

Research Article

Sign-Changing Solutions for Nonlinear Operator Equations

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The existence of six solutions for nonlinear operator equations is obtained by using the topological degree and fixed point index theory. These six solutions are all nonzero. Two of them are positive, the other two are negative, and the fifth and sixth ones are both sign-changing solutions. Furthermore, the theoretical results are applied to elliptic partial differential equations.

1. Introduction

In recent years, motivated by some ecological problems, much attention has been attached to the existence of sign-changing solutions for nonlinear partial differential equations (see [1–4] and the references therein). We note that the proofs of main results in [1–4] depend upon critical point theory. However, some concrete nonlinear problems have no variational structures [5]. To overcome this difficulty, in [6], Zhang studied the existence of sign-changing solution for nonlinear operator equations by using the cone theory and combining uniformly positive condition.

Xu [7] studied multiple sign-changing solutions to the following m -point boundary value problems:

$$\begin{aligned} y''(t) + f(y(t)) &= 0, \quad 0 \leq t \leq 1, \\ y(0) &= 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i), \end{aligned} \quad (1)$$

where $0 < \alpha_i, i = 1, 2, \dots, m-2, 0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$.

We list some assumptions as follows.

(A₁) Suppose that the sequence of positive solutions to the equation

$$\sin \sqrt{x} = \sum_{i=1}^{m-2} \alpha_i \sin \eta_i \sqrt{x} \quad (2)$$

is $\lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots$;

(A₂) $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $f(0) = 0$, and $xf(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$;

(A₃) let $\beta_0 = \lim_{x \rightarrow 0} (f(x)/x)$ and $\beta_\infty = \lim_{|x| \rightarrow \infty} (f(x)/x)$. There exist positive integers n_0 and n_1 such that

$$\begin{aligned} \lambda_{2n_0} &< \beta_0 < \lambda_{2n_0+1}, \\ \lambda_{2n_1} &< \beta_\infty < \lambda_{2n_1+1}; \end{aligned} \quad (3)$$

(A₄) there exists $C_0 > 1$ such that

$$|f(x)| < \frac{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)}{5 - \sum_{i=1}^{m-2} \alpha_i \eta_i} C_0 \quad (4)$$

for all $x \in \mathbb{R}$ with $|x| \leq C_0$.

Theorem 1 (see [7]). *Suppose that conditions (A₁)–(A₄) are satisfied. Then the problem (1) has at least two sign-changing solutions. Moreover, the problem (1) also has at least two positive solutions and two negative solutions.*

Based on [7], many authors studied the sign-changing solutions of differential and difference equations. For example, Yang [8] considered the existence of multiple sign-changing solutions for the problem (1). Compared with Theorem 1, Yang employed the following assumption which is different from (A₄).

(A'₄) There exists $T > 0$ such that

$$|f(x)| < 2(1 - \alpha\eta)|x|, \quad |x| \leq T. \quad (5)$$

Pang et al. [9] investigated multiple sign-changing solutions of fourth-order differential equation boundary value problems. Moreover, Wei and Pang [10] established the existence theorem of multiple sign-changing solutions for fourth-order boundary value problems. Y. Li and F. Li [11] studied two sign-changing solutions of a class of second-order integral boundary value problems by computing the eigenvalues and the algebraic multiplicities of the corresponding linear problems. He et al. [12] discussed the existence of sign-changing solutions for a class of discrete boundary value problems, and a concrete example was also given. Very recently, Yang [13] investigated the following discrete fourth Neumann boundary value problems

$$\Delta^4 u(t-2) - \alpha \Delta^2 u(t-1) + \beta(t) = f(t, u(t)),$$

$$t \in [2, T]_{\mathbb{Z}}, \quad (6)$$

$$\Delta u(1) = \Delta u(T) = \Delta^3 u(0) = \Delta^3(T-1) = 0.$$

The author employed similar conditions with (A_1) – (A_4) and obtained a similar result to Theorem 1 (see Theorem 5.1 in [13]).

The main purpose of this paper is to abstract more general conditions from (A_1) – (A_4) of Theorem 1, obtain the existence theorem of sign-changing solutions for general operator equations, and, then, apply the abstract result obtained in this paper to nonlinear elliptic partial differential equations.

2. Preliminaries and Some Lemmas

For the discussion of the following sections, we state here preliminary definitions and known results on cones, partial orderings, and topological degree theory, which can be found in [14–18].

