Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 781394, 19 pages
http://dx.doi.org/10.1155/2014/781394

Research Article

Dynamical Behavior of the Stochastic Delay Mutualism System

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Received 11 February 2014; Accepted 9 April 2014; Published 29 May 2014

Academic Editor: Juntao Sun

We discuss the dynamical behavior of the stochastic delay three-specie mutualism system. We develop the technique for stochastic differential equations to deal with the asymptotic property. Using it we obtain the existence of the unique positive solution, the asymptotic properties, and the nonpersistence. Finally, we give the numerical examinations to illustrate our results.

1. Introduction

The classical Lotka-Volterra model for two mutualistic species is described by the ordinary differential equation (ODE)

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[r_1 - a_{11}x_1(t) + a_{12}x_2(t)\right],\\ \dot{x}_2(t) &= x_2(t)\left[r_2 + a_{21}x_1(t) - a_{22}x_2(t)\right].\end{aligned} \tag{1}$$

There are many extensive literatures concerned with the dynamics of this model and we do not mention them here except [1]. Goh [1] showed that if $a_{11}a_{22} > a_{12}a_{21}$ holds, system (1) has a stable and globally attractive equilibrium point $x^* = (x_1^*, x_2^*)$ with the following property:

$$x_1(t) \longrightarrow x_1^*, \quad x_2(t) \longrightarrow x_2^*, \quad \text{as } t \longrightarrow \infty, \tag{2}$$

where

$$x_1^* = \frac{r_1a_{22} + r_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} > 0, \qquad x_2^* = \frac{r_2a_{11} + r_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} > 0. \tag{3}$$

In fact, in many physical as well as biological systems, many studies indicate that time delay widely exists in nature, for examples, in [2–7]. When the growth rate of each specie is affected by the time delay, as a result, (1) becomes a delay differential equation (DDE)

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[r_1 - a_{11}x_1(t-\tau) + a_{12}x_2(t-\tau)\right],\\ \dot{x}_2(t) &= x_2(t)\left[r_2 + a_{21}x_1(t-\tau) - a_{22}x_2(t-\tau)\right].\end{aligned} \tag{4}$$

In [4], He and Gopalsamy obtained a supercritical Hopf-bifurcation of (4) at $\tau = \tau^*$ (a constant) and proved that the equilibrium (x_1^*, x_2^*) is no longer asymptotically stable as the delay increases to τ^*. If only the interspecific positive feedback terms are affected by the delay, (1) becomes

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[r_1 - a_{11}x_1(t) + a_{12}x_2(t-\tau)\right],\\ \dot{x}_2(t) &= x_2(t)\left[r_2 + a_{21}x_1(t-\tau) - a_{22}x_2(t)\right].\end{aligned} \tag{5}$$

The positive equilibrium (x_1^*, x_2^*) of (5) is globally attractive if $a_{11}a_{22} > a_{12}a_{21}$ holds, **a**which implies the delay is harmless.

Population systems are often subject to environmental noise and many authors have investigated the dynamical behaviors of stochastic population systems, for examples, in [8–26]. May [27] revealed that the parameters of the stochastic systems always fluctuate around their average values and the solution also fluctuates around its average value. If we still use r_i to denote the average growth rate, then the intrinsic growth rate becomes

$$r_i \longrightarrow r_i + \sigma_i\dot{B}_i(t), \tag{6}$$

where $\dot{B}_i(t)$ is white noise and σ_i is a positive constant representing the intensity of the noise. As a result, the mutualism system (1) becomes a stochastic differential equation (SDE)

$$\dot{x}_1(t) = x_1(t)\left[\left(r_1 - a_{11}x_1(t) + a_{12}x_2(t)\right)dt + \sigma_1 dB_1(t)\right],$$
$$\dot{x}_2(t) = x_2(t)\left[\left(r_2 + a_{21}x_1(t) - a_{22}x_2(t)\right)dt + \sigma_2 dB_2(t)\right]. \quad (7)$$

Ji et al. in [14] analyzed the long-time asymptotic behavior of the system (7) and obtained the ergodic property and its stationary distribution.

