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Research Article

Generalized Bilinear Differential Operators Application in a (3+1)-Dimensional Generalized Shallow Water Equation

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The relations between D_p-operators and multidimensional binary Bell polynomials are explored and applied to construct the bilinear forms with D_p-operators of nonlinear equations directly and quickly. Exact periodic wave solution of a (3+1)-dimensional generalized shallow water equation is obtained with the help of the D_p-operators and a general Riemann theta function in terms of the Hirota method, which illustrate that bilinear D_p-operators can provide a method for seeking exact periodic solutions of nonlinear integrable equations. Furthermore, the asymptotic properties of the periodic wave solutions indicate that the soliton solutions can be derived from the periodic wave solutions.

1. Introduction

The studies of exact solutions of nonlinear partial differential equations (NPDEs) have received considerable attention in connection with the important problems that arise in scientific applications. Many powerful methods have been proposed to obtain exact solutions of (NPDEs); a series of methods have been proposed, such as Painléve test [1], Bäcklund transformation method [2, 3], Darboux transformation [4], inverse scattering transformation method [5], Lie group method [6, 7], Hamiltonian method [8, 9], and the Hirota method [10, 11].

In order to seek the periodic solutions of nonlinear evolution equations, Porubov and Parker proposed Weierstrass elliptic function expansion method [12]; Liu et al. proposed Jacobi elliptic sine function expansion methods [13, 14] and obtained some exact periodic solutions of some nonlinear evolution equations. They pointed out that their method can be applied to solve the nonlinear evolution equations in which the odd- and even-order derivative terms do not coexist. Zhang [15] developed Jacobi elliptic function expansion method to solve some nonlinear evolution equations in which the odd- and even-order derivative term coexist and obtained some exact periodic solutions of the equations. The bilinear method developed by Hirota have proved to be particularly powerful in obtaining the soliton solutions, quasiperiodic wave solutions, and periodic wave solutions [16, 17]. As we all know, once the bilinear forms of nonlinear differential equations are obtained, the multisoliton solutions, the bilinear Bäcklund transformation, and Lax pairs of NPDEs can be constructed easily. It is clear that the key of Hirota direct method is finding the bilinear forms of the given differential equations by the Hirota differential D-operators. However, Hirota bilinear equations are special and there are many other bilinear differential equations which are not written in the Hirota bilinear form.

In fact, solving nonlinear equations (especially nonlinear partial differential equations) is very difficult, and there is no unified method. The present methods can only be applied to a certain equation or some equations. So the work of continuing to find some effective method of solving nonlinear equations is important and meaningful. Recently, Ma put forward generalized bilinear differential operators named D_p-operators in [18], which are used to create bilinear differential equations. Furthermore, different symbols are also used to furnish relations with Bell polynomials in [19] and even for trilinear equations in [20]. In this paper, we would like to explore how to construct the bilinear forms with D_p-operators and how to obtain the exact solutions of nonlinear equation with the help of D_p bilinear operators method.

The paper is structured as follows. In Section 2, we will give a brief introduction about the bilinear D_p-operators. In Section 3, we explore the relations between multivariate binary Bell polynomials and the D_p-operators. The D_p bilinear forms of some nonlinear evolutions are given quickly and easily from the relations. In Section 4, we will use the relation in Section 2 to seek the bilinear form with D_p-operators of the (3+1)-dimensional generalized shallow water equation and then take advantage of the D_p-operators and the Riemann theta function [21, 22] to obtain its exact periodic wave solution which can be reduced to the soliton solution via asymptotic analysis.

2. Bilinear D_p-Operators

It is known to us that Hirota bilinear D-operators play a significant role in Hirota direct method. The D-operators are defined as follows:

$$D_x^m D_y^k f \cdot g = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^m \left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right)^k \cdot \left[f(x_1, y_1) g(x_2, y_2)\right], \tag{1}$$

where the right-hand side is computed in

$$x_1 = x_2 = x, \quad y_1 = y_2 = y. \tag{2}$$

According to the definition of Hirota bilinear D-operators, we have

$$\begin{aligned} D_x f \cdot g &= f_x g - f g_x, \\ D_x^2 f \cdot g &= f_{xx} g - 2 f_x g_x + f g_{xx}, \\ D_x D_t f \cdot g &= f_{xt} g - f_x g_t - f_t g_x + f g_{xt}, \\ D_x^3 f \cdot g &= f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx}, \\ D_t D_x^3 f \cdot g &= f_{3xt} g - f_{3x} g_t - 3 f_{2xt} g_x + 3 f_{2x} g_{xt} \\ &\quad + 3 f_{xt} g_{2x} - 3 f_x g_{2xt} - f_t g_{3x} + f g_{3xt}. \end{aligned} \tag{3}$$

Based on the Hirota D-operators, Professor Ma put forward a kind of bilinear D_p-operators in [18]:

$$\begin{aligned} &D_{p,x}^m D_{p,y}^k \left[f(x_1, y_1) g(x_2, y_2)\right] \\ &= \left(\frac{\partial}{\partial x_1} + \alpha \frac{\partial}{\partial x_2}\right)^m \left(\frac{\partial}{\partial y_1} + \alpha \frac{\partial}{\partial y_2}\right)^k \Bigg|_{x_1 = x_2 = x,\ y_1 = y_2 = y}, \end{aligned} \tag{4}$$

where the powers of α are determined by

$$\alpha^i = (-1)^{r(i)}, \tag{5}$$

where $i = r(i) \bmod p$ with $0 \le r(i) < p$; $i \ge 0$.

