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Research Article

Convergent Analysis of Energy Conservative Algorithm for the Nonlinear Schrödinger Equation

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Using average vector field method in time and Fourier pseudospectral method in space, we obtain an energy-preserving scheme for the nonlinear Schrödinger equation. We prove that the proposed method conserves the discrete global energy exactly. A deduction argument is used to prove that the numerical solution is convergent to the exact solution in discrete L_2 norm. Some numerical results are reported to illustrate the efficiency of the numerical scheme in preserving the energy conservation law.

1. Introduction

The nonlinear Schrödinger (NLS) equation describes a wide range of physical phenomena, such as hydrodynamics, plasma physics, nonlinear optics, self-focusing in laser pulses, propagation of heat pulses in crystals, and description of the dynamics of Bose-Einstein condensate at extremely low temperature [1, 2]. It plays an essential role in mathematical and physical context, and more and more focus is concentrated upon its numerical solvers in recent years [3, 4]. For the NLS equation, construction and theoretical analysis of numerical algorithms have achieved fruitful results [5–14].

The general form of the NLS equation with the initial value and the periodic boundary condition is

$$\begin{gathered} i\psi_t + \psi_{xx} + a|\psi|^2\psi = 0, \\ \psi(0,t) = \psi(2\pi,t), \end{gathered} \tag{1}$$

where a is a real parameter. Now using $\psi = p + iq$, we can rewrite (1) as a pair of real-valued equations as follows:

$$\begin{gathered} p_t + q_{xx} + a\left(p^2+q^2\right)q = 0, \\ q_t - p_{xx} - a\left(p^2+q^2\right)p = 0. \end{gathered} \tag{2}$$

Equations (2) can be expressed in the Hamiltonian form. Consider

$$\frac{dz}{dt} = J\frac{\delta H(z)}{\delta z}, \tag{3}$$

where $z = (p,q)^T \in \mathbb{R}^2$ and the Hamiltonian function, which is system energy, is

$$\begin{gathered} H(z) = \int_0^{2\pi} \frac{1}{2}\left(p_x^2 + q_x^2 - \frac{a}{2}\left(p^2+q^2\right)^2\right)dx, \\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{gathered} \tag{4}$$

The NLS equation (1) admits the energy conservation law. Consider

$$\varepsilon(t) = \int_0^{2\pi}\left(\frac{a}{4}|\psi|^4 - \frac{1}{2}|\psi_x|^2\right)dx = \varepsilon(0). \tag{5}$$

Quispel and McLaren [15] proposed the average vector field (AVF) method, which is a second-order energy-preserving method, and they also provided the corresponding high-order method which is of fourth-order accuracy. The second-order energy-preserving method has been applied to solve the partial differential equation [16]. However, to our knowledge, the current papers are most concentrated on construction of energy-preserving scheme, and very few papers discussed convergent analysis of the energy-preserving scheme. In this paper, we develop an energy conservative algorithm for the NLS equation by using AVF method in time and Fourier pseudospectral method in space and analyze the proposed method.

The paper is organized as follows. In Section 2, a new conservative scheme is proposed for the NLS equation. We prove that the method preserves the energy conservation law. In Section 3, a deduction argument is used to prove that the numerical solution is convergent to the exact solution in discrete L_2 norm. The solitary wave behaviors for the NLS equation are simulated by the new scheme in Section 4. In Section 5, it is devoted to the conclusions.

2. Construction of Conservative Algorithm for the NLS Equation

In this section, we apply the Fourier pseudospectral method in space and the AVF method in time to construct an energy-preserving algorithm for the NLS equation.

One usually uses second-order Fourier spectral differentiation matrix D_2 to approximate the second-order differential operator ∂_{xx}. For the ordinary differential equation $u_{xx} = f$, we set $u_x = v$ and $v_x = f$. Applying the Fourier pseudospectral method to the two equations leads to $D_1 u = v$ and $D_1 v = f$. Eliminating vector v gives $D_1^2 u = f$. In this work, we use $D_1^2 u$ to approximate u_{xx} instead of $D_2 u$ and obtain the corresponding Fourier pseudospectral semidiscretization for the NLS equation (1) as follows:

$$i\frac{d}{dt}\psi_j + \left(D_1^2\Psi\right)_j + a\left|\psi_j\right|^2\psi_j = 0, \quad j = 0,1,2,\ldots,N-1, \tag{6}$$

where $\Psi = (\psi_0, \psi_1, \psi_2, \ldots, \psi_{N-1})^T$. Equations (6) can be rewritten as

$$\begin{aligned} &\frac{d}{dt}p_j + \left(D_1^2\mathbf{q}\right)_j + a\left(p_j^2 + q_j^2\right)q_j = 0, \\ &\frac{d}{dt}q_j - \left(D_1^2\mathbf{p}\right)_j - a\left(p_j^2 + q_j^2\right)p_j = 0, \end{aligned} \tag{7}$$

where $\mathbf{p} = (p_0, p_1, \ldots, p_{N-1})^T$ and $\mathbf{q} = (q_0, q_1, \ldots, q_{N-1})^T$. Since D_1^2 is symmetric, (7) is regarded as a Hamiltonian system with Hamiltonian. Consider

$$H(\mathbf{p},\mathbf{q}) = \frac{1}{2}\left(\mathbf{p}^T D_1^2\mathbf{p} + \mathbf{q}^T D_1^2\mathbf{q}\right) + \frac{a}{4}\sum_{j=0}^{N-1}\left(\left(p_j\right)^2 + \left(q_j\right)^2\right)^2. \tag{8}$$

Now we discretize (6) with respect to time by the AVF method and obtain

$$\begin{aligned} &i\delta_t\psi_j^n + \left(D_1^2 A_t\Psi^n\right)_j \\ &\quad + a\left(A_t\left(\left|\psi_j^n\right|^2\right)A_t\psi_j^n\right. \\ &\quad\quad \left. + \frac{1}{6}\left(\psi_j^{n+1} - \psi_j^n\right)\left(\psi_j^{n+1}p_j^n - \psi_j^n p_j^{n+1}\right)\right) = 0, \end{aligned} \tag{9}$$

where $\delta_t\psi_j^n = (\psi_j^{n+1} - \psi_j^n)/\tau$, $A_t\psi_j^n = (\psi_j^{n+1} + \psi_j^n)/2$, and $p_j^n = \mathrm{Re}(\psi_j^n) = (\psi_j^n + \overline{\psi_j^n})/2$. Obviously, scheme (9) can be reformed as a vector form. Consider

$$\begin{aligned} &i\delta_t\Psi^n + D_1^2 A_t\Psi^n \\ &\quad + a\left(A_t\left(\left|\Psi^n\right|^2\right)\cdot A_t\Psi^n + \frac{1}{6}\left(\Psi^{n+1} - \Psi^n\right)\right. \\ &\quad\quad \left.\cdot\left(\Psi^{n+1}\cdot P^n - \Psi^n\cdot P^{n+1}\right)\right) = 0, \end{aligned} \tag{10}$$

where $|\Psi^n|^2 = |\Psi^n|\cdot|\Psi^n|$ and "$\cdot$" denotes point multiplication between vectors; that is,

$$\mathbf{p}^n\cdot\mathbf{q}^n = \left(p_0^n q_0^n, p_1^n q_1^n, p_2^n q_2^n, \ldots, p_{N-1}^n q_{N-1}^n\right)^T. \tag{11}$$

