}$ is given by

$$K(\tilde{U},U\tilde{U}) = \frac{R(\tilde{U},U\tilde{U},\tilde{U},U\tilde{U})}{\tilde{g}(\tilde{U},\tilde{U})\tilde{g}(U\tilde{U},U\tilde{U})-(\tilde{g}(\tilde{U},U\tilde{U}))^2} \tag{3.27}$$

where

$$R(\tilde{U},U\tilde{U},\tilde{U},U\tilde{U}) = \tilde{g}(R(\tilde{U},U\tilde{U})U\tilde{U},\tilde{U}). \tag{3.28}$$

Making use of (3.5), (3.6), and (3.19), one finds

$$[\tilde{U},U\tilde{U}] = 4(\tilde{U}+2\xi) \tag{3.29}$$

and

$$R(\tilde{U},U\tilde{U})U\tilde{U} = 4(5\tilde{U}+8\xi) . \tag{3.30}$$

Hence by (3.27) and (3.28) one gets $K(\tilde{U},U\tilde{U}) = -\frac{1}{5}$. Now referring to (2.10) and (3.12) one finds after some calculations

$$\tilde{\Theta}^a_a = \tilde{v}_S \wedge \tilde{v}_{S^*} + \tilde{v}_S \wedge \tilde{t}_a \tilde{\omega}^{a^*} - \tilde{v}_{S^*} \wedge \tilde{t}_{a^*} \tilde{\omega}^a + \tilde{t}_a \tilde{t}_{a^*} \tilde{\Omega} + \tilde{\omega}^{a^*} \wedge \tilde{\omega}^a \quad \text{(no summation)} \tag{3.31}$$

where we have set

$$\begin{cases} \tilde{v}_S = \sum_a \tilde{t}_{a^*} \tilde{\omega}^a \in \Lambda^1 \tilde{S}, \\ \tilde{v}_{S^*} = \sum_a \tilde{t}_a \tilde{\omega}^{a^*} \in \Lambda^1 \tilde{S}^*. \end{cases} \tag{3.32}$$

As is known (see Libermann [11]), the components of the *Ricci tensor* are given by $\tilde{\Theta}^a_a = \tilde{R}_{bc^*} \tilde{\omega}^b \wedge \tilde{\omega}^{c^*}$ ($\tilde{\Theta}^a_a + \tilde{\Theta}^{a^*}_{a^*} = 0$). Because of this, we get from (3.31) that

$$\begin{cases} \tilde{R}_{bc^*} = \tilde{t}_{b^*} \tilde{t}_c, \\ \tilde{R}_{aa^*} = 2\tilde{t}_a \tilde{t}_{a^*} - 1. \end{cases} \tag{3.33}$$

It follows from (3.33) that the components of the Ricci tensor are *disjoint* (see Rosca [16]). In addition, since the *scalar curvature* $\tilde{C}_s$ is the trace of the Ricci tensor with respect to $\tilde{g}$, one finds by (2.7) and (3.3) that $\tilde{C}_s = 4-m$ $(m > 4)$. Therefore we conclude that the pseudo-Sasakian manifold $\tilde{M}$ under consideration is a *space form* $\tilde{M}(4-m)$ of *hyperbolic type*.

THEOREM 1. Let $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$ be a pseudo-Sasakian manifold endowed with a c.c. connection and let $\tilde{U}$ (resp. $\tilde{\Omega} = d\tilde{\eta}/2$) be the principal vector field associated with this connection (resp. the fundamental 2-form on $\tilde{M}$). One has the following properties:

(i) $\tilde{U}$ is divergence free, and $U\tilde{U}$ defines an infinitesimal homothety on $\tilde{M}$;

(ii) all the connection forms on $\tilde{M}$ are integral relations of invariance for $U\tilde{U}$;

(iii) $\tilde{U}$ and $U\tilde{U}$ define an U-contact concircular pairing, and $\{\tilde{U},U\tilde{U},\xi\}$ defines a 3-foliation on $\tilde{M}$;