Obviously, the case of $p = 1$ gives the normal derivatives, and the cases of $p = 2k$, $k \in N$, reduce to Hirota bilinear operators.

In particular, when $m = 0$, we have

$$\begin{aligned} D_{p,x}^n (f \cdot g)(x) &= \left(\partial_x + \alpha \partial_{x'}\right)^n f(x) g(x') \big|_{x' = x} \\ &= \sum_{i=0}^{n} \alpha^i C_n^i \partial_x^{n-i} f \partial_x^i g. \end{aligned} \tag{6}$$

According to the definition of D_p-operator, when $p = 3$, we have

$$\begin{aligned} \alpha^0 &= 1, \\ \alpha &= -1, \\ \alpha^2 &= \alpha^3 = 1, \\ \alpha^4 &= -1, \\ \alpha^5 &= \alpha^6 = 1, \\ \alpha^7 &= -1, \\ \alpha^8 &= \alpha^9 = 1, \dots, \end{aligned} \tag{7}$$

$$\begin{aligned} D_{3,x}^4 f \cdot g &= \sum_{i=0}^{4} \alpha^i C_4^i \partial_x^{4-i} f \partial_x^i g \\ &= f_{4x} g - 4 f_{3x} g_x + 6 f_{2x} g_{2x} + 4 f_x g_{3x} \\ &\quad - f g_{4x}; \end{aligned}$$

when $p = 5$, we have

$$\begin{aligned} \alpha^0 &= 1, \\ \alpha^1 &= -1, \\ \alpha^2 &= 1, \\ \alpha^3 &= -1, \\ \alpha^4 &= \alpha^5 = 1, \\ \alpha^6 &= -1, \\ \alpha^7 &= 1, \\ \alpha^8 &= -1, \dots, \end{aligned}$$

$$\begin{aligned} D_{5,x} f \cdot g &= f_x g - f g_x, D_{5,t} D_{5,x} f \cdot g \\ &= D_{5,t} (f_x g - f g_x) \\ &= f_{xt} g - f_x g_t - f_t g_x + f g_{xt}, \end{aligned}$$

$$D_{5,x}^2 f \cdot g = \sum_{i=0}^{2} \alpha^i C_2^i \partial_x^{2-i} f \partial_x^i g = f_{xx} g - 2 f_x g_x + f g_{xx},$$

$$D_{5,x}^4 f\cdot g=\sum_{i=0}^{4}\alpha^i C_4^i\partial_x^{4-i}f\partial_x^i g$$
$$=f_{4x}g-4f_{3x}g_x+6f_{xx}g_{xx}-4f_xg_{xx}$$
$$+fg_{4x},$$
$$D_{5x}^5 f\cdot g=\sum_{i=0}^{5}\alpha^i C_5^i\partial_x^{5-i}f\partial_x^i g$$
$$=f_{5x}g-5f_{4x}g_x+10f_{3x}g_{2x}-10f_{2x}g_{3x}$$
$$+5f_xg_{4x}+fg_{5x}. \tag{8}$$

Now, under $u=2(\ln f)_{xx}$, for Kdv equation,

$$u_t+6uu_x+u_{xxx}=0, \tag{9}$$

we have

$$\frac{\partial}{\partial x}\left[\frac{\left(f_{xt}f-f_xf_t+f_{4x}f-4f_{3x}f_x+3f_{2x}^2\right)}{f^2}\right]=0; \tag{10}$$

we can get its bilinear form with D_p-operators:

$$\left(D_{5,x}D_{5,t}+D_{5x}^4\right)f\cdot f=0. \tag{11}$$

In fact, if we seek the bilinear form with D_p-operators of nonlinear integrable differential equations according to the definition of D_p-operators, this needs some special skills and complex computations. So we would like to explore the relations between D_p-operators and multivariate binary Bell polynomials. The bilinear forms with D_p-operators of nonlinear integrable differential equations are obtained quickly and easily by applying the relations.

3. Relations with Bell Exponential Polynomials

3.1. Relations with Bell Exponential Polynomials. As we all know, Bell proposed three kinds of exponent form polynomials. Later, Wang and Chen generalized the third type of Bell polynomials in [23, 24] which is used mainly in this paper. The multidimensional binary Bell polynomials which we will use are defined as follows:

$$Y_{n_1x_1,\dots,n_lx_l}(y)=Y_{n_1,\dots,n_l}\left(y_{r_1x_1},\dots,y_{r_lx_l}\right)$$
$$=e^{-y}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}e^{y}\quad\left(n_1,\dots,n_l\geq 0\right), \tag{12}$$

with $y_{r_1x_1,\dots,r_lx_l}=\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}$, $r_1=0,\dots,n_1,\dots,r_l=0,\dots,n_l$.