Equations (9) can also be rewritten as

$$\begin{aligned} &\frac{p_j^{n+1} - p_j^n}{\tau} + \left(D_1^2\mathbf{q}^{n+1/2}\right)_j \\ &\quad + a\left(\frac{1}{4}\left(q_j^{n+1} + q_j^n\right)\right. \\ &\quad\quad \cdot\left(\left(p_j^{n+1}\right)^2 + \left(p_j^n\right)^2 + \left(q_j^{n+1}\right)^2 + \left(q_j^n\right)^2\right) \\ &\quad\quad \left. + \frac{1}{6}\left(p_j^{n+1} - p_j^n\right)\left(q_j^{n+1}\cdot p_j^n - q_j^n\cdot p_j^{n+1}\right)\right) = 0, \\ &\frac{q_j^{n+1} - q_j^n}{\tau} - \left(D_1^2\mathbf{p}^{n+1/2}\right)_j \\ &\quad - a\left(\frac{1}{4}\left(p_j^{n+1} + p_j^n\right)\right. \\ &\quad\quad \cdot\left(\left(p_j^{n+1}\right)^2 + \left(p_j^n\right)^2 + \left(q_j^{n+1}\right)^2 + \left(q_j^n\right)^2\right) \\ &\quad\quad \left. + \frac{1}{6}\left(q_j^{n+1} - q_j^n\right)\left(p_j^{n+1}\cdot q_j^n - p_j^n\cdot q_j^{n+1}\right)\right) = 0. \end{aligned} \tag{12}$$

Next, we prove that scheme (10) conserves the discrete total energy. Let $X_N = \{u \mid u = (u_0, u_1, \ldots, u_{N-1})\} \subseteq \mathbb{C}^N$ and define discrete inner product and discrete L_2 norm over X_N as

$$(u, v)_N = h\sum_{j=0}^{N-1} u_j\overline{v_j}, \quad \|u\|_N = (u, u)_N^{1/2}. \tag{13}$$

Theorem 1. *With periodic boundary condition $\psi(0,t) = \psi(2\pi,t)$, scheme (10) possesses the discrete global energy conservation law; namely,*

$$\varepsilon^{n+1} = \varepsilon^{n} = \cdots = \varepsilon^{1} = \varepsilon^{0}, \tag{14}$$

where $\varepsilon^n = (a/4)\|\Psi^n\|_{N,4}^4 - (1/2)\|D_1\Psi^n\|_N^2$ and $\|\Psi^n\|_{N,4}^4 = (|\Psi^n|^2, |\Psi^n|^2)_N$.

Proof. Taking the inner product of (10) with $\Psi^{n+1} - \Psi^n$ yields

$$\begin{aligned} & i\left(\delta_t\Psi^n, \Psi^{n+1} - \Psi^n\right)_N + \left(D_1^2 A_t\Psi^n, \Psi^{n+1} - \Psi^n\right)_N \\ &\quad + a\Big(A_t\left(|\Psi^n|^2\right)\cdot A_t\Psi^n + \frac{1}{6}\left(\Psi^{n+1} - \Psi^n\right) \\ &\qquad \cdot\left(\Psi^{n+1}\cdot P^n - \Psi^n\cdot P^{n+1}\right), \Psi^{n+1} - \Psi^n\Big)_N = 0. \end{aligned} \tag{15}$$

The first term becomes

$$\frac{i}{\tau}\left(\Psi^{n+1} - \Psi^n, \Psi^{n+1} - \Psi^n\right)_N = \frac{i}{\tau}\left\|\Psi^{n+1} - \Psi^n\right\|_N^2, \tag{16}$$

which is purely imaginary. The second term can be reduced to

$$\begin{aligned} & \frac{1}{2}\left(D_1^2\left(\Psi^{n+1} + \Psi^n\right), \Psi^{n+1} - \Psi^n\right)_N \\ &\quad = -\frac{1}{2}\left(\left\|D_1\Psi^{n+1}\right\|_N^2 - \left\|D_1\Psi^n\right\|_N^2\right) \\ &\qquad + i\,\mathrm{Im}\left(D_1\Psi^{n+1}, D_1\Psi^n\right)_N. \end{aligned} \tag{17}$$

Noticing that $P^n = (\Psi^n + \overline{\Psi^n})/2$, we have

$$\begin{aligned} & a\Big(A_t\left(|\Psi^n|^2\right)\cdot A_t\Psi^n + \frac{1}{6}\left(\Psi^{n+1} - \Psi^n\right) \\ &\qquad \cdot\left(\Psi^{n+1}\cdot P^n - \Psi^n\cdot P^{n+1}\right), \Psi^{n+1} - \Psi^n\Big)_N \\ &\quad = a\Big\{\frac{1}{4}\left(\left\|\Psi^{n+1}\right\|_{N,4}^4 - \left\|\Psi^n\right\|_{N,4}^4\right) \\ &\qquad - \frac{i}{2}\left(\left|\Psi^{n+1}\right|^2 + \left|\Psi^n\right|^2, \mathrm{Im}\left(\Psi^{n+1}\cdot\overline{\Psi^n}\right)\right)_N \\ &\qquad + \frac{i}{6}\left(\mathrm{Im}\left(\Psi^{n+1}\cdot\overline{\Psi^n}\right), \left|\Psi^{n+1} - \Psi^n\right|^2\right)_N\Big\}. \end{aligned} \tag{18}$$

Therefore, the real part of (15) is

$$\begin{aligned} & -\frac{1}{2}\left(\left\|D_1\Psi^{n+1}\right\|_N^2 - \left\|D_1\Psi^n\right\|_N^2\right) \\ &\quad + \frac{a}{4}\left(\left\|\Psi^{n+1}\right\|_{N,4}^4 - \left\|\Psi^n\right\|_{N,4}^4\right) = 0. \end{aligned} \tag{19}$$

So (19) gives the energy conservation law (14). □

3. Convergence Analysis

Let $I = [0, 2\pi]$, $L^2(I)$ with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. For any positive integer r, the seminorm and the norm of $H^r(I)$ are denoted by $|\cdot|_r$ and $\|\cdot\|_r$, respectively. Let $C_{(p)}^{\infty}(I)$ be the set of infinitely differentiable functions with period 2π, defined on $\mathbb{R}$. $H_{(p)}^r(I)$ is the closure of $C_{(p)}^{\infty}(I)$ in $H^r(I)$. In this section, let C be a generic positive constant which may be dependent on the regularity of exact solution and the initial data but independent of the time step τ and the grid size h.