(iv) any contact extension $\tilde{U}_f = \tilde{U} + \tilde{f}\xi$ of $\tilde{U}$ is a relative contact infinitesimal transformation of $\tilde{\eta}$;

(v) $\tilde{U}$ and $U\tilde{U}$ define both an infinitesimal automorphism of all $(2q+1)$-forms $\tilde{\beta}_q = L^q\tilde{u}$ where $\tilde{u}$ is the dual form of $U\tilde{U}(q<m)$;

(vi) the Ricci 1-form of $\tilde{M}$ is exterior recurrent, and the Ricci tensor is disjoint;

(vii) $\tilde{M}$ is a space-form of hyperbolic type ;

(viii) any such submanifold $\tilde{M}$ is defined by a completely integrable system of differential equations whose solution depends on $2m$ arbitrary constants.

4. CO-ISOTROPIC FOLIATION ON $\tilde{M}(U,\xi,\tilde{\eta},\tilde{g})$.

We shall consider on M the following three distributions:

a) An *invariant* distribution D^T (i.e. $UD^T \subseteq D^T$) of dimension $2(m-\ell)+1$ defined by $D^T = \{h_i, h_{i^*}, \xi;\ i=1,\dots,m-\ell;\ i^*=i+m\}$.

b) An *isotropic* distribution $D^\perp$ (i.e. $D^\perp \subseteq \text{orth}\, D^\perp$) of dimension ℓ defined by $D^\perp = \{h_r;\ r=m-\ell+1,\dots,m\}$.

c) A transversal distribution $D_t = \mathcal{C}_{S^*}(D^{\top} \oplus D^{\perp}) \cap S^*$ of dimension ℓ defined by $D_t = \{h_{r^*};\ r^*=2m-\ell+1,\dots,2m\}$.

These three distributions have no common direction and they define on $\tilde{M}$ a *f-structure of rank* 2ℓ (see Sinha [17]).

Accordingly we shall split the principal vector field $\tilde{U}$ as follows:

$$\tilde{U} = \tilde{U}^{\top} \oplus \tilde{U}^{\perp} \oplus \tilde{U}_t \tag{4.1}$$

where $\tilde{U}^{\top} \in D^{\top}$, $\tilde{U}^{\perp} \in D^{\perp}$, $\tilde{U}_t \in D_t$.

Denote now by

$$\tilde{\psi} = \tilde{\omega}^{2m-\ell+1} \wedge \dots \wedge \tilde{\omega}^{2m} \tag{4.2}$$

the simple unit form which corresponds to D_t. Because D_t is orientable, $\tilde{\psi}$ is a well-defined global form. Since $\tilde{\psi}$ annihilates $D^{\top} \oplus D^{\perp}$, the necessary and sufficient condition for $D^{\top} \oplus D^{\perp}$ to be a *co-isotropic foliation* F_c is that $\tilde{\psi}$ be exterior recurrent (see Lichnerowicz [18] and Yano and Kon [7]).

Hence one must write $d\tilde{\psi} = \tilde{\gamma} \wedge \tilde{\psi}$ and if $H^1(F_c,R)$ represent the *1-cohomology class* of F_c, then the recurrence 1-form $\tilde{\gamma}$ defines an element of $H^1(F_c,R)$ (see Lichnerowicz [6]). In the case under discussion one finds (compare with Yano and Kon [7]) that the necessary and sufficient condition for $\tilde{M}$ to receive a co-isotropic foliation $F_c = D^{\top} \oplus D^{\perp}$ is that the component $\tilde{U}_t$ of $\tilde{U}$ vanishes. In this case the recurrence 1-form $\tilde{\gamma}$ of $\tilde{\psi}$ is given by

$$\tilde{\gamma} = \ell(\tilde{u}-\tilde{\eta}). \tag{4.3}$$

Denote by M_c a $(2m-\ell+1)$-dimensional leaf of F_c and supress $\sim$ for the induced elements on M_c.