For example,

$$Y_x=y_x,$$
$$Y_{2x}=y_x^2+y_{2x},$$
$$Y_{3x}=y_x^3+3y_xy_{2x}+y_{3x},$$
$$Y_{4x}=y_x^4+4y_xy_{3x}+6y_x^2y_{2x}+3y_{2x}^2+y_{4x},$$
$$Y_{5x}=y_x^5+5y_xy_{4x}+15y_xy_{2x}^2+10y_x^2y_{3x}+10y_{2x}y_{3x}$$
$$+10y_x^3y_{xx}+y_{5x},$$
$$Y_{6x}=\left(y_x^5+5y_xy_{4x}+15y_xy_{2x}^2+10y_x^2y_{3x}+10y_{2x}y_{3x}\right.$$
$$\left.+10y_x^3y_{xx}+y_{5x}\right)_x+\left(y_x^5+5y_xy_{4x}+15y_xy_{2x}^2\right.$$
$$\left.+10y_x^2y_{3x}+10y_{2x}y_{3x}+10y_x^3y_{xx}+y_{5x}\right)_x y_x;$$
$$Y_{x,t}=y_x^2+y_{xt},$$
$$Y_{2x,t}=2y_xy_{xt}+y_{2x,t}+y_x^2y_t+y_{2x}y_t,$$
$$Y_{3x,t}=3y_x^2y_t+3y_{xt}y_{2x}+3y_xy_{2x,t}+y_{3x,t}+\left(y_x^3\right.$$
$$\left.+3y_xy_{2x}+y_{3x}\right)y_t. \tag{13}$$

For the sake of computational convenience, we assume that

$$f=e^{\xi(x_1,\dots,x_l)},$$
$$g=e^{\eta(x_1,\dots,x_l)}; \tag{14}$$

we have

$$(fg)^{-1}D_{p,x_1}^{n_1},\dots,D_{p,x_l}^{n_l}f\cdot g$$
$$=\sum_{k_1=0}^{n_1}\cdots\sum_{k_l=0}^{n_1}\prod_{i=1}^{l}\alpha^{k_i}\binom{n_i}{k_i}\left(e^{-\xi}\partial_{x_1}^{n_1-k_1}\cdots\partial_{x_l}^{n_l-k_l}e^{\xi}\right)$$
$$\cdot\left(e^{-\eta}\partial_{x_1}^{n_1-k_1}\cdots\partial_{x_l}^{n_l-k_l}e^{\eta}\right)$$
$$=\sum_{k_1=0}^{n_1}\cdots\sum_{k_l=0}^{n_1}\prod_{i=1}^{l}\alpha^{k_i}\binom{n_i}{k_i}Y_{(n_1-k_1)x_1,\dots,(n_l-k_l)x_l}(\xi)$$
$$\cdot Y_{k_1x_1,\dots,k_lx_l}(\eta)=\sum_{k_1=0}^{n_1}\cdots\sum_{k_l=0}^{n_1}\prod_{i=1}^{l}\alpha^{k_i}\binom{n_i}{k_i}$$
$$\cdot Y_{(n_1-k_1)x_1,\dots,(n_l-k_l)x_l}\left(\xi_{r_1,\dots,r_l}\right)$$
$$\cdot Y_{k_1x_1,\dots,k_lx_l}\left(\alpha^{r_1+\cdots+r_l}\eta_{r_1,\dots,r_l}\right)=Y_{n_1,\dots,n_l}\left(y_{r_1,\dots,r_l}\right.$$
$$\left.=\xi_{r_1,\dots,r_l}+\alpha^{r_1+\cdots+r_l}\eta_{r_1,\dots,r_l}\right)=\overline{Y}_{p;n_1x_1,\dots,n_lx_l}(v,w), \tag{15}$$

where

$$w=\xi+\eta,$$
$$v=\xi-\eta. \tag{16}$$

We find that the link between $\overline{Y}$-polynomials and the D_p-operator can be given in the following through the above deduction:

$$\begin{aligned}(fg)^{-1} D_{p,x_1}^{n_1}, \ldots, D_{p,x_l}^{n_l} f \cdot g &= \overline{Y}_{p;n_1x_1,\ldots,n_lx_l}\left(v = \ln\frac{f}{g}, w = \ln fg\right) \\ &= Y_{n_1,\ldots,n_l}\left(y_{r_1,\ldots,r_l} = \xi_{r_1,\ldots,r_l} + \alpha^{r_1+\cdots+r_l}\eta_{r_1,\ldots,r_l}\right).\end{aligned} \tag{17}$$

In particular, when $f = g$, (17) becomes

$$\begin{aligned}f^{-2} D_{p,x_1}^{n_1}, \ldots, D_{p,x_l}^{n_l} f \cdot f &= \overline{P}_{p;n_1x_1,\ldots,n_lx_l}(q) \\ &= \overline{Y}_{p;n_1x_1,\ldots,n_lx_l}(v = 0, w = 2\ln f = q).\end{aligned} \tag{18}$$

Equation (18) give the relations between D_p-operators and multivariate binary Bell polynomials.