Let us take a further step by considering a 3-dimensional mutualism system

$$\dot{x}_1(t) = x_1(t)\left[r_1 - a_{11}x_1(t) + a_{12}x_2(t-\tau) + a_{13}x_3(t-\tau)\right],$$
$$\dot{x}_2(t) = x_2(t)\left[r_2 + a_{21}x_1(t-\tau) - a_{22}x_2(t) + a_{23}x_3(t-\tau)\right],$$
$$\dot{x}_3(t) = x_3(t)\left[r_3 + a_{31}x_1(t-\tau) + a_{32}x_2(t-\tau) - a_{33}x_3(t)\right], \quad (8)$$

subject to the white noise. As a result, it becomes a stochastic delay differential equation (SDDE)

$$\begin{aligned} dx_1(t) &= x_1(t)\left[\left(r_1 - a_{11}x_1(t) + a_{12}x_2(t-\tau) + a_{13}x_3(t-\tau)\right)dt + \sigma_1 dB_1(t)\right],\\ dx_2(t) &= x_2(t)\left[\left(r_2 + a_{21}x_1(t-\tau) - a_{22}x_2(t) + a_{23}x_3(t-\tau)\right)dt + \sigma_2 dB_2(t)\right],\\ dx_3(t) &= x_3(t)\left[\left(r_3 + a_{31}x_1(t-\tau) + a_{32}x_2(t-\tau) - a_{33}x_3(t)\right)dt + \sigma_2 dB_2(t)\right]. \end{aligned} \quad (9)$$

Our aim is to investigate the long-time asymptotic behavior of SDDE (9). This paper is organized as follows. In order to obtain better dynamic properties of SDDE (9), we show that there exists a unique global positive solution with any initial positive value under some assumptions in Section 2. Then, we estimate the expectation in time average of the distance between the solution of (9) and the positive equilibrium point of the deterministic model (8); namely,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E\left\|x(t) - x^*\right\| \le K(\sigma), \quad (10)$$

where x^* is the unique positive equilibrium point of system (8). In Section 3, we prove that system (9) is persistent in time average as the intensity of the white noise is small and yields the limit of the solution in time average. In Section 4, we obtain the nonpersistence of system (9) as the intensity of noise is big. Finally, in Section 5, we illustrate our results by some numerical examinations.

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathcal{F}_t\}_{t\ge 0}, P)$ denote a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Denote by R_+^3 the positive cone in R^3; namely, $R_+^3 = \{x = (x_1, x_2, x_3)^T \in R^3 : x_i > 0,\ i = 1, 2, 3\}$. Let $\tau > 0$ and denote by $C([-\tau, 0]; R_+^3)$ the family of the continuous functions from $[-\tau, 0]$ to R_+^3. For $x \in R^3$, its norm is denoted by $|x| = |x_1| + |x_2| + |x_3|$.

To discuss the dynamical behavior of (9), we impose the following assumption.

Assumption 1. Consider $a_{ij} > 0$, $i, j = 1, 2, 3$ and $a_{11} > a_{12} + a_{13}$, $a_{22} > a_{21} + a_{23}$, $a_{33} > a_{31} + a_{32}$.

Assumption 2. Consider $r_i > \sigma_i^2/2$, $i = 1, 2, 3$.

2. Existence and Uniqueness of the Positive Solution

In population dynamics, the existence of the global positive solution is necessary. In order for a SDE to have a unique global (i.e., no explosion at any finite time) solution for any given initial value, its coefficients are generally required to satisfy the linear growth condition and local Lipschitz condition (Arnold et al. [28], Mao [23]). However, the coefficients of system (9) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of system (9) may explode at a finite time. So we prepare the following useful lemma and then yield the existence of the positive solution by using it.