According to the considerations of Section 1, it follows that M_c is a *CICR submanifold*. By definition we have $du = 0$. Because of this and (3.1), the exterior differentiation of (4.3) gives

$$d\gamma = -2\ell\Omega. \tag{4.4}$$

Equation (4.4) shows that the restriction $\Omega = \tilde{\Omega}|_{M_c}$ is an *exact form*.

On the other hand, the form of Godbillon-Vey (see Lichnerowicz [6]) on M_c is the $(2\ell+1)$-form $w_G \in \Lambda^{2\ell+1}(M_c)$ given by

$$w_G = \gamma\wedge(\wedge d\gamma)^{\ell}. \tag{4.5}$$

One knows (see Lichnerowicz [18]) that the class of cohomology of w_G which is an element of $H^{2\ell+1}(M_c;R)$ is an invariant of the foliation. Using the same notation as in section 3 and applying (4.4), we may write

$$w_G = c(L^{\ell}u-L^{\ell}\eta) = c(\beta_{\ell}-L^{\ell}\eta) \tag{4.6}$$

where we have set $c = -2^{\ell}\ell^{\ell+1}$.

Thus it follows from (3.22) that

$$\mathcal{L}_U w_G = -c\mathcal{L}_U(L^{\ell}\eta). \tag{4.7}$$

By means of (2.13) and (3.9) one has

$$d(L^{\ell}\eta) = 2(\wedge\Omega)^{\ell+1} \tag{4.8}$$

and

$$di_U(L^{\ell}\eta) = 4\ell u(\Lambda\Omega)^{\ell} . \tag{4.9}$$

Therefore we get

$$\mathcal{L}_U(L^{\ell}\eta) = -u \wedge (\Lambda\Omega)^{\ell} = -\beta_{\ell} \tag{4.10}$$

and finally

$$\mathcal{L}_U w_G = c\beta_{\ell} . \tag{4.11}$$

Since β_{ℓ} is closed, the above equation gives $d\mathcal{L}_U w_G = 0$ and allows us to say that w_G is a *relative integral invariant* of U.

Further since the submanifold M_c is co-isotropic, it follows from this that the normal bundle $T^{\perp}M_c$ of M_c coincides with $D^{\perp}$.

Since M_c is defined by $\omega^{r^*} = 0$, $r^* = 2m-\ell+1,\dots,2m$, we derive from (2.8) and (3.12) that the covariant derivatives ∇h_r of the null normal sections h_r satisfy

$$\nabla h_r = \frac{v}{2} \otimes h_r . \tag{4.12}$$

Since h_r are null vector fields, equation (4.12) shows that h_r are *geodesic* directions. Hence according to the definition of Rosca [19], one may say that $D^{\perp}$ has the *geodesic property*.

Further if X and Y are any vector fields of $D^{\perp}$, one has $\nabla_Y X \in D^{\perp}$. Thus according to a known definition, the distribution $D^{\perp}$ is *autoparallel*.

Setting $\ell_r = -\langle dp, \nabla h_r \rangle$ for the *second fundamental quadratic forms* associated with the improper immersion $x: M_c \to \tilde{M}$ (ℓ_r is a field of symmetric covariant tensors of order 2 on M_c), we derive by a simple argument that all ℓ_r vanish. Therefore according to a well-known definition, we agree to say that the improper immersion $x: M_c \to \tilde{M}$ is *improper totally geodesic*.

It was proved by Goldberg and Rosca [2] that the distribution $D^{\perp}$ is always involutive. If $M^{\perp}$ are the leaves of $D^{\perp}$, then in a similar manner as for M_c one easily finds that the improper immersion $x: M^{\perp} \to \tilde{M}$ is *improper totally geodesic*. Since $x: M^{\top} \to \tilde{M}$ is a proper immersion, it is *totally geodesic*.