Let E be a real Banach space. Given a cone $P \subset E$, we define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is said to be normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; the smallest N is called the normal constant of P . P is called solid if it contains interior, that is, $\text{int } P \neq \emptyset$. If $x \leq y$ and $x \neq y$, we write $x < y$; if cone P is solid and $y - x \in \text{int } P$, we write $x \ll y$. P is reproducing if $P - P = E$ and total if $\overline{P - P} = E$. Let $B : E \rightarrow E$ be a bounded linear operator. B is said to be positive if $B(P) \subset P$. An operator A is strongly increasing; that is, $x < y$ implies $Ax \ll Ay$. If A is a linear operator, A is strongly increasing which implies A is strongly positive. A fixed point u of operator A is said to be a sign-changing fixed point if $u \notin P \cup (-P)$. If $x_0 \in E \setminus \{\theta\}$ satisfies $\lambda Ax_0 = x_0$, where λ is some real number, then λ is called an eigenvalue of A and x_0 is called an eigenfunction belonging to the eigenvalue λ .

Definition 2 (see [16]). Let E_1, E_2 be real Banach spaces and let $D \subset E_1$ contain the outside of a ball $\{x : \|x\| \leq r\}$, and $A : D \rightarrow E_2$. The operator A is called asymptotically linear if there is a bounded linear operator $B : E_1 \rightarrow E_2$ such that

$$\lim_{\|x\| \rightarrow +\infty} \frac{\|Ax - Bx\|}{\|x\|} = 0. \quad (7)$$

The operator B involved in the definition of an asymptotically linear operator A is uniquely determined. It is called the derivative of A at infinity and is denoted by A'_{∞} .

Definition 3 (see [16, 18]). Let X be a retract of E , and let $U \subset X$ be a relatively bounded open set of X . Suppose that $A : \overline{U} \rightarrow X$ is completely continuous and has no fixed point on ∂U . Let the positive integer $i(A, U, X)$ be defined by

$$i(A, U, X) = \deg(I - Ar, T_R \cap r^{-1}(U), \theta), \quad (8)$$

where $r : E \rightarrow X$ is an arbitrary retraction, and R is a large enough positive number such that $\overline{U} \subset T_R = \{x \mid x \in E, \|x\| < R\}$. Then $i(A, U, X)$ is called the fixed point index of A on U with respect to X .

Lemma 4 (see [14, 18]). *Let P be a reproducing cone and let B be a positive completely continuous linear operator, with $r(B) > 0$, where $r(B)$ denotes the spectral radius of B . Then B has a positive eigenfunction in $P \setminus \{\theta\}$ corresponding to the eigenvalue $(r(B))^{-1}$.*

Lemma 5 (see [18]). *Let P be a cone in E and let Ω be a bounded open set of E , and $\theta \in \Omega$. Assume that $A : P \cap \overline{\Omega} \rightarrow P$ is a condensing operator. If $Ax \not\leq x$ for every $x \in P \cap \partial\Omega$, then $i(A, P \cap \Omega, P) = 1$.*

Lemma 6 (see [19]). *Let P be a normal and total cone in E , and let $A : E \rightarrow E$ be a completely continuous increasing operator. Then the following assertions hold*

- (a) $A(\theta) = \theta$, A is Fréchet differentiable at θ . If $A'_\theta x \neq x$, $\forall x \in P \setminus \{\theta\}$, and there exist $\lambda_0 < 1$ and $x_0 \in P \setminus \{\theta\}$ such that $\lambda_0 A'_\theta x_0 = x_0$, then there exists $\rho_0 > 0$ such that $i(A, B_\rho \cap P, P) = 0$ for all $\rho \in (0, \rho_0]$, where $B_\rho = \{x \in E \mid \|x\| < \rho\}$;
- (b) A is an asymptotically linear operator. If $A'_\infty x \neq x$, $\forall x \in P \setminus \{\theta\}$, and there exist $\tilde{\lambda}_0 < 1$ and $\tilde{x}_0 \in P \setminus \{\theta\}$ such that $\tilde{\lambda}_0 A'_\infty \tilde{x}_0 = \tilde{x}_0$, then there exists $\rho_\infty > 0$ such that $i(A, B_\rho \cap P, P) = 0$ for all $\rho \geq \rho_\infty$.