Lemma 3. *Under Assumption 1, then we have*

$$\begin{aligned}
&(1)\ a_{22}a_{33} - a_{23}a_{32} > 0, \quad a_{11}a_{33} - a_{13}a_{31} > 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0,\\
&(2)\ \left(a_{22}a_{33} - a_{23}a_{32}\right) + \left(a_{33}a_{21} + a_{31}a_{23}\right) + \left(a_{32}a_{21} + a_{31}a_{22}\right) > 0,\\
&(3)\ \left(a_{11}a_{33} - a_{13}a_{31}\right) + \left(a_{33}a_{12} + a_{32}a_{13}\right) + \left(a_{11}a_{32} + a_{31}a_{12}\right) > 0,\\
&(4)\ \left(a_{11}a_{22} - a_{12}a_{21}\right) + \left(a_{12}a_{23} + a_{22}a_{13}\right) + \left(a_{11}a_{23} + a_{21}a_{13}\right) > 0,\\
&(5)\ \begin{vmatrix} -a_{11} & a_{12} & a_{13} \\ a_{21} & -a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{33} \end{vmatrix} = -a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{22}a_{13}a_{31} + a_{33}a_{12}a_{21} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} < 0.
\end{aligned} \quad (11)$$

Proof. The assertions (1)–(5) are obviously obtained from $a_{11} > a_{12} + a_{13}$, $a_{22} > a_{21} + a_{23}$, $a_{33} > a_{31} + a_{32}$, and $a_{ij} > 0$, $i, j = 1, 2, 3$. Here we omit the proof. □

Theorem 4. *Under Assumption 1, for any given initial value $x(\cdot) \in C([-\tau, 0]; R_+^3)$, there is a unique positive solution $x(t)$ to system (9) on $t \ge -\tau$ and the solution will remain in R_+^3 with probability 1; namely, $x(t) \in R_+^3$ for all $t \ge -\tau$ a. s.*

Proof. Firstly, define a C^2 function $V_1 : R_+^3 \to R_+$ by

$$V_1(x_1, x_2, x_3) = c_1[x_1(t) - 1 - \log x_1(t)] + c_2[x_2(t) - 1 - \log x_2(t)] + c_3[x_3(t) - 1 - \log x_3(t)], \tag{12}$$

where

$$\begin{aligned} c_1 &= ((a_{22}a_{33} - a_{23}a_{32}) + (a_{33}a_{21} + a_{31}a_{23}) + (a_{32}a_{21} + a_{31}a_{22})) \\ &\quad \times (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21} - a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13})^{-1}, \\ c_2 &= ((a_{11}a_{33} - a_{13}a_{31}) + (a_{33}a_{12} + a_{32}a_{13}) + (a_{11}a_{32} + a_{31}a_{12})) \\ &\quad \times (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21} - a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13})^{-1}, \\ c_3 &= ((a_{11}a_{22} - a_{12}a_{21}) + (a_{12}a_{23} + a_{22}a_{13}) + (a_{11}a_{23} + a_{21}a_{13})) \\ &\quad \times (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21} - a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13})^{-1}. \end{aligned} \tag{13}$$

From Lemma 3, $c_i > 0$, $i = 1, 2, 3$. So, $V_1(x_1, x_2, x_3)$ is nonnegative. By Itô's formula, we obtain

$$dV_1(x_1, x_2, x_3) = LV_1(x_1, x_2, x_3)\,dt + \sum_{i=1}^{3} c_i[x_i(t) - 1]\sigma_i dB_i(t), \tag{14}$$

where

$$\begin{aligned} LV_1(x_1, x_2, x_3) &= c_1[x_1(t) - 1][r_1 - a_{11}x_1(t) + a_{12}x_2(t-\tau) + a_{13}x_3(t-\tau)] \\ &\quad + c_2[x_2(t) - 1][r_2 + a_{21}x_1(t-\tau) - a_{22}x_2(t) + a_{23}x_3(t-\tau)] \\ &\quad + c_3[x_3(t) - 1][r_3 + a_{31}x_1(t-\tau) + a_{32}x_2(t-\tau) - a_{33}x_3(t)] \\ &\quad + \frac{c_1\sigma_1^2 + c_2\sigma_2^2 + c_3\sigma_3^2}{2}. \end{aligned} \tag{15}$$