For even N, set

$$\begin{aligned} V_N &= \left\{u \mid u(x) = \sum_{|k|\le N/2}\hat{u}_k e^{ikx},\ \overline{\hat{u}_k} = \hat{u}_{-k},\ |k| \le \frac{N}{2}\right\}, \\ V_N'' &= \Big\{u \mid u(x) = \sum_{|k|\le N/2}{}''\hat{u}_k e^{ikx},\ \overline{\hat{u}_k} = \hat{u}_{-k}, \\ &\qquad |k| \le \frac{N}{2},\ \hat{u}_{N/2} = \hat{u}_{-N/2}\Big\}, \end{aligned} \tag{20}$$

where the summation $\sum''$ is defined by

$$\sum_{|k|\le N/2}{}''\psi_k = \frac{1}{2}\psi_{-N/2} + \sum_{|k|<N/2}\psi_k + \frac{1}{2}\psi_{N/2}. \tag{21}$$

It is obviously that $V_N'' \subseteq V_N$. Denote the orthogonal projection operator $P_N : L^2(I) \to V_N$ and the interpolation operator $I_N : L^2(I) \to V_N''$. Note that P_N and I_N satisfy the following properties:

(1) $P_N\partial_x u = \partial_x P_N u$, $I_N\partial_x u \ne \partial_x I_N u$;

(2) $(P_{N-2}u, v)_N = (P_{N-2}u, v)$, $\forall v \in V_N$;

(3) $P_N u = u$, $\forall u \in V_N$; $I_N u = u$, $\forall u \in V_N''$.

Lemma 2. *For $u \in V_N''$, $\|u\| \le \|u\|_N \le \sqrt{2}\|u\|$.*

Lemma 3 (see [17]). *If $0 \le l \le r$ and $u \in H_{(p)}^r(I)$, then*

$$\begin{aligned} \left\|P_N u - u\right\|_l &\le CN^{l-r}|u|_r, \\ \left\|P_N u\right\|_l &\le C\|u\|_l, \end{aligned} \tag{22}$$

and if $r > 1/2$, then

$$\begin{aligned} \left\|I_N u - u\right\|_l &\le CN^{l-r}|u|_r, \\ \left\|I_N u\right\|_l &\le C\|u\|_l. \end{aligned} \tag{23}$$

Lemma 4. *Suppose $u^* = P_{N-2}u$, $u \in H_{(p)}^r(I)$, and $r > 1/2$; then $\|u^* - u\|_N \le CN^{-r}|u|_r$.*

Proof. According to Lemmas 2 and 3, we have

$$\begin{aligned} \left\|u^* - u\right\|_N &= \left\|I_N\left(u^* - u\right)\right\|_N \\ &= \left\|u^* - I_N u\right\|_N \\ &\le \sqrt{2}\left\|u^* - I_N u\right\| \\ &\le \sqrt{2}\left(\left\|u^* - u\right\| + \left\|u - I_N u\right\|\right) \\ &\le CN^{-r}|u|_r. \end{aligned} \tag{24}$$

□

Lemma 5 (Gronwall's inequality [18]). *Suppose that the discrete function $\{w^n \mid n = 0, 1, 2, \ldots, M;\ M\tau = T\}$ satisfies the following inequality:*

$$w^n - w^{n-1} \le A\tau w^n + B\tau w^{n-1} + C_n\tau, \tag{25}$$

where A, B, and C_n $(n = 0, 1, 2, \ldots, M)$ are nonnegative constants. Then

$$\max_{1\le n\le M} |w^n| \le \left(w^0 + \sum_{l=1}^{M} C_l \right) e^{2(A+B)T}, \tag{26}$$

where τ is sufficiently small, such that $(A + B)\tau \le (M - 1)/2M, (M > 1)$.

An equivalent form of full-discrete Fourier pseudospectral scheme (12) is to find $(p_c^n, q_c^n)^T \in (V_N'')^2$, so that, for any $\Phi^n = (\Phi_1^n, \Phi_2^n)^T \in (V_N'')^2$, then

$$\begin{aligned}\left(\frac{p_c^{n+1} - p_c^n}{\tau}, \Phi_1^n\right)_N &- \left(\partial_x q_c^{n+1/2}, \partial_x \Phi_1^n\right)_N \\ &+ \left(f\left(p_c^n, p_c^{n+1}, q_c^n, q_c^{n+1}\right), \Phi_1^n\right)_N = 0,\end{aligned} \tag{27}$$

$$\begin{aligned}\left(\frac{q_c^{n+1} - q_c^n}{\tau}, \Phi_2^n\right)_N &+ \left(\partial_x p_c^{n+1/2}, \partial_x \Phi_2^n\right)_N \\ &- \left(f\left(q_c^n, q_c^{n+1}, p_c^n, p_c^{n+1}\right), \Phi_2^n\right)_N = 0,\end{aligned} \tag{28}$$

where

$$\begin{aligned} f(x, y, u, v) &= a\left(\frac{1}{4}(u + v)\cdot\left(x^2 + y^2 + u^2 + v^2\right)\right. \\ &\quad \left. + \frac{1}{6}(y - x)(v\cdot x - u\cdot y)\right).\end{aligned} \tag{29}$$

Proposed scheme (10) conserves the energy exactly, which can be regarded as the energy stable algorithm. So we assume that the numerical solution is bounded; that is,

$$\max_{1\le n\le N} \|p_c^n\|_\infty \le C, \qquad \max_{1\le n\le N} \|q_c^n\|_\infty \le C. \tag{30}$$

Theorem 6. *Suppose that the exact solutions $p, q \in H^1(0, T; H_p^r(I)) \cap H^3(0, T; L^2(I))$, $r > 1/2$, and τ are small enough; then the solution of full-discrete Fourier pseudospectral scheme (12) converges to the solution of problems (3) with order $O(N^{-r} + \tau^2)$ in discrete L_2 norm.*