Lemma 7 (see [18]). *Let D be an open set of E , $A : D \rightarrow E$ be completely continuous, $x_0 \in D$, and $Ax_0 = x_0$. Assume that A is Fréchet differentiable at x_0 and 1 is not an eigenvalue of A'_{x_0} , then x_0 is an isolated fixed point, and*

$$\text{ind}(I - A, x_0) = \text{ind}(I - A'_{x_0}, \theta) = (-1)^\eta, \quad (9)$$

where η is the sum of algebraic multiplicities of the real eigenvalues of A'_{x_0} in $(0, 1)$.

Lemma 8 (see [18]). *Suppose that $A : E \rightarrow E$ is a completely continuous and asymptotically linear operator. If 1 is not an eigenvalue of the linear operator A'_{∞} , then there exists $R_0 > 0$ such that*

$$\deg(I - A, B_R, \theta) = (-1)^\gamma, \quad (10)$$

for all $R \geq R_0$, where γ is the sum of the algebraic multiplicities of the real eigenvalues of A'_{∞} in $(0, 1)$.

Lemma 9 (see [9]). *Let X be a solid cone in E , Ω be a relatively bounded open subset of X , and $A : X \rightarrow X$ be a completely continuous operator. If any fixed point of A in Ω is an interior point of X , then there exists an open subset O of E ($O \subset \Omega$) such that*

$$\deg(I - A, O, \theta) = i(A, \Omega, X). \quad (11)$$

Lemma 10 (see [18]). *Let Ω be a bounded open set of E , and let $\theta \in \Omega$. Assume that $A : \overline{\Omega} \rightarrow E$ is a condensing operator. If*

$$Ax \neq \mu x, \quad \forall x \in \partial\Omega, \mu \geq 1, \quad (12)$$

then $\deg(I - A, \Omega, \theta) = 1$.

3. Multiple Sign-Changing Solutions for Nonlinear Operator Equations

Theorem 11. *Let P be a normal solid cone in E , $A : E \rightarrow E$ be a completely continuous operator, $A(P \setminus \{\theta\}) \subset P^\circ$, $A(-P \setminus \{\theta\}) \subset -P^\circ$, and $A\theta = \theta$. Suppose that*

- (H₁) A'_∞ exists and is an increasing operator; $r(A'_\infty) > 1$; 1 is not an eigenvalue of A'_∞ , and the sum of algebraic multiplicities for the real eigenvalues of A'_∞ in $(0, 1)$ is an even number;
- (H₂) A'_θ exists and is an increasing operator; $r(A'_\theta) > 1$; 1 is not an eigenvalue of A'_θ , and the sum of algebraic multiplicities for the real eigenvalues of A'_θ in $(0, 1)$ is an even number;
- (H₃) $\lim_{\|x\| \rightarrow 0} \sup(\|Ax\|/\|x\|) < 1$.

Then A has at least two sign-changing fixed points, two positive fixed points, and two negative fixed points.

Proof. From condition (H₃), we obtain that there exists $\tilde{r} > 0$ such that $\|Ax\| < \|x\|$ for all $0 < \|x\| < \tilde{r}$. By Lemma 5, there exists $r < \tilde{r}$ such that

$$i(A, P \cap T_r, P) = 1, \quad (13)$$

$$i(A, -P \cap T_r, -P) = 1, \quad (14)$$

where $T_r = \{x \in E \mid \|x\| < r\}$.

Since $A(P \setminus \{\theta\}) \subset P^\circ$, $A(-P \setminus \{\theta\}) \subset -P^\circ$, and $A\theta = \theta$, which together with (H₂) imply that $A'_\theta(P) \subset P$ and $r(A'_\theta) > 1 > 0$. According to Lemma 4, we know that there exists $\bar{\varphi} \in P \setminus \{\theta\}$ such that $A'_\theta \bar{\varphi} = r(A'_\theta) \bar{\varphi} > \bar{\varphi}$. Since 1 is not an eigenvalue of A'_θ , $A'_\theta x \neq x$ and $\forall x \in P \setminus \{\theta\}$. By Lemma 6, there exists $\rho_0 \in (0, r)$ such that