Then we have

$$\begin{aligned}D_{p,x_1}^{n_1}, \ldots, D_{p,x_l}^{n_l} f \cdot f &= f^2 \overline{P}_{p;n_1x_1,\ldots,n_lx_l}(q) \\ &= f^2 \overline{Y}_{p;n_1x_1,\ldots,n_lx_l}(v = 0, w = 2\ln f = q).\end{aligned} \tag{19}$$

From (13) and (18), we have

$$\begin{aligned}\overline{P}_{3;2x} &= \overline{P}_{5;2x} = \overline{P}_{7;2x} = q_{xx}, \\ \overline{P}_{3;4x} &= 3q_{2x}^2, \\ \overline{P}_{3;x,t} &= \overline{P}_{5;x,t} = \overline{P}_{7;x,t} = q_{xt}, \\ \overline{P}_{5;4x} &= \overline{P}_{7;4x} = q_{4x} + 3q_{2x}^2, \\ \overline{P}_{5;3x,y} &= \overline{P}_{7;3x,y} = q_{3x,y} + 3q_{xx}q_{xy}, \\ \overline{P}_{5;2x,t} &= 0, \\ \overline{P}_{5;6x} &= 15q_{2x}^3 + 15q_{2x}q_{4x}, \\ \overline{P}_{7;6x} &= 15q_{2x}^3 + 15q_{2x}q_{4x} + q_{6x}.\end{aligned} \tag{20}$$

3.2. Bilinear Form with D_p-Operators. In this section, we will construct the bilinear forms for Kdv equation, (2+1)-dimensional Kdv equation, and (2+1)-dimensional Sawada-Kotera equation with the D_p-operators quickly and easily by utilizing the relations between D_p-operators and multidimensional bilinear Bell polynomials.

Example 1 (Kdv equation). Consider

$$u_t + 6uu_x + u_{xxx} = 0. \tag{21}$$

Setting $u = q_{2x}$, substituting it into (21), and integrating with respect to x yield

$$q_{xt} + 3q_{2x}^2 + q_{4x} - \lambda_1 = 0, \tag{22}$$

where λ_1 is an arbitrary function of t.

Based on (20) and (22), (21) can be written as follows:

$$\overline{P}_{5,x,t}(q) + \overline{P}_{5,4x}(q) - \lambda_1 = 0. \tag{23}$$

From (19) and (23), we get the bilinear form with D_p-operators of (21)

$$\left(D_{5,x}D_{5,t} + D_{5,x}^4\right) f \cdot f - \lambda_1 f^2 = 0. \tag{24}$$

Example 2 ((2+1)-dimensional Kdv equation). Consider

$$u_t + 3uu_y + u_{xxy} + 3u_x \int u_y dx = 0. \tag{25}$$

Setting $u = q_{2x}$, substituting it into (25), and integrating with respect to x yield

$$q_{xt} + 3q_{xy}q_{2x}^2 + q_{3x,y} - \lambda_2 = 0, \tag{26}$$

where λ_2 is an arbitrary function of y, t. Based on (20) and (26), (25) can be written as follows:

$$\overline{P}_{5,x,t}(q) + \overline{P}_{5,3x,y}(q) - \lambda_2 = 0. \tag{27}$$

From (19) and (27), we get the bilinear form with D_p-operators of (25):

$$\left(D_{5,x}D_{5,t} + D_{5,x}^3 D_{5,y}\right) f \cdot f - \lambda_2 f^2 = 0. \tag{28}$$

Example 3 ((2+1)-dimensional Sawada-Kotera equation). Consider

$$\begin{aligned}&u_t - \left(u_{4x} + 5uu_{2x} + \frac{5}{3}u^3 + 5u_{xy}\right)_x + 5\int u_{2y} dx \\ &\quad - 5uu_y - 5u_x \int u_y dx = 0.\end{aligned} \tag{29}$$

Setting $u = 3q_{2x}$, substituting it into (29), and integrating with respect to x yield

$$\begin{aligned}&q_{xt} + 5q_{2y} - \left(q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3\right) \\ &\quad - 5\left(q_{3xy} + 3q_{2x}q_{xy}\right) - \lambda_3 = 0,\end{aligned} \tag{30}$$

where λ_3 is an arbitrary function of y, t. Based on (20) and (30), (29) can be written as follows:

$$\begin{aligned}&\overline{P}_{7,x,t}(q) + 5\overline{P}_{7,2y}(q) - 5\overline{P}_{7,3x,y}(q) - \overline{P}_{7,6x}(q) - \lambda_3 \\ &\quad = 0.\end{aligned} \tag{31}$$

From (19) and (31), we get the bilinear form with D_p-operators of (29):

$$\begin{aligned}&\left(D_{7,x}D_{7,t} + 5D_{7,y}^2 - 5D_{7,x}^3 D_{7,y} - D_{7,x}^6\right) f \cdot f - \lambda_3 f^2 \\ &\quad = 0.\end{aligned} \tag{32}$$

From the above computation process for seeking the bilinear forms of three nonlinear equation, we can find that the bilinear forms with D_p-operators of nonlinear integrable differential equations are obtained quickly and easily by appling the relations between D_p-operators and multidimensional bilinear Bell polynomials.

4. Periodic Wave Solution of the (3+1)-Dimensional Generalized Shallow Water Equation

In this section, firstly, we will give the bilinear form of a (3+1)-dimensional generalized shallow water equation with the help of $\overline{P}$-polynomials and the D_p-operators. And then, we construct the exact periodic wave solution of the (3+1)-dimensional generalized shallow water equation with the aid of the Riemann theta function, D_p-operators, and the special property of the D_p-operators when acting on exponential functions.