Proof. Let $p^* = P_{N-2}p$ and $q^* = P_{N-2}q$; we have from (2)

$$\begin{aligned} p_t^* + q_{xx}^* + P_{N-2}\left[a\left(p^2 + q^2\right)q\right] &= 0, \\ q_t^* - p_{xx}^* - P_{N-2}\left[a\left(p^2 + q^2\right)p\right] &= 0,\end{aligned} \tag{31}$$

and then

$$\begin{aligned} &\frac{p^{*n+1} - p^{*n}}{\tau} + \partial_{xx}\left(q^*\right)^{n+1/2} + P_{N-2}\left[a\left(p^2 + q^2\right)q\right]^{n+1/2} \\ &\quad = \frac{p^{*n+1} - p^{*n}}{\tau} - \left(p_t^*\right)^{n+1/2} \triangleq \xi_1^n, \\ &\frac{q^{*n+1} - q^{*n}}{\tau} - \partial_{xx}\left(p^*\right)^{n+1/2} - P_{N-2}\left[a\left(p^2 + q^2\right)p\right]^{n+1/2} \\ &\quad = \frac{q^{*n+1} - q^{*n}}{\tau} - \left(q_t^*\right)^{n+1/2} \triangleq \xi_2^n,\end{aligned} \tag{32}$$

where $(p^*)^{n+1/2} = (p^{*n} + p^{*n+1})/2$ and so forth. Using Taylor's expansion, we obtain

$$\|\xi_1^n\|_N \le C\tau^2, \qquad \|\xi_2^n\|_N \le C\tau^2. \tag{33}$$

For any $\Phi^n = (\Phi_1^n, \Phi_2^n)^T \in (V_N'')^2$, (32) are equivalent to the following equations:

$$\begin{aligned} &\left(\frac{p^{*n+1} - p^{*n}}{\tau}, \Phi_1^n\right)_N + \left(\partial_{xx}\left(q^*\right)^{n+1/2}, \Phi_1^n\right)_N \\ &\quad + \left(P_{N-2}\left[a\left(p^2 + q^2\right)q\right]^{n+1/2}, \Phi_1^n\right)_N = \left(\xi_1^n, \Phi_1^n\right)_N, \\ &\left(\frac{q^{*n+1} - q^{*n}}{\tau}, \Phi_2^n\right)_N - \left(\partial_{xx}\left(p^*\right)^{n+1/2}, \Phi_2^n\right)_N \\ &\quad - \left(P_{N-2}\left[a\left(p^2 + q^2\right)p\right]^{n+1/2}, \Phi_2^n\right)_N = \left(\xi_2^n, \Phi_2^n\right)_N.\end{aligned} \tag{34}$$

According to $(P_{N-2}u, v)_N = (P_{N-2}u, v), \forall v \in V_N$ and $P_N\partial_x u = \partial_x P_N u$, we can deduce

$$\begin{aligned} &\left(\frac{p^{*n+1} - p^{*n}}{\tau}, \Phi_1^n\right)_N - \left(\partial_x\left(q^*\right)^{n+1/2}, \partial_x\Phi_1^n\right)_N \\ &\quad + \left(P_{N-2}\left[a\left(p^2 + q^2\right)q\right]^{n+1/2}, \Phi_1^n\right)_N = \left(\xi_1^n, \Phi_1^n\right)_N,\end{aligned} \tag{35}$$

$$\begin{aligned} &\left(\frac{q^{*n+1} - q^{*n}}{\tau}, \Phi_2^n\right)_N + \left(\partial_x\left(p^*\right)^{n+1/2}, \partial_x\Phi_2^n\right)_N \\ &\quad - \left(P_{N-2}\left[a\left(p^2 + q^2\right)p\right]^{n+1/2}, \Phi_2^n\right)_N = \left(\xi_2^n, \Phi_2^n\right)_N.\end{aligned} \tag{36}$$

Let $\varepsilon^n = p^{*n} - p_c^n$ and $\eta^n = q^{*n} - q_c^n$. Subtracting (27)-(28) from (35)-(36), respectively, we obtain the error equations:

$$\begin{aligned} &\left(\frac{\varepsilon^{n+1} - \varepsilon^n}{\tau}, \Phi_1^n\right)_N - \left(\partial_x\eta^{n+1/2}, \partial_x\Phi_1^n\right)_N \\ &\quad + \left(P_{N-2}\left[a\left(p^2 + q^2\right)q\right]^{n+1/2}\right. \\ &\qquad \left. - f\left(p_c^n, p_c^{n+1}, q_c^n, q_c^{n+1}\right), \Phi_1^n\right)_N = \left(\xi_1^n, \Phi_1^n\right)_N, \\ &\left(\frac{\eta^{n+1} - \eta^n}{\tau}, \Phi_2^n\right)_N + \left(\partial_x\varepsilon^{n+1/2}, \partial_x\Phi_2^n\right)_N \\ &\quad - \left(P_{N-2}\left[a\left(p^2 + q^2\right)p\right]^{n+1/2}\right. \\ &\qquad \left. - f\left(q_c^n, q_c^{n+1}, p_c^n, p_c^{n+1}\right), \Phi_2^n\right)_N = \left(\xi_2^n, \Phi_2^n\right)_N.\end{aligned} \tag{37}$$

We take $\Phi_1^n = \varepsilon^{n+1/2}$ and $\Phi_2^n = \eta^{n+1/2}$, and then

$$\left(\frac{\varepsilon^{n+1}-\varepsilon^n}{\tau}, \frac{\varepsilon^{n+1}+\varepsilon^n}{2}\right)_N - \left(\partial_x \eta^{n+1/2}, \partial_x \varepsilon^{n+1/2}\right)_N + \left(G_1, \varepsilon^{n+1/2}\right)_N = \left(\xi_1^n, \varepsilon^{n+1/2}\right)_N, \tag{38}$$

$$\left(\frac{\eta^{n+1}-\eta^n}{\tau}, \frac{\eta^{n+1}+\eta^n}{2}\right)_N + \left(\partial_x \varepsilon^{n+1/2}, \partial_x \eta^{n+1/2}\right)_N - \left(G_2, \eta^{n+1/2}\right)_N = \left(\xi_2^n, \eta^{n+1/2}\right)_N, \tag{39}$$

where

$$\begin{aligned} G_1 &= P_{N-2}\left[a\left(p^2+q^2\right)q\right]^{n+1/2} - f\left(p_c^n, p_c^{n+1}, q_c^n, q_c^{n+1}\right), \\ G_2 &= P_{N-2}\left[a\left(p^2+q^2\right)p\right]^{n+1/2} - f\left(q_c^n, q_c^{n+1}, p_c^n, p_c^{n+1}\right). \end{aligned} \tag{40}$$