$$i(A, P \cap T_{\rho_0}, P) = 0 \quad (15)$$

for all $\rho \in (0, \rho_0]$. Similarly, we get that

$$i(A, -P \cap T_{\rho_0}, -P) = 0. \quad (16)$$

In the same sense, we know that there exists $\rho_\infty > r$ such that

$$i(A, P \cap T_{\rho_\infty}, P) = 0, \quad (17)$$

$$i(A, -P \cap T_{\rho_\infty}, -P) = 0,$$

for all $\rho \geq \rho_\infty$. Further, combining Lemmas 7 and 8 with (H₁) and (H₂), we get that there exist $\rho_1 \in (0, \rho_0)$ and $\rho_2 > \rho_\infty$ such that

$$\deg(I - A, T_{\rho_1}, \theta) = 1, \quad (18)$$

$$\deg(I - A, T_{\rho_2}, \theta) = 1. \quad (19)$$

By (15)–(17), we have

$$i(A, P \cap T_{\rho_1}, P) = 0, \quad (20)$$

$$i(A, -P \cap T_{\rho_1}, -P) = 0, \quad (21)$$

$$i(A, P \cap T_{\rho_2}, P) = 0, \quad (22)$$

$$i(A, -P \cap T_{\rho_2}, -P) = 0. \quad (23)$$

It follows from (13), (20), (22), and the additivity property of fixed point index that

$$\begin{aligned} i(A, P \cap (T_r \setminus \overline{T_{\rho_1}}), P) &= i(A, P \cap T_r, P) \\ &\quad - i(A, P \cap T_{\rho_1}, P) = 1 - 0 = 1, \\ i(A, P \cap (T_{\rho_2} \setminus \overline{T_r}), P) &= i(A, P \cap T_{\rho_2}, P) \\ &\quad - i(A, P \cap T_r, P) = 0 - 1 = -1. \end{aligned} \quad (24)$$

Hence A has at least two fixed points u_1 and u_2 in $P \cap (T_r \setminus \overline{T_{\rho_1}})$ and $P \cap (T_{\rho_2} \setminus \overline{T_r})$, respectively. It is obvious that u_1 and u_2 are both positive. Moreover, it follows from (14), (21), (23), and the additivity property of fixed point index that

$$\begin{aligned} i(A, -P \cap (T_r \setminus \overline{T_{\rho_1}}), -P) &= 1, \\ i(A, -P \cap (T_{\rho_2} \setminus \overline{T_r}), -P) &= -1. \end{aligned} \quad (25)$$

Consequently, A also has at least two fixed points u_3 and u_4 in $-P \cap (T_r \setminus \overline{T_{\rho_1}})$ and $-P \cap (T_{\rho_2} \setminus \overline{T_r})$, respectively. Evidently, u_3 and u_4 are both negative.

Since $A(P \setminus \{\theta\}) \subset P^\circ$, $A(-P \setminus \{\theta\}) \subset -P^\circ$. Let

$$\begin{aligned} \Sigma_1 &= \{u \in P \cap (T_r \setminus \overline{T_{\rho_1}}) : Au = u\}, \\ \Sigma_2 &= \{u \in P \cap (T_{\rho_2} \setminus \overline{T_r}) : Au = u\}, \\ \Sigma_3 &= \{u \in -P \cap (T_r \setminus \overline{T_{\rho_1}}) : Au = u\}, \\ \Sigma_4 &= \{u \in -P \cap (T_{\rho_2} \setminus \overline{T_r}) : Au = u\}. \end{aligned} \quad (26)$$

By Lemma 9 and (24)–(25), we get that there exist open sets Ω_i ($i = 1, 2, 3, 4$) of E such that

$$\begin{aligned} \Sigma_1 &\subset \Omega_1 \subset P \cap (T_r \setminus \bar{T}_{\rho_1}), \\ \Sigma_2 &\subset \Omega_2 \subset P \cap (T_{\rho_2} \setminus \bar{T}_r), \\ \Sigma_3 &\subset \Omega_3 \subset -P \cap (T_r \setminus \bar{T}_{\rho_1}), \\ \Sigma_4 &\subset \Omega_4 \subset -P \cap (T_{\rho_2} \setminus \bar{T}_r), \end{aligned} \tag{27}$$