Next as it was proved (Goldberg and Rosca [2]) the necessary and sufficient condition for the manifold M_c to be *foliate* is that the simple unit form ψ which corresponds to $D^{\perp}$ be exterior recurrent.

Since obviously one has $\phi = \omega^{m-\ell+1}\wedge\dots\wedge\omega^{m}$, then by (3.1) one finds that the property of exterior recurrency for ϕ is equivalent to the condition $U^{\perp} = 0$.

Since by definition in this case $D^{\top}$ is involutive, let us denote by $M^{\top}$ a $(2(m-\ell)+1)$-dimensional leaf of $D^{\top}$. Because M_c is a CICR submanifold, $M^{\top}$ is as is known an invariant submanifold of $\tilde{M}$, and this implies (see Rosca [1]) that $M^{\top}$ is *minimal*.

Coming back to the case under discussion, using (2.8), (3.12) and the fact that on M one has $U_t = 0$, $U^{\perp} = 0$, we can show by means of a simple calculation that $M^{\top}$ is also totally geodesic.

Hence M_c is foliated by two families of orthogonal totally geodesic submanifolds $M^{\perp}$ and $M^{\top}$.

On the other hand, let $X \in \mathfrak{X} M_c$ be any vector field on M_c. According to Rosca [1], one has $UX = PX + FX$ where PX (resp. FX) is the tangential (resp. the normal) component of UX. By virtue of the total geodesicity of M^T, one easily finds that $\nabla PX \in M^T$.

Therefore the tangential component PX of X is *parallel*. According to Yano and Kon [7], it follows from this that M_c is a CR product i.e. $M_c = M^{\perp} \times M^T$.

Since M_c is connected, this property can be checked by *de Rham decomposition theorem*.

It is worth to note that this situation is quite similar to that of co-isotropic CR submanifolds of a para Kaehlerian manifold structured by a *geodesic connection* (Rosca [20]).

THEOREM 2. Let M be a pseudo-Sasakian manifold structured by a c.c. connection and let $\tilde{U}$ be the principal vector field associated with this connection. Then the necessary and sufficient condition for $\tilde{M}$ to receive a co-isotropic foliation F_c is that the transversal component $\tilde{U}_t$ of $\tilde{U}$ vanishes. In this case the leaves M_c of F_c are CICR submanifolds of M_c, and if codim $M_c = \ell$, the form of Godbillon-Vey on M_c is a $(2\ell+1)$-form w_{G} which is a relative integral invariant of $U = \tilde{U}|_{M_c}$.

In addition, one has the following properties:

(i) the improper immersion $x: M_c \to \tilde{M}$ is improper totally geodesic;

(ii) M_c is foliated by anti-invariant submanifolds $M^{\perp}$ which are improper totally geodesic and have the geodesic property.

Further the necessary and sufficient condition for M_c to be foliate is that the vertical (or isotropic) component $U^{\perp}$ of $U = \tilde{U}|_{M_c}$ vanishes. In this case M_c is a CR product.

5. TANGENT BUNDLE MANIFOLD $T\tilde{M}$.

Let $T\tilde{M}$ be the *tangent bundle manifold* having the pseudo-Sasakian manifold discussed in Section 3 as a basis.

Denote by $\tilde{V}_L(\tilde{v}^{\Lambda})$ the *canonical vector field* (or *the vector field of Liouville*) on $T\tilde{M}$. Accordingly we may consider the set $B^* = \{\tilde{\omega}^{\Lambda}, d\tilde{v}^{\Lambda}\}$ as an adapted cobasis on $T\tilde{M}$. Following Godbillon [21], we shall designate by $d_{\boldsymbol{v}}$ and $i_{\boldsymbol{v}}$ the *vertical differentiation* and the *vertical derivation operators*, respectively taken with respect to B^* ($d_{\boldsymbol{v}}$ is an *antiderivation* of degree 1 of $\Lambda T\tilde{M}$ and $i_{\boldsymbol{v}}$ is a *derivative of degree* 0 of $\Lambda T\tilde{M}$).