Adding (38) and (39), we obtain

$$\begin{aligned} &\frac{1}{2\tau}\left(\left\|\varepsilon^{n+1}\right\|_N^2 + \left\|\eta^{n+1}\right\|_N^2 - \left\|\varepsilon^n\right\|_N^2 - \left\|\eta^n\right\|_N^2\right) \\ &\quad = \left(\xi_1^n, \varepsilon^{n+1/2}\right)_N + \left(\xi_2^n, \eta^{n+1/2}\right)_N \\ &\qquad - \left(G_1, \varepsilon^{n+1/2}\right)_N + \left(G_2, \eta^{n+1/2}\right)_N. \end{aligned} \tag{41}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left|\left(\xi_1^n, \varepsilon^{n+1/2}\right)_N\right| &\le \left\|\xi_1^n\right\|_N \cdot \left\|\varepsilon^{n+1/2}\right\|_N \\ &\le \frac{1}{2}\left\|\xi_1^n\right\|_N^2 + \frac{1}{8}\left\|\varepsilon^n + \varepsilon^{n+1}\right\|_N^2 \\ &\le C\tau^4 + \frac{1}{4}\left(\left\|\varepsilon^n\right\|_N^2 + \left\|\varepsilon^{n+1}\right\|_N^2\right). \end{aligned} \tag{42}$$

Similarly, we have

$$\begin{aligned} \left|\left(\xi_2^n, \eta^{n+1/2}\right)_N\right| &\le C\tau^4 + \frac{1}{4}\left(\left\|\eta^n\right\|_N^2 + \left\|\eta^{n+1}\right\|_N^2\right), \\ \left|\left(G_1, \varepsilon^{n+1/2}\right)_N\right| &\le \frac{1}{2}\left\|G_1\right\|_N^2 + \frac{1}{4}\left(\left\|\varepsilon^n\right\|_N^2 + \left\|\varepsilon^{n+1}\right\|_N^2\right), \\ \left|\left(G_2, \eta^{n+1/2}\right)_N\right| &\le \frac{1}{2}\left\|G_2\right\|_N^2 + \frac{1}{4}\left(\left\|\eta^n\right\|_N^2 + \left\|\eta^{n+1}\right\|_N^2\right). \end{aligned} \tag{43}$$

Using the triangle inequality, we obtain

$$\begin{aligned} &\left\|G_1\right\|_N \\ &\quad = \left\|P_{N-2}\left[a\left(p^2+q^2\right)q\right]^{n+1/2} - f\left(p_c^n, p_c^{n+1}, q_c^n, q_c^{n+1}\right)\right\|_N \\ &\quad \le \left\|P_{N-2}\left[a\left(p^2+q^2\right)q\right]^{n+1/2} - \left[a\left(p^2+q^2\right)q\right]^{n+1/2}\right\|_N \\ &\qquad + \left\|\left[a\left(p^2+q^2\right)q\right]^{n+1/2} - f\left(p^n, p^{n+1}, q^n, q^{n+1}\right)\right\|_N \\ &\qquad + \left\|f\left(p^n, p^{n+1}, q^n, q^{n+1}\right) - f\left(p^{*n}, p^{*n+1}, q^{*n}, q^{*n+1}\right)\right\|_N \\ &\qquad + \left\|f\left(p^{*n}, p^{*n+1}, q^{*n}, q^{*n+1}\right) - f\left(p_c^n, p_c^{n+1}, q_c^n, q_c^{n+1}\right)\right\|_N \\ &\quad \triangleq I + II + III + IV. \end{aligned} \tag{44}$$

According to Lemma 4, $I \le CN^{-r}$. Using Taylor's expansion, $II \le C\tau^2$. Using the inequality

$$|u \cdot v - \tilde{u} \cdot \tilde{v}| \le |u - \tilde{u}| \cdot |v| + |\tilde{u}| \cdot |v - \tilde{v}| \tag{45}$$

and Lemma 4, $III \le CN^{-r}$. According to inequality (45) and the boundedness of numerical solution (30), $IV \le C(\|\varepsilon^n\|_N + \|\varepsilon^{n+1}\|_N + \|\eta^n\|_N + \|\eta^{n+1}\|_N)$.

Therefore, we can deduce

$$\begin{aligned} \left\|G_1\right\|_N^2 &\le C\left(N^{-2r} + \tau^4\right) \\ &\quad + C\left(\left\|\varepsilon^n\right\|_N^2 + \left\|\varepsilon^{n+1}\right\|_N^2 + \left\|\eta^n\right\|_N^2 + \left\|\eta^{n+1}\right\|_N^2\right). \end{aligned} \tag{46}$$

Similarly, we have

$$\begin{aligned} \left\|G_2\right\|_N^2 &\le C\left(N^{-2r} + \tau^4\right) \\ &\quad + C\left(\left\|\varepsilon^n\right\|_N^2 + \left\|\varepsilon^{n+1}\right\|_N^2 + \left\|\eta^n\right\|_N^2 + \left\|\eta^{n+1}\right\|_N^2\right). \end{aligned} \tag{47}$$

Thus, we obtain

$$\begin{aligned} &\frac{1}{2\tau}\left(\left\|\varepsilon^{n+1}\right\|_N^2 + \left\|\eta^{n+1}\right\|_N^2 - \left\|\varepsilon^n\right\|_N^2 - \left\|\eta^n\right\|_N^2\right) \\ &\quad \le C\left(N^{-2r} + \tau^4\right) \\ &\qquad + C\left(\left\|\varepsilon^n\right\|_N^2 + \left\|\varepsilon^{n+1}\right\|_N^2 + \left\|\eta^n\right\|_N^2 + \left\|\eta^{n+1}\right\|_N^2\right). \end{aligned} \tag{48}$$

Let $\omega^n = \|\varepsilon^n\|_N^2 + \|\eta^n\|_N^2$, and (48) can be rewritten as

$$\omega^{n+1} - \omega^n \le C\tau\left(\omega^{n+1} + \omega^n\right) + C\tau\left(N^{-2r} + \tau^4\right). \tag{49}$$

According to Lemma 5, we have

$$\omega^n \le \left(\omega^0 + \tau \sum_{l=1}^{M} C\left(N^{-2r} + \tau^4\right)\right) e^{4CT}. \tag{50}$$