The following is (3+1)-dimensional generalized shallow water equation:

$$u_{xxxy} + 3u_{xx}u_y + 3u_x u_{xy} - u_{yt} - u_{xz} = 0. \quad (33)$$

Setting $u = q_x$, inserting it into (33), and integrating with respect to x yield

$$q_{3x,y} + 3q_{2x}q_{x,y} - q_{y,t} - q_{x,z} - \lambda = 0, \quad (34)$$

where λ is an arbitrary function of y, z, t. Based on (20) and (34), (33) can be expressed as

$$-\overline{P}_{5;y,t} - \overline{P}_{5;x,z} + \overline{P}_{5;3x,y} - \lambda = 0. \quad (35)$$

From the above, we can get the bilinear form of (33):

$$\left(-D_{5;y}D_{5;t} - D_{5;x}D_{5;z} + D_{5;x}^3 D_{5;y}\right) f \cdot f - \lambda \cdot f^2 = 0 \quad (36)$$

with $q = 2\ln f$. When acting on exponential functions, we find that D_p-operators have a good property:

$$\begin{aligned} &G\left(D_{p,x_1}, \dots, D_{p,x_l}\right) e^{\xi_1} \cdot e^{\xi_2} \\ &\quad = G\left(k_1 + \alpha k_2, l_1 + \alpha l_2, h_1 + \alpha h_2, \omega_1 + \alpha\omega_2\right) e^{\xi_1+\xi_2}, \end{aligned} \quad (37)$$

$$\xi_i = k_i x + l_i y + h_i z + \omega_i t + \xi_i^{(0)}, \quad i = 1, 2, \dots \quad (38)$$

In order to construct periodic wave solutions of (33), we study the multidimensional Riemann theta function with genus N given by

$$f(\xi) = f(\xi, \tau) = \sum_{n\in z^N} e^{-\pi i\langle \tau n, n\rangle + 2\pi i\langle \xi, n\rangle} \quad (39)$$

in which $n = (n_1, n_2, \dots, n_N)^T \in z^N$ denotes the integer value vector and $\xi = (\xi_1, \xi_2, \dots, \xi_N)$ is complex phase variable. In addition, for the given two vectors $h = (h_1, h_2, \dots, h_N)$ and $g = (g_1, g_2, \dots, g_N)$ their inner product can be written by

$$\langle h, g\rangle = h_1 g_1 + h_2 g_2 + \cdots + h_N g_N. \quad (40)$$

$-i\tau = (-i\tau_{ij})$ in (39) is a positive definite and real-valued symmetric $N \times N$ matrix, which can be called the period matrix of the theta function. The entries τ_{ij} of τ are free parameters of the theta function (39); we consider that Riemann's (39) converges to a real-valued function with an arbitrary vector $\xi \in C^N$.

In what follows we construct the one-periodic wave solutions of (33). For $N = 1$, Riemann theta function (39) reduces Fourier series in n as follows:

$$f = \sum_{n=-\infty}^{+\infty} e^{\pi i n^2 \tau + 2\pi i n\eta}, \quad (41)$$

where $n \in Z$, $\tau \in C$, $\operatorname{Im}\tau > 0$, and $\eta = kx + ly + hz + \omega t$, with k, l, h, and ω being constants to be determined.

Riemann theta function (41) satisfying the bilinear equation (36) yields the sufficient conditions for obtaining periodic waves. Substituting the theta function (41) into the left of (36) and using the property (37), we have

$$\begin{aligned} G\left(D_{p,x}, D_{p,y}, D_{p,z}, D_{p,t}\right) f \cdot f &= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} G\left(D_{p,x}, D_{p,y}, D_{p,z}, D_{p,t}\right) e^{2\pi i n\eta + \pi i n^2\tau} e^{2\pi i m\eta + \pi i m^2\tau} \\ &= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} G\left(2\pi i(n+\alpha m)k, 2\pi i(n+\alpha m)l, 2\pi i(n+\alpha m)h, 2\pi i(n+\alpha m)\omega\right) e^{2\pi i(n+m)\eta + \pi i(n^2+m^2)\tau} \\ &= \sum_{\delta=-\infty}^{+\infty}\left\{\sum_{n=-\infty}^{+\infty} G\left(\left(2\pi i\left((1-\alpha)n - \alpha^2\delta\right)\right)k, \left(2\pi i\left((1-\alpha)n - \alpha^2\delta\right)\right)l, \left(2\pi i\left((1-\alpha)n - \alpha^2\delta\right)\right)h, \left(2\pi i\left((1-\alpha)n - \alpha^2\delta\right)\right)\omega\right)\right. \\ &\quad \left.\cdot e^{\pi i(n^2 + (n+\alpha\delta)^2)\tau}\right\} e^{2\pi i(-\alpha\delta)\eta} = \sum_{\delta=-\infty}^{+\infty} \overline{G}(\delta) e^{2\pi i(-\alpha\delta)\eta}, \end{aligned} \quad (42)$$

where $\delta = -(1/\alpha)(m+n)$. To the bilinear form of (33), $\overline{G}(\delta)$ satisfies the period characters when $p = 5$. The powers of α obey rule (5), noting that

$$\begin{aligned} \overline{G}(\delta) = \sum_{n=-\infty}^{+\infty} G&\left(\left(2\pi i\left((1-\alpha)n - \alpha^2\delta\right)\right)\right. \\ &\cdot k, \left(2\pi i\left((1-\alpha)n - \alpha^2\delta\right)\right) \end{aligned}$$