$$\deg(I - A, \Omega_1, \theta) = 1, \tag{28}$$

$$\deg(I - A, \Omega_2, \theta) = -1, \tag{29}$$

$$\deg(I - A, \Omega_3, \theta) = 1, \tag{30}$$

$$\deg(I - A, \Omega_4, \theta) = -1. \tag{31}$$

By Lemma 10, we have

$$\deg(I - A, T_r, \theta) = 1. \tag{32}$$

According to (32), (28), (30), (18), and the additivity property of Leray-Schauder degree, we obtain

$$\begin{aligned} \deg(I - A, T_r \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_3 \cup \bar{T}_{\rho_1}), \theta) \\ = 1 - 1 - 1 - 1 = -2, \end{aligned} \tag{33}$$

which yields that A has at least one fixed point u_5 in $T_r \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_3 \cup \bar{T}_{\rho_1})$, and then u_5 is a sign-changing fixed point. It follows from (19), (29), (31), (32), and the additivity property of Leray-Schauder degree that we have

$$\begin{aligned} \deg(I - A, T_{\rho_2} \setminus (\bar{\Omega}_2 \cup \bar{\Omega}_4 \cup \bar{T}_r), \theta) \\ = 1 + 1 + 1 - 1 = 2, \end{aligned} \tag{34}$$

which implies that A has at least one fixed point u_6 in $T_{\rho_2} \setminus (\bar{\Omega}_2 \cup \bar{\Omega}_4 \cup \bar{T}_r)$, and then u_6 is also a sign-changing fixed point. The proof is completed. \square

4. Example

The main purpose of this section is to apply our theorem to nonlinear differential equations.

We consider the following boundary value problem for elliptic partial differential equations

$$\begin{aligned} L\varphi(x) &= f(x, \varphi(x)), \quad x \in \Omega, \\ B\varphi &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{35}$$

where Ω is a bounded open domain in \mathbb{R}^n , $\partial\Omega \in C^{2+\mu}$, and $0 < \mu < 1$; $f(x, \varphi) : \bar{\Omega} \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is continuous;

$$L\varphi = - \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial \varphi}{\partial x_i} + c(x) \varphi \tag{36}$$

is a uniformly elliptic operator; that is, $a_{ij}(x) = a_{ji}(x)$, $b_i(x), c(x) \in C^\mu(\bar{\Omega})$, $c(x) > 0$ and there exists a constant number $\mu_0 > 0$ such that $\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \mu_0 |\xi|^2$ for all $x \in \bar{\Omega}$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$. Consider

$$B\varphi = b(x) \varphi + \delta \sum_{i=1}^n \beta_i(x) \frac{\partial \varphi}{\partial x_i} \tag{37}$$

which is a boundary operator, where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a vector field on $\partial\Omega$ of $C^{1+\mu}$ satisfying $\beta \cdot \mathbf{n} > 0$ (\mathbf{n} denotes the outer unit normal vector on $\partial\Omega$) and $b(x) \in C^{1+\mu}(\partial\Omega)$, and assume that one of the following cases holds:

- (i) $\delta = 0$ and $b(x) \equiv 1$;
- (ii) $\delta = 1$ and $b(x) \equiv 0$;
- (iii) $\delta = 1$ and $b(x) > 0$.

According to the theory of elliptic partial differential equations (see [20, 21]), we know that for each $u \in C(\bar{\Omega})$, the linear boundary value problem

$$\begin{aligned} L\varphi(x) &= u(x), \quad x \in \Omega, \\ B\varphi &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{38}$$

has a unique solution $\varphi_u \in C^2(\bar{\Omega})$. Define the operator K by

$$(Ku)(x) = \varphi_u(x), \quad x \in \Omega. \tag{39}$$

Then $K : C(\bar{\Omega}) \rightarrow C^2(\bar{\Omega})$ is a linear completely continuous operator and has an unbounded sequence of eigenvalues:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots, \lambda_n \rightarrow +\infty \tag{40}$$

and the spectral radius $r(K) = \lambda_1^{-1}$.

Let

$$E = C(\bar{\Omega}), \quad P = \{\varphi \in E \mid \varphi(x) \geq 0, x \in \bar{\Omega}\}. \tag{41}$$

Then E is an ordered Banach space with the norm $\|\varphi\| = \sup_{x \in \bar{\Omega}} |\varphi(x)|$ and P is a normal solid cone in E and $K(P) \subset P$.