By Young's inequality, we have

$$\begin{aligned} LV_1(x_1, x_2, x_3) &= c_1\Big[(r_1 + a_{11})x_1(t) - a_{12}x_2(t-\tau) - a_{13}x_3(t-\tau) - a_{11}x_1^2(t) + a_{12}x_1(t)x_2(t-\tau) \\ &\qquad + a_{13}x_1(t)x_3(t-\tau) - r_1 + \frac{\sigma_1^2}{2}\Big] \\ &\quad + c_2\Big[(r_2 + a_{22})x_2(t) - a_{21}x_1(t-\tau) - a_{23}x_3(t-\tau) - a_{22}x_2^2(t) + a_{21}x_2(t)x_1(t-\tau) \\ &\qquad + a_{23}x_2(t)x_3(t-\tau) - r_2 + \frac{\sigma_2^2}{2}\Big] \\ &\quad + c_3\Big[(r_3 + a_{33})x_3(t) - a_{31}x_1(t-\tau) - a_{32}x_2(t-\tau) - a_{33}x_3^2(t) + a_{31}x_3(t)x_1(t-\tau) \\ &\qquad + a_{32}x_3(t)x_2(t-\tau) - r_3 + \frac{\sigma_3^2}{2}\Big] \\ &\le c_1\Big[(r_1 + a_{11})x_1(t) - a_{12}x_2(t-\tau) - a_{13}x_3(t-\tau) + \left(-a_{11} + \frac{a_{12}}{2} + \frac{a_{13}}{2}\right)x_1^2(t) \\ &\qquad + \frac{a_{12}}{2}x_2^2(t-\tau) + \frac{a_{13}}{2}x_3^2(t-\tau) - r_1 + \frac{\sigma_1^2}{2}\Big] \\ &\quad + c_2\Big[(r_2 + a_{22})x_2(t) - a_{21}x_1(t-\tau) - a_{23}x_3(t-\tau) + \left(-a_{22} + \frac{a_{21}}{2} + \frac{a_{23}}{2}\right)x_2^2(t) \\ &\qquad + \frac{a_{21}}{2}x_1^2(t-\tau) + \frac{a_{23}}{2}x_3^2(t-\tau) - r_2 + \frac{\sigma_2^2}{2}\Big] \\ &\quad + c_3\Big[(r_3 + a_{33})x_3(t) - a_{31}x_1(t-\tau) - a_{32}x_2(t-\tau) + \left(-a_{33} + \frac{a_{31}}{2} + \frac{a_{32}}{2}\right)x_3^2(t) \\ &\qquad + \frac{a_{31}}{2}x_1^2(t-\tau) + \frac{a_{32}}{2}x_2^2(t-\tau) - r_3 + \frac{\sigma_3^2}{2}\Big]. \end{aligned} \tag{16}$$

Secondly, define

$$V_2(x_1,x_2,x_3)=\int_t^{t+\tau}\left[\left(\frac{a_{12}c_1}{2}+\frac{a_{32}c_3}{2}\right)x_2^2(s-\tau)+\left(\frac{a_{13}c_1}{2}+\frac{a_{23}c_2}{2}\right)x_3^2(s-\tau)+\left(\frac{a_{21}c_2}{2}+\frac{a_{31}c_3}{2}\right)x_1^2(s-\tau)\right]ds. \tag{17}$$