Let $T^r_s\tilde{M}$ be the set of all tensor fields of type (r,s) on $\tilde{M}$. In general the *vertical* and *complete* lifts are linear mappings of $T^r_s\tilde{M}$ into $T^r_s T\tilde{M}$, and for complete lifts one has:

$$(T_1 \otimes T_2)^C = T_1^V \otimes T_2^C + T_1^C \otimes T_2^V .$$

With respect to B^* the complete lift of the fundamental form $\tilde{\Omega} = d\tilde{\eta}/2$ is given by

$$\tilde{\Omega}^C = \sum_a (d\tilde{v}^a \wedge \tilde{\omega}^{a^*} + \tilde{\omega}^a \wedge d\tilde{v}^{a^*}) . \tag{5.1}$$

The exterior differentiation of (5.1) by means of (3.1) gives

$$d\tilde{\Omega}^C = \tilde{u} \wedge \tilde{\Omega}^C + \sum_a (\tilde{t}_a d\tilde{v}^{a^*} - \tilde{t}_{a^*} d\tilde{v}^a) \wedge \tilde{\Omega}$$
$$+ \tilde{\eta} \wedge (\tilde{\omega}^a \wedge d\tilde{\omega}^{a^*} + \tilde{\omega}^{a^*} \wedge d\tilde{v}^a) . \tag{5.2}$$

Using (5.2), we find

$$\mathcal{L}_{\tilde{V}_L} \tilde{\Omega}^C = \tilde{\Omega}^C . \tag{5.3}$$

As is known (see Godbillon [21]), equation (5.3) shows that $\tilde{\Omega}^C$ is *homogeneous of degree 1.*

We will now take the complete lift $\tilde{u}^C$ of the principal Pfaffian $\tilde{u}$ associated with the c.c. connection with structures $\tilde{M}$. For this purpose we shall denote by $\partial_B(\tilde{t}_A) = h_B(\tilde{t}^*_A)$ the Pfaffian derivatives of $\tilde{t}^*_A$ (A=0,1,...,2m) with respect to cobasis $\tilde{W}^*$. Then according to the general theory (Yano and Ishihara [7]) one has

$$\tilde{u}^C = \tilde{u}_A d\tilde{v}^A + \partial_B(\tilde{u}_A)\tilde{v}^B\tilde{\omega}^A \tag{5.4}$$

where we have set $\tilde{u} = \tilde{u}_A\tilde{\omega}^A$. Referring to (3.4) and (3.5) (c=2), after some calculations one finds

$$\tilde{u}^C = \frac{1}{2}\sum_a(\tilde{t}_{a^*} d\tilde{v}^a - \tilde{t}_a d\tilde{v}^{a^*}) + \frac{1}{2}\sum (\tilde{t}_{a^*}\tilde{v}^a - \tilde{t}_a\tilde{v}^{a^*})\tilde{u}$$
$$+ \sum_a (\tilde{v}^a\tilde{\omega}^{a^*} + \tilde{v}^{a^*}\tilde{\omega}^a) - \frac{1}{2}\tilde{v}^0\tilde{v} . \tag{5.5}$$

The exterior differentiation of (5.5) by means of (3.1) gives

$$d\tilde{u}^C = \frac{1}{2}(\sum_a (\tilde{t}_a\tilde{v}^{a^*} + \tilde{t}_{a^*}\tilde{v}^a)\tilde{u} + \sum_a (\tilde{v}^a\tilde{\omega}^{a^*} - \tilde{v}^{a^*}\tilde{\omega}^a)$$
$$+ \sum_a (\tilde{t}_a d\tilde{v}^a + \tilde{t}_a d\tilde{v}^{a^*})) \wedge \tilde{\eta} . \tag{5.6}$$

Using (5.5) and (5.6), one finds $\mathcal{L}_{\tilde{V}_L} \tilde{u}^C = \tilde{u}^C$. Hence u^C is also a homogeneous form of degree 1.