For $\varphi \in E$, define Nemytskii operator by

$$(F\varphi)(x) = f(x, \varphi(x)), \quad x \in \bar{\Omega}. \tag{42}$$

Clearly, $F : E \rightarrow E$ is continuous. Let $A = KF$. Then $A : E \rightarrow E$ is completely continuous.

By the proof of Lemma 4.1 in [22], we have that $A'_\theta = f_\theta K$.

Let $e = e(x)$ be the solution of the following boundary value problem

$$\begin{aligned} L\varphi(x) &= 1, \quad x \in \Omega, \\ B\varphi &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{43}$$

In order to obtain multiple sign-changing solutions of (35), we give the following assumptions.

- (E₁) $f(x, 0) \equiv 0$ and $\forall x \in \bar{\Omega}; f(x, \varphi)\varphi > 0, \forall x \in \bar{\Omega}, \varphi \neq 0,$
and $\varphi \in (-\infty, +\infty)$;
- (E₂) $\lim_{\varphi \rightarrow 0}(f(x, \varphi)/\varphi) = f_0$ uniformly for $x \in \bar{\Omega}$, and
 $\lambda_n < f_0 < \lambda_{n+1}$, where n is an even number;
- (E₃) $\lim_{\varphi \rightarrow \infty}(f(x, \varphi)/\varphi) = f_\infty$ uniformly for $x \in \bar{\Omega}$, and
 $\lambda_{n_0} < f_\infty < \lambda_{n_0+1}$, where n_0 is an even number;
- (E₄) there exists a constant number $r < 1$ such that
 $\lim_{\varphi \rightarrow 0}(f(x, \varphi)/\varphi) = r(1/\|e\|)$ uniformly for $x \in \bar{\Omega}$.

Theorem 12. *Suppose that (E₁)–(E₄) are satisfied. Then the problem (35) has at least two sign-changing solutions. Moreover, problem (35) has at least two positive solutions and two negative solutions.*

Proof. From condition (E₁), we know that

$$\begin{aligned} f(x, \varphi) &> 0, \quad \forall \varphi > 0; \\ f(x, \varphi) &< 0, \quad \forall \varphi < 0. \end{aligned} \tag{44}$$

Copy the proof proceed of Theorem 3.4 in [23], we have that $A((E_{e_0} \cap P) \setminus \{\theta\}) \subset \text{int}(E_{e_0} \cap P)$, and $A(-(E_{e_0} \cap P) \setminus \{\theta\}) \subset \text{int}(-(E_{e_0} \cap P))$, where

$$\begin{aligned} E_{e_0} &= \{\varphi \in E : \text{there exists } \mu > 0 \\ &\text{such that } -\mu e_1(x) \leq \varphi \leq \mu e_1(x)\}, \end{aligned}$$

$$\begin{aligned} \text{int}(E_{e_0} \cap P) &= \{\varphi \in (E_{e_0} \cap P) : \text{there exist } \alpha > 0 \\ &\text{and } \beta > 0 \text{ such that } \alpha e_1(x) \\ &\leq \varphi(x) \leq \beta e_1(x)\}, \end{aligned} \tag{45}$$

where e_1 is the first normalized eigenfunction of K corresponding to its first eigenvalue λ_1 .

It follows from (E₂) and (E₃) that conditions (H₁) and (H₂) of Theorem II hold.

In the following, we prove that (H₃) of Theorem II is satisfied. It follows from (E₄) that there exists $\delta > 0$ such that

$$\frac{f(x, \varphi)}{\varphi} < \frac{1+r}{2\|e\|}, \quad \forall 0 < |\varphi| < \delta, x \in \bar{\Omega}. \tag{46}$$

That is,

$$|f(x, \varphi)| < \frac{1+r}{2\|e\|} |\varphi|, \quad x \in \bar{\Omega}. \tag{47}$$

Thus

$$\|A\varphi\| \leq \frac{1+r}{2\|e\|} \|e\| \|\varphi\|. \tag{48}$$

Therefore,

$$\lim_{\|\varphi\| \rightarrow 0} \frac{\|A\varphi\|}{\|\varphi\|} \leq \frac{1+r}{2} < 1. \tag{49}$$

The proof is completed. □

Remark 13. It follows from conditions (E₂) and (E₄) that $f_0\|e\| < 1$. We should point out that the initial ideas of condition (E₄) and the general one (H₃) are motivated by [24].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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