$$\begin{aligned}
&\cdot l, \left(2\pi i\left((1-\alpha)n-\alpha^2\delta\right)\right) \\
&\cdot h, \left(2\pi i\left((1-\alpha)n-\alpha^2\delta\right)\right)\omega) e^{\pi i(n^2+(n+\alpha\delta)^2)\tau} \\
&= \sum_{n=-\infty}^{+\infty} G\left(2\pi i\left(2n-\delta\right)k, 2\pi i\left(2n-\delta\right)\right. \\
&\cdot l, 2\pi i\left(2n-\delta\right)h, 2\pi i\left(2n-\delta\right)\omega) e^{\pi i(n^2+(\delta-n)^2)\tau} \\
&= \sum_{n=-\infty}^{+\infty} G\left(2\pi i\left(2s-(\delta-2)\right)k, 2\pi i\left(2s-(\delta-2)\right)\right. \\
&\cdot l, 2\pi i\left(2s-(\delta-2)\right)h, 2\pi i\left(2s-(\delta-2)\right)\omega) \\
&\cdot e^{\pi i(s^2+(\delta-s-2)^2)\tau} e^{2\pi i(\delta-1)\tau} = \overline{G}(\delta-2)\, e^{2\pi i(\delta-1)\tau},
\end{aligned} \tag{43}$$

where $s = n + \alpha$. From (43) we can infer that

$$\overline{G}(\delta) = \begin{cases} \overline{G}(0)\, e^{\pi i n\delta\tau}, & \delta = 2n; \\ \overline{G}(1)\, e^{\pi i(2n+2n^2)(\delta+1)\tau}, & \delta = 2n+1, \end{cases} \tag{44}$$

$$\begin{aligned}
\overline{G}(0) &= \sum_{n=-\infty}^{\infty} \left\{-\left[2\pi i(1-\alpha)n\right]^2 l\omega \right. \\
&\quad - \left[2\pi i(1-\alpha)n\right]^2 kh \\
&\quad \left. + \left[2\pi i(1-\alpha)nk\right]^3 2\pi i(1-\alpha)nl - \lambda\right\} e^{2\pi i n^2\tau} \\
&= \sum_{n=-\infty}^{\infty} \left(16\pi^2 n^2 l\omega + 16\pi^2 n^2 kh + 256\pi^4 n^4 k^3 l - \lambda\right) \\
&\quad \cdot e^{2\pi i n^2\tau} = 0,
\end{aligned} \tag{45}$$

$$\begin{aligned}
\overline{G}(1) &= \sum_{n=-\infty}^{\infty} \left\{-2\pi i\left((1-\alpha)n-\alpha^2\right)l \right. \\
&\quad \cdot 2\pi i\left((1-\alpha)n-\alpha^2\right)\omega \\
&\quad - 2\pi i\left((1-\alpha)n-\alpha^2\right)k 2\pi i\left((1-\alpha)n-\alpha^2\right)h \\
&\quad + \left[2\pi i\left((1-\alpha)n-\alpha^2\right)k\right]^3 2\pi i\left((1-\alpha)n-\alpha^2\right)l \\
&\quad \left. -\lambda\right\} e^{2\pi i(n^2-2n+1)\tau} = \sum_{n=-\infty}^{\infty} \left[4\pi^2(2n-1)^2 l\omega \right. \\
&\quad \left. + 4\pi^2(2n-1)^2 kh + 16\pi^4(2n-1)^4 k^3 l - \lambda\right] \\
&\quad \cdot e^{\pi i(2n^2-2n+1)\tau} = 0.
\end{aligned} \tag{46}$$

Also, the powers of α obey rule (5). For the sake of computational convenience, we denote that

$$\begin{aligned}
\rho_1(n) &= e^{2\pi i n^2\tau}, \\
a_{11} &= \sum_{n=-\infty}^{\infty} 16\pi^2 l n^2 \rho_1(n), \\
a_{12} &= \sum_{n=-\infty}^{\infty} \rho_1(n), \\
b_1 &= -\sum_{n=-\infty}^{\infty} \left\{16\pi^2 n^2 kh + 256\pi^4 n^4 k^3 l\right\} \rho_1(n), \\
\rho_2(n) &= e^{\pi i(2n^2-2n+1)\tau}, \\
a_{21} &= \sum_{n=-\infty}^{\infty} n^2 4\pi^2 (2n-1)^2 l\rho_2(n), \\
a_{22} &= \sum_{n=-\infty}^{\infty} \rho_2(n), \\
b_2 &= -\sum_{n=-\infty}^{\infty} \left(4\pi^2(2n-1)^2 n^2 kh + 16\pi^4(2n-1)^4 k^3 l\right) \\
&\quad \cdot \rho_2(n).
\end{aligned} \tag{47}$$

Then (45) and (46) can be written as

$$\begin{aligned}
a_{11}\omega + a_{12}\lambda - b_1 &= 0, \\
a_{21}\omega + a_{22}\lambda - b_2 &= 0.
\end{aligned} \tag{48}$$

Solving this system, we get

$$\begin{aligned}
\omega &= \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \\
\lambda &= \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}.
\end{aligned} \tag{49}$$

Finally, we get one-periodic wave solution:

$$u = 2\left(\ln f\right)_x, \tag{50}$$

where f is given by (41) and ω, λ are satisfied with (49). And if we assume that $k = 0.01$, $l = 0.01$, $h = 0.01$, and $\tau = i$ to (50), the solution (50) of (33) can be shown in Figure 1.