Consider now the following scalar field on $T\tilde{M}$:

$$\tilde{T} = \sum \tilde{v}^a\tilde{v}^{a^*} + (\tilde{v}^0)^2/2 \tag{5.7}$$

and apply the vertical differentiation of $\tilde{T}$. According to Godbillon [21], one has

$$\tilde{\nu} = d_v\tilde{T} = \sum_a (\tilde{v}^a\tilde{\omega}^{a^*} + \tilde{v}^{a^*}\tilde{\omega}^a) + \tilde{v}^0\tilde{\eta} \tag{5.8}$$

and by means of (3.1) one gets

$$\tilde{\Pi} = d\tilde{\nu} = \iota\tilde{v}\tilde{\Omega} + \tilde{u} \wedge \sum_a (\tilde{v}^a\tilde{\omega}^{a^*} + \tilde{v}^{a^*}\tilde{\omega}^a)$$
$$+ \tilde{\eta} \wedge (\sum (\tilde{v}^{a^*}\tilde{\omega}^a - \tilde{v}^a\tilde{\omega}^{a^*}) - d\tilde{v}^0) + 2\tilde{v}^0\tilde{\Omega} \tag{5.9}$$

In (5.9) $\iota : \Lambda^1\tilde{M} \to C^\infty T\tilde{M}$ is the operator of Yano and Ishihara [7], that is with respect to $\mathcal{B}^*$ one has by (3.6)

$$\iota\tilde{v} = \sum_a (\tilde{t}_{a^*}\tilde{v}^a + \tilde{t}_a\tilde{v}^{a^*}). \tag{5.10}$$

One quickly finds

$$i_{\tilde{V}_L}\tilde{\Pi} = \tilde{\nu} , \tag{5.11}$$

and since $\tilde{\Pi}$ is closed, it follows from (5.11) that

$$\mathcal{L}_{\tilde{V}_L}\tilde{\Pi} = \tilde{\Pi} \tag{5.12}$$

i.e. $\tilde{\Pi}$ is homogeneous of degree 1. Moreover, taking the vertical derivation of $\tilde{\Pi}$, one has (see Godbillon [21]):

$$i_{\mathcal{V}}\tilde{\Pi} = 0 . \tag{5.13}$$

On the other hand, it is easy to see from (5.9) that $\tilde{\Pi}$ is of maximal rank (see Godbillon [21]) on $T\tilde{M}$. Accordingly, as is known, equations (5.11) and (5.13) prove that $\tilde{\Pi}$ is a *Finslerian form* (See Klein and Voutier [22]). Since the vertical differentiation $d_{\mathcal{V}}$ is an anti-derivation of square zero, one easily derives from (5.8) that

$$d_{\mathcal{V}}\tilde{\nu} = 0, \qquad i_{\tilde{V}_L}\tilde{\nu} = 0 . \tag{5.14}$$

Thus according to Godbillon [21], $\tilde{\nu}$ is a *semibasic form*.

In the following we shall call $\tilde{T}$ (resp. $\tilde{\nu}$) the *Liouville function* (resp. *the Liouville 1-form*) on $T\tilde{M}$ (see Rosca [16]). Further one may call $\tilde{\Pi}$ the *2-form of Cartan* on $T\tilde{M}$ (see Rosca [19]).

Denote now by $\mathcal{B} = \{h_A, \frac{\partial}{\partial \tilde{v}^A}\}$ the vectorial basis dual to $\mathcal{B}^*$ on $\tilde{M}$. Then as is known (see Yano and Ishihara [9] or Godbillon [21]) the vertical lift $(\tilde{Z})^V$ of $\tilde{V}$ is expressed by

$$(\tilde{Z})^V = \tilde{z}^A \frac{\partial}{\partial \tilde{v}^A} . \tag{5.15}$$