By Itô's formula, we have

$$\begin{aligned} dV_2(x_1,x_2,x_3) &= \left[\left(\frac{a_{12}c_1}{2}+\frac{a_{32}c_3}{2}\right)x_2^2(t)+\left(\frac{a_{13}c_1}{2}+\frac{a_{23}c_2}{2}\right)x_3^2(t)\right. \\ &\quad+\left(\frac{a_{21}c_2}{2}+\frac{a_{31}c_3}{2}\right)x_1^2(t)-\left(\frac{a_{12}c_1}{2}+\frac{a_{32}c_3}{2}\right)x_2^2(t-\tau) \\ &\quad-\left(\frac{a_{13}c_1}{2}+\frac{a_{23}c_2}{2}\right)x_3^2(t-\tau) \\ &\quad\left.-\left(\frac{a_{21}c_2}{2}+\frac{a_{31}c_3}{2}\right)x_1^2(t-\tau)\right]dt. \end{aligned} \tag{18}$$

Therefore, let

$$V(x_1,x_2,x_3)=V_1(x_1,x_2,x_3)+V_2(x_1,x_2,x_3). \tag{19}$$

Equalities (16) and (18) imply that

$$dV(x_1,x_2,x_3)=LV(x_1,x_2,x_3)+\sum_{i=1}^{3}c_i\left[x_i(t)-1\right]\sigma_i dB_i(t), \tag{20}$$

where

$$\begin{aligned} LV(x_1,x_2,x_3) &\le c_1\left[-x_1^2(t)(a_{11}-a_{12}-a_{13})+(r_1+a_{11})x_1(t)\right. \\ &\quad\left.-a_{12}x_2(t-\tau)-a_{13}x_3(t-\tau)-r_1+\frac{\sigma_1^2}{2}\right] \\ &\quad+c_2\left[-x_2^2(t)(a_{22}-a_{21}-a_{23})+(r_2+a_{22})x_2(t)\right. \\ &\quad\left.-a_{21}x_1(t-\tau)-a_{23}x_3(t-\tau)-r_2+\frac{\sigma_2^2}{2}\right] \\ &\quad+c_3\left[-x_3^2(t)(a_{33}-a_{31}-a_{32})+(r_3+a_{33})x_3(t)\right. \\ &\quad\left.-a_{31}x_1(t-\tau)-a_{32}x_2(t-\tau)-r_3+\frac{\sigma_3^2}{2}\right]\le K, \end{aligned} \tag{21}$$

where K is a positive constant. By the method similar to that in [24], the proof is therefore completed. □

Under Assumption 1, DDE (8) has a positive equilibrium $x^*=(x_1^*,x_2^*,x_3^*)$ which is globally attractive while the system with the stochastic perturbation has a unique global positive solution. It is natural to ask how to estimate the distance between the solution of the deterministic system and the solution of the stochastic system. The following theorem gives us an answer.

Theorem 5. *Under Assumption 1, system* (9) *has the following property:*

$$\begin{aligned} \limsup_{t\to\infty}\frac{1}{t}E\int_0^t&\left\{\frac{(a_{11}-a_{12}-a_{13})c_1+1}{2}\left[x_1(s)-x_1^*\right]^2\right. \\ &+\frac{(a_{22}-a_{21}-a_{23})c_2+1}{2}\left[x_2(s)-x_2^*\right]^2 \\ &\left.+\frac{(a_{33}-a_{31}-a_{32})c_3+1}{2}\left[x_3(s)-x_3^*\right]^2\right\}ds \\ \le&\ \frac{\sum_{i=1}^{3}c_ix_i^*\sigma_i^2}{2}, \end{aligned} \tag{22}$$

where $x(t)$ is a solution on $t\ge\tau$ to (9) *with an initial value $x(\cdot)\in C([-\tau,0];R_+^3)$, c_i $(i=1,2,3)$ is defined in the proof of Theorem 4, and $x^*=(x_1^*,x_2^*,x_3^*)$ is the unique positive equilibrium of system* (8)*; namely,*