To this end, the soliton solution of (33) can be obtained when we consider limit of the periodic solution (50). Then, assuming $e^{\pi i\tau} = \gamma$, we can obtain that

$$\begin{aligned}
a_{11} &= \sum_{n=-\infty}^{\infty} 16\pi^2 l n^2 e^{2\pi i n^2\tau} = 32\pi^2 l\left(\gamma^2 + 4\gamma^8 + \cdots\right), \\
a_{12} &= \sum_{n=-\infty}^{\infty} e^{2\pi i n^2\tau} = 1 + 2\gamma^2 + 2\gamma^8 + 2\gamma^{18} + \cdots, \\
a_{21} &= \sum_{n=-\infty}^{\infty} n^2 4\pi^2 (2n-1)^2 l e^{\pi i(2n^2-2n+1)\tau} = 8\pi^2 l\left(\gamma \right. \\
&\quad \left. + 9\gamma^5 + \cdots\right), \\
a_{22} &= \sum_{n=-\infty}^{\infty} e^{\pi i(2n^2-2n+1)\tau} = \gamma + \gamma^5 + \cdots,
\end{aligned}$$

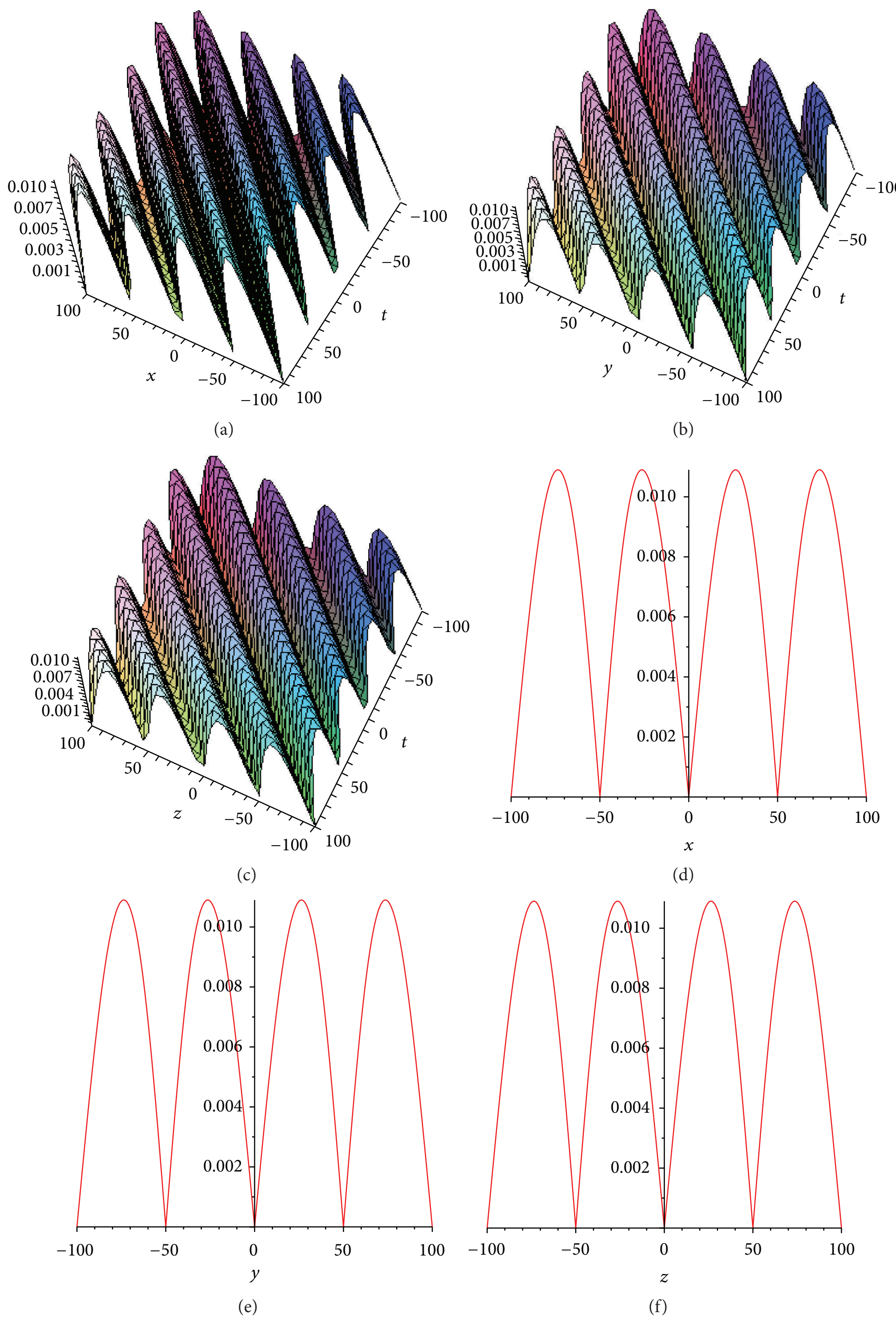

Figure 1: A one-periodic wave (50) of the (3+1)-dimensional shallow water wave equation (33) with parameters $k = 0.01$, $l = 0.01$, $h = 0.01$, and $\tau = i$. This figure shows that every one-periodic wave is one-dimensional, and it can be viewed as a superposition of overlapping solitary waves, placed one period apart. (a) Perspective view of the periodic wave Abs(u) on xot-plane. (b) Perspective view of the periodic wave Abs(u) on yot-plane. (c) Perspective view of the periodic wave Abs(u) on zot-plane. (d) Wave propagation pattern of the wave along the x-axis. (e) Wave propagation pattern of the wave along the y-axis. (f) Wave propagation pattern of the wave along the z-axis.