Coming back to the case under consideration and using that $U\tilde{U} = b^{-1}(\tilde{u})$ (see Section 3), we find by (5.15) that

$$(U\tilde{U})^V = \sum_a (\tilde{t}_a \frac{\partial}{\partial \tilde{v}^a} - \tilde{t}_{a*} \frac{\partial}{\partial \tilde{v}^{a*}}). \tag{5.16}$$

Now, taking the dual $\mu(U\tilde{U})^V$ of $(U\tilde{U})^V$ with respect to $\tilde{\Pi}$ and referring to (3.5) (c=2), we quickly find

$$\mu(U\tilde{U})^V = 2\tilde{u} . \tag{5.17}$$

Since $\tilde{u}$ and $\tilde{\Pi}$ are both closed, it follows from this that $\mathcal{L}_{(U\tilde{U})^V}\tilde{\Pi} = 0$, i.e. $(U\tilde{U})^V$ is an infinitesimal automorphism of $\tilde{\Pi}$.

Consider now on $T\tilde{M}$ the *mechanical system* $\mathcal{M} = \{\tilde{M}, \tilde{T}, \tilde{\pi}\}$ where $\tilde{T}$ and

$$\tilde{\pi} = 2\tilde{T}\tilde{u} \tag{5.18}$$

are the *kinetic energy* and the *field of forces* of $\mathcal{M}$ (see Godbillon [21]).

Since $\tilde{u}$ is closed, one has $d\tilde{\pi} = \frac{d\tilde{T}}{\tilde{T}} \wedge \tilde{\eta}$ and referring to (5.7), one quickly finds

$$\begin{aligned} \mathcal{V}_{\tilde{V}_L} \tilde{T} &= 2\tilde{T} , \\ \mathcal{V}_{\tilde{V}_L} \tilde{\pi} &= 2\tilde{\pi} . \end{aligned} \tag{5.19}$$

Equations (5.19) show that $\tilde{T}$ and $\tilde{\pi}$ are *homogeneous of degree 2*. On the other hand, since $\tilde{\pi}$ is an exact 2-form of maximal rank, it defines a *potential symplectic structure* on $T\tilde{M}$. Hence, according to the definition given by Klein (see Godbillon [21]) the system $\mathcal{M}$ is *regular*.

Denote now by $\tilde{Z}_d$ the *dynamical system* associated with $\mathcal{M}$. As is known, $\tilde{Z}_d$

is defined via formula

$$i_{\tilde{Z}_d} \tilde{\Pi} = d(\tilde{T} - \tilde{V}_L \tilde{T}) + \tilde{\pi} \,. \tag{5.20}$$

Then:

a) Since $\tilde{T}$ and $\tilde{\pi}$ are both homogeneous and of the same degree, $\tilde{Z}_d$ is a *spray* on $\tilde{M}$, i.e. $[\tilde{V}_L, \tilde{Z}_d] = \tilde{Z}_d$.

b) Since $\tilde{T}$ is of degree 2, the 2-form $\tilde{\Pi} - (d\tilde{T} - \tilde{\pi}) \wedge dt \in \Lambda^2(T\tilde{M} \times R)$ is an integral relation of invariance for $\tilde{Z}_d + \frac{\partial}{\partial t}$ (Lichnerowicz [5]).

THEOREM 3. Let $T\tilde{M}$ be the tangent bundle manifold having as a basis the manifold $\tilde{M}(U, \xi, \tilde{\eta}, \tilde{g})$ defined in Section 3 and let $\tilde{U}$ (resp. $\tilde{\Omega}$) be the principal vector field (resp. the fundamental 2-form) on $\tilde{M}$. Then:

(i) the complete lifts $\tilde{\Omega}^C$ and $\tilde{u}^C$ of $\tilde{\Omega}$ and $\tilde{u} = \flat(U\tilde{U})$ are homogeneous of degree one;

(ii) the 2-form of Cartan $\tilde{\Pi}$ on $T\tilde{M}$ is a Finslerian form;

(iii) one may associate with $\tilde{\Pi}$ a regular mechanical system whose dynamical system is a spray on $\tilde{M}$.

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