According to Lemma 4 and noticing $p_c^0 = p^0$ and $q_c^0 = q^0$, we have

$$\omega^0 = \left\|\varepsilon^0\right\|_N^2 + \left\|\eta^0\right\|_N^2 = \left\|p^{*0} - p^0\right\|_N^2 + \left\|q^{*0} - q^0\right\|_N^2 \le CN^{-2r}. \tag{51}$$

Therefore, we get

$$\omega^n \le C\left(N^{-2r} + \tau^4\right). \tag{52}$$

Moreover, we have

$$\|p^{*n} - p_c^n\|_N \le C\left(N^{-r} + \tau^2\right),$$
$$\|q^{*n} - q_c^n\|_N \le C\left(N^{-r} + \tau^2\right). \tag{53}$$

Using the triangle inequality and Lemma 4, we obtain

$$\|p^n - p_c^n\|_N \le \|p^n - p^{*n}\|_N + \|p^{*n} - p_c^n\|_N \le C\left(N^{-r} + \tau^2\right),$$
$$\|q^n - q_c^n\|_N \le \|q^n - q^{*n}\|_N + \|q^{*n} - q_c^n\|_N \le C\left(N^{-r} + \tau^2\right). \tag{54}$$

This completes the proof. □

4. Numerical Experiments

In this section, we conduct some tentative numerical experiments for this new scheme (10) to verify the theoretical conclusions, including the accuracy, the ability to preserve the first integrals of the nonlinear Schrödinger equation for long-time integration.

First we take the parameter $a = 2$. Then, we get the following:

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0. \tag{55}$$

We consider nonlinear Schrödinger equation (55) with the one-soliton solution as follows:

$$\psi(x,t) = \operatorname{sech}(x - 4t)\exp\left(2i\left(x - \frac{3}{2}t\right)\right). \tag{56}$$

In order to analyze new scheme (10), the problem is solved in $[-15, 15]$ with the initial condition

$$u(x,0) = \operatorname{sech}(x)\exp(2xi). \tag{57}$$

We take $N = 200$ and the time step $\tau = 10^{-3}$ for the new scheme (10). We check the ability of this new scheme preserving the first integral which is one of the important criteria to judge numerical schemes. The nonlinear Schrödinger equation with periodic boundary condition has the energy conservation law:

$$F(\psi) = \int_0^L \left[\frac{a}{4}|\psi|^4 - \frac{1}{2}|\psi_x|^2\right]dx. \tag{58}$$

If the approximate solution of $\psi(x, t = j\tau)$ is $\psi^j = (\psi_0, \psi_1, \dots, \psi_N)^T$, then the discrete conservation law F is

$$F^h(\psi) = \sum_{j=1}^{N}\frac{1}{2}\left(|\psi_j|^4 - |D_1\Psi|_j^2\right)h. \tag{59}$$

We define the errors of the discrete conservation law on the jth time level as

$$\mathrm{Err}_F(j\tau) = F^h\left(\psi^j\right) - F^h\left(\psi^0\right), \tag{60}$$

where ψ^j is the numerical solution on the jth time level and ψ^0 is the discrete initial value. Numerical solutions and exact solutions at different time levels and the changes of the errors between the exact solutions and the numerical solutions and Err_F with time are shown in Figure 1.

Table 1: Time accuracy of new scheme (10) with initial condition (57) ($N = 100$).

τ	L_2 error	Order
0.004	$1.1351e-004$	—
0.002	$2.8338e-005$	2.0020
0.001	$7.0510e-006$	2.0068
0.0005	$1.7305e-006$	2.0266
0.00025	$4.0445e-007$	2.0972
0.000125	$9.4398e-008$	2.0991

The discrete L^2 norm of complex-valued function ψ is defined as

$$\|\psi\|_2 = \left(\sum_{j=1}^{N}|\psi_j|^2 h\right)^{1/2}. \tag{61}$$

We consider that the problem is solved in $[-15, 15]$ till time $t = 1$ for accuracy test. Note that in Table 1 the spatial error ($N = 100$) is negligible and the error is dominated by the time discretization error. It shows that accuracy of space is very large. Table 1 clearly indicates that new scheme (10) is of second order in time.

We also test our new scheme on the following initial condition $\psi(x, 0) = 0.5 + 0.025\cos(\mu x)$ with the periodic boundary condition $\psi(0,t) = \psi(4\sqrt{2}\pi, t)$. We take $L = 4\sqrt{2}\pi$, $\mu = 2\pi/L$. The initial condition is in the vicinity of the homoclinic orbit in [19].

In this case, we also take $N = 200$ and the time step $\tau = 10^{-3}$ for new scheme (10). The corresponding waveforms at different time levels and the changes of errors of discrete conservation law F with time are showed in Figure 2. We find that the numerical results we presented in the paper show that the new scheme is very robust and stable. Thus, our new scheme provides a new choice for solving the nonlinear Schrödinger equation.

5. Conclusions

In this paper, we derive a new method for the nonlinear Schrödinger system. We prove the proposed method preserves the energy conservation law exactly. A deduction argument is used to prove that the numerical solution is second-order convergent to the exact solutions in $\|\cdot\|_2$ norm. Some numerical results are reported to illustrate the efficiency of the numerical scheme in preserving the energy conservation laws. Therefore, it will be a good choice for solving the nonlinear Schrödinger equation computation.