$$b_1 = -\sum_{n=-\infty}^{\infty} \left\{16\pi^2 n^2 kh + 256\pi^4 n^4 k^3 l\right\} e^{2\pi i n^2 \tau}$$
$$= -32\pi^2 k \left[\left(h + 16\pi^2 k^2 l\right)\gamma^2 + 4\left(h + 16\pi^2 k^2 l\right)\gamma^8 + \cdots\right],$$
$$b_2 = -\sum_{n=-\infty}^{\infty} \left(4\pi^2 (2n-1)^2 n^2 kh + 16\pi^2 (2n-1)^4 k^3 l\right) \cdot e^{\pi i(2n^2-2n+1)\tau} = -8\pi^2 k \left[\left(h + 4\pi^2 k^2 l\right)\gamma + 9\left(h + 4\pi^2 k^2 l\right)\gamma^5 + \cdots\right], \tag{51}$$

which lead to

$$a_{11}a_{22} - a_{12}a_{21} = 8\pi^2 l\gamma + o(\gamma),$$
$$a_{22}b_1 - a_{12}b_2 = 8\pi^2 k \left(h + 4\pi^2 k^2 l\right)\gamma + o(\gamma). \tag{52}$$

So, we have $\omega \to (hk + 4\pi^2 k^3 l)/l$ as $\operatorname{Im}\tau \to +\infty (\gamma \to 0)$.

And we can write f as

$$f = \sum_{n=-\infty}^{+\infty} e^{\pi i n^2 \tau + 2\pi i n \eta} = \sum_{n=-\infty}^{+\infty} e^{\pi i n^2 \tau} e^{2\pi i n \eta}$$
$$= 1 + \sum_{n=1}^{+\infty} e^{\pi i n^2 \tau}\left(e^{2\pi i n \eta} + e^{-2\pi i n \eta}\right)$$
$$= 1 + e^{\pi i \tau}\left(e^{2\pi i \eta} + e^{-2\pi i \eta}\right) + e^{4\pi i n \tau}\left(e^{4\pi i \eta} + e^{-4\pi i \eta}\right) + e^{9\pi i n \tau}\left(e^{6\pi i \eta} + e^{-6\pi i \eta}\right) + \cdots \tag{53}$$
$$= 1 + e^{\pi i \tau} e^{2\pi i \eta} + \left(e^{\pi i \tau} e^{-2\pi i \eta} + e^{4\pi i n \tau} e^{4\pi i \eta}\right) + \left(e^{4\pi i n \tau} e^{-4\pi i \eta} + e^{9\pi i n \tau} e^{6\pi i \eta}\right) + \cdots$$
$$= 1 + e^{2\pi i \eta + \pi i \tau} + e^{2\pi i \tau}\left(e^{-2\pi i \eta - \pi i \tau} + e^{4\pi i \eta + 2\pi i n \tau}\right) + e^{6\pi i n \tau}\left(e^{-4\pi i \eta - 2\pi i \tau} + e^{6\pi i \eta + 3\pi i \tau}\right) + \cdots$$

It is interesting that if we set $\eta' = 2\pi i \eta + \pi i \tau$, (53) can be rewritten as

$$f = 1 + e^{\eta'} + \gamma^2\left(e^{-\eta'} + e^{2\eta'}\right) + \gamma^6\left(e^{-2\eta'} + e^{3\eta'}\right) + \cdots, \tag{54}$$

From (54), we have $f \to 1 + e^{\eta'}$ as

$$\operatorname{Im}\tau \longrightarrow +\infty\,(\gamma \longrightarrow 0). \tag{55}$$

Then the periodic wave solution (50) of (33) turns to the soliton

$$u = 2(\ln f)_x,$$
$$f = 1 + e^{\eta'} = 1 + e^{\pi i(2kt + 2ly + 2hz + 2\omega t + \tau)}. \tag{56}$$

5. Conclusions and Remarks

In this paper, we investigate a (3+1)-dimensional generalized shallow water wave equation (33). Its bilinear form is given by applying the relations D_p-operators and binary Bell polynomials, which has proved to be a quick and direct method. Then, we successfully get the exact periodic wave solution with the help of D_p-operators and Riemann theta function in terms of Hirota direct method. Furthermore, we obtain the corresponding soliton solutions via asymptotic analysis for their periodic wave solutions.

There are many other interesting questions on bilinear differential equations; for example, can the approach be generalized to solve trilinear equations with trilinear differential operators? How to apply the D_p-operators into the discrete equations? Besides, we will try to explore how to construct more nonlinear evolution equations with other operators simply and directly. We will continue to explore these problems in the near future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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