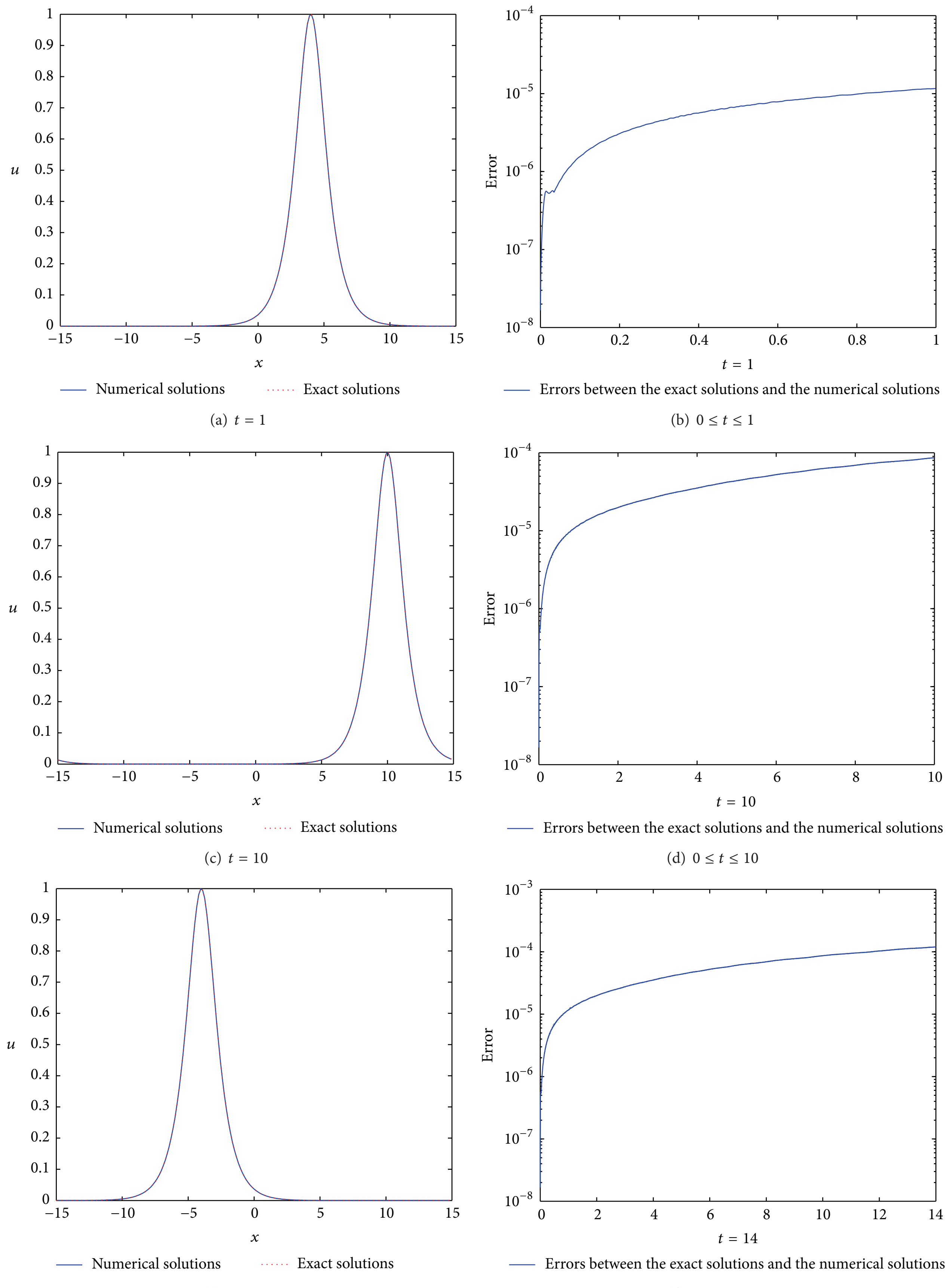


Figure 1: Continued.

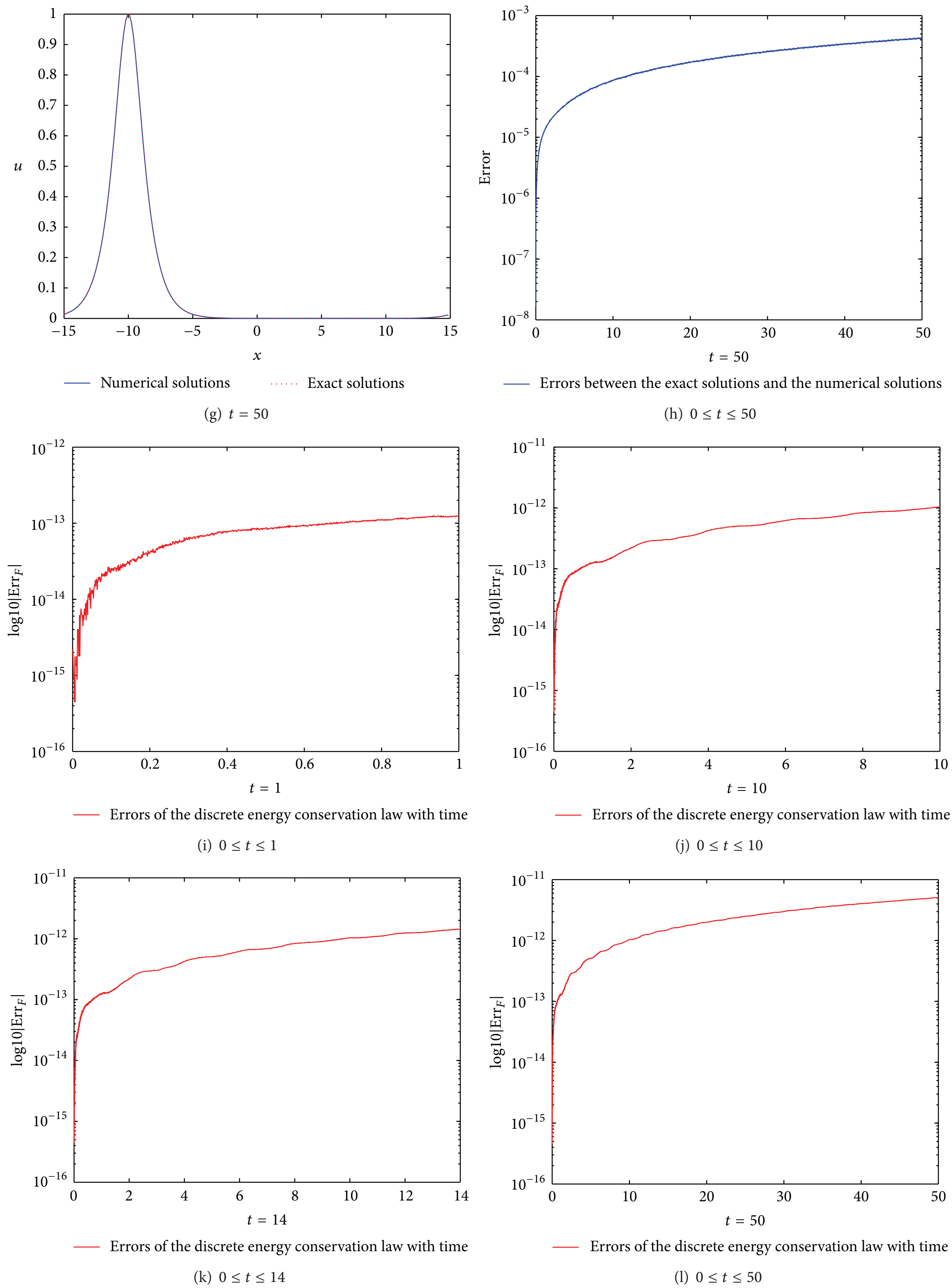
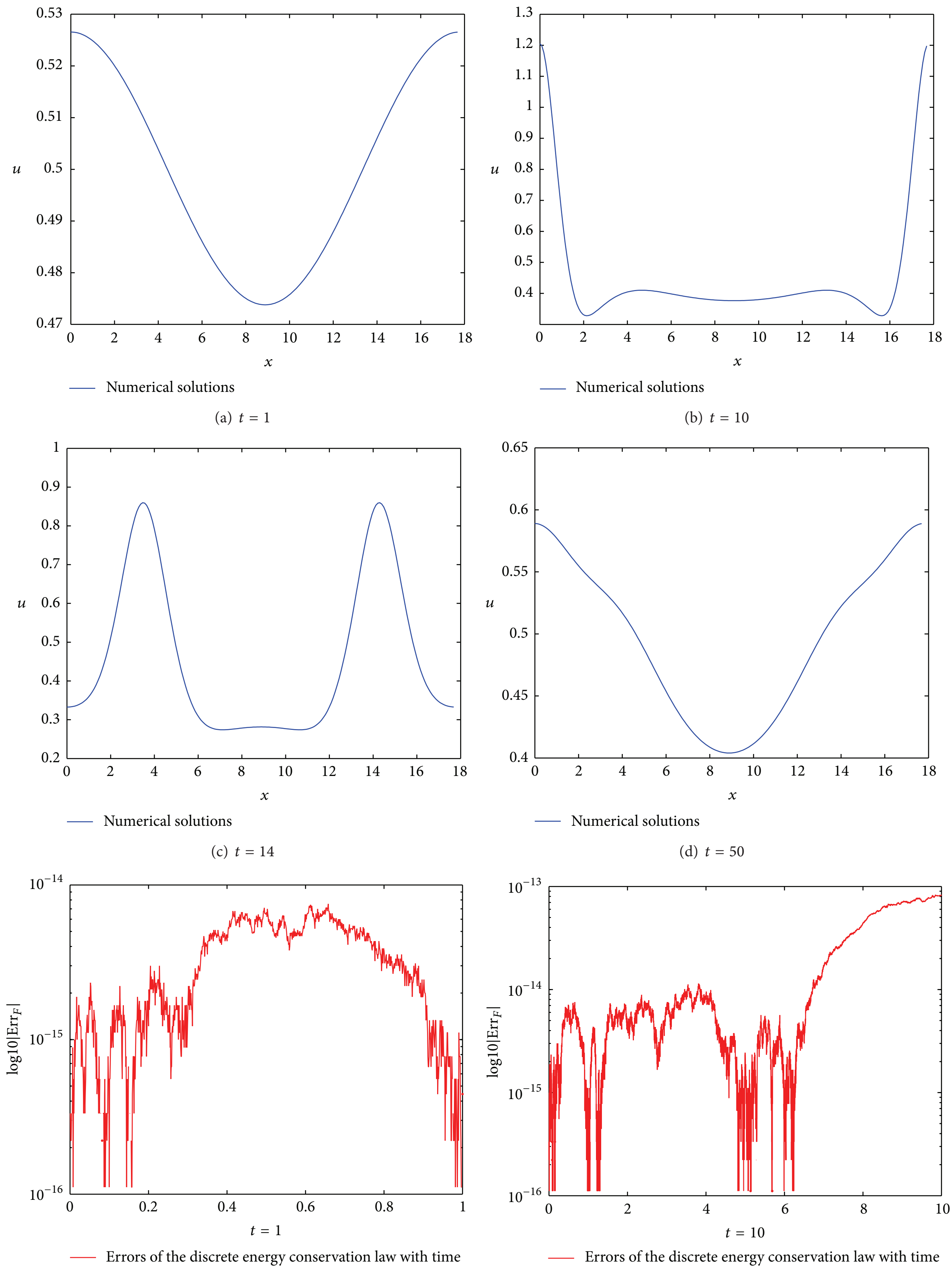


FIGURE 1: The numerical solutions and the exact solutions at $t = 1, 10, 14, 50$ and the changes of the errors between the exact solutions and the numerical solutions and Err_F with time.



(a) $t = 1$

(b) $t = 10$

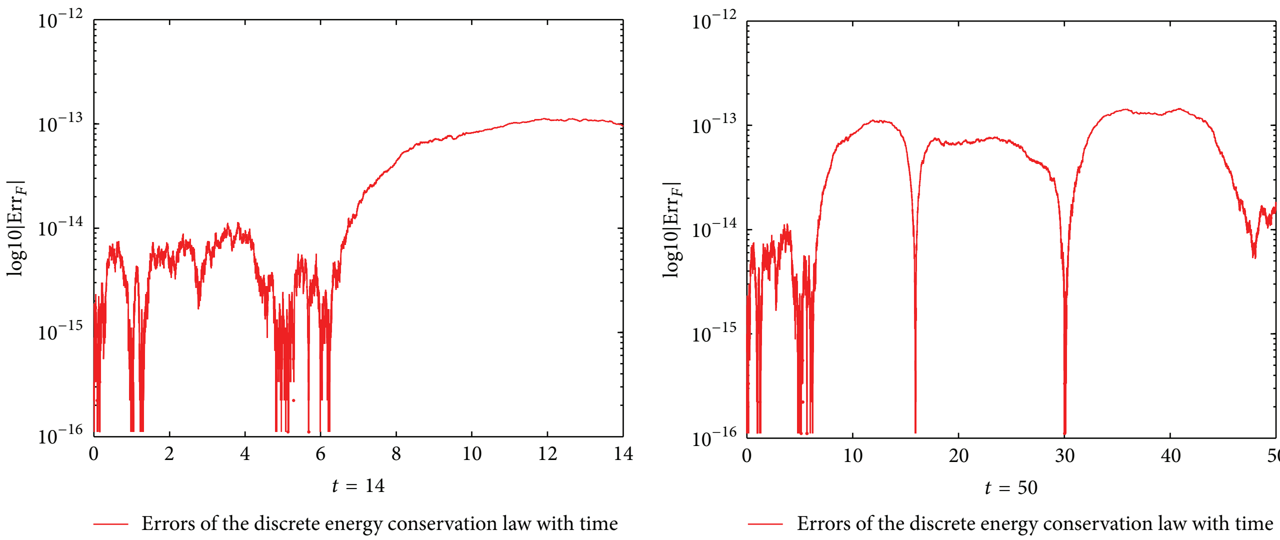
(c) $t = 14$

(d) $t = 50$

(e) $0 \leq t \leq 1$

(f) $0 \leq t \leq 10$

FIGURE 2: Continued.



(g) $0 \leq t \leq 14$

(h) $0 \leq t \leq 50$

FIGURE 2: The numerical solutions at $t = 1, 10, 14, 50$ and the changes of Err_F with time.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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