$$\begin{aligned} x_1^* &= ((a_{22}a_{33}-a_{23}a_{32})r_1+(a_{12}a_{33}+a_{13}a_{32})r_2 \\ &\quad+(a_{12}a_{23}+a_{13}a_{22})r_3) \\ &\quad\times(a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{22}a_{13}a_{31} \\ &\quad-a_{33}a_{12}a_{21}-a_{12}a_{23}a_{31}-a_{21}a_{32}a_{13})^{-1}, \\ x_2^* &= ((a_{21}a_{33}+a_{23}a_{31})r_1+(a_{11}a_{33}-a_{13}a_{31})r_2 \\ &\quad+(a_{11}a_{23}+a_{13}a_{21})r_3) \\ &\quad\times(a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{22}a_{13}a_{31} \\ &\quad-a_{33}a_{12}a_{21}-a_{12}a_{23}a_{31}-a_{21}a_{32}a_{13})^{-1}, \\ x_3^* &= ((a_{21}a_{32}+a_{22}a_{31})r_1+(a_{11}a_{32}+a_{12}a_{31})r_2 \\ &\quad+(a_{11}a_{22}-a_{12}a_{21})r_3) \\ &\quad\times(a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{22}a_{13}a_{31} \\ &\quad-a_{33}a_{12}a_{21}-a_{12}a_{23}a_{31}-a_{21}a_{32}a_{13})^{-1}. \end{aligned} \tag{23}$$

For brevity, we will give the proof in the appendix.

3. Persistence in Time Average

For convenience, we denote the unique global solution of system (9) by $x(t,\xi)$ with an initial data $\xi\in C([-\tau,0];R_3^+)$. Theorem 4 shows that the solution of system (9) will remain positive under Assumption 1. This property gives us an opportunity to investigate how the solution varies in R_3^+. In population dynamics, one of the most attractive properties is persistence which means all species will coexist. Now we give the definition of persistence in time average.

Definition 6. System (9) is persistent in time average, if, for any initial data $\xi \in C([-\tau, 0]; R_+^3)$, the solution $x(t,\xi)$ has the property that

$$0 < \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)\,ds < +\infty \quad \text{a.s. } i = 1,2,3. \tag{24}$$

To prove that system (9) is persistent in time average, we will cite a lemma. Jiang and Shi in [17] discussed a stochastic nonautonomous logistic equation

$$dN(t) = N(t)\left[(a(t) - b(t)N(t))\,dt + \alpha(t)\,dB(t)\right], \tag{25}$$

where $B(t)$ is 1-dimensional standard Brownian motion, $N(0) = N_0$, and N_0 is independent of $B(t)$. They obtained the following result.

Lemma 7 (see [17]). *Assume that $a(t), b(t)$, and $\alpha(t)$ are bounded continuous functions defined on $[0,\infty)$, $a(t) > 0$, and $b(t) > 0$. Then there exists a unique continuous positive solution of* (25) *for any initial value $N(0) = N_0 > 0$, which is global and represented by*

$$N(t) = \left(\exp\left\{\int_0^t\left[a(s) - \frac{\alpha^2(s)}{2}\right]ds + \alpha(s)\,dB(s)\right\}\right) \times \left(\frac{1}{N_0} + \int_0^t b(s)\exp\left\{\int_0^s\left[a(\tau) - \frac{\alpha^2(\tau)}{2}\right]d\tau + \alpha(\tau)\,dB(\tau)\right\}ds\right)^{-1}, \quad t \ge 0. \tag{26}$$

Moreover, the solution $N(t)$ has the property that

$$\lim_{t\to\infty}\frac{\log N(t)}{t} = 0, \quad \text{a.s.} \tag{27}$$

Remark 8. Let $\phi_i(t)$, $i = 1,2,3$, be the solution of the following equation:

$$d\phi_i(t) = \phi_i(t)\left[\left(r_i - a_{ii}\phi_i(t)\right)dt + \sigma_i dB_i(t)\right], \quad t \ge 0, \quad i = 1,2,3, \tag{28}$$

with initial value $\phi_i(0) = x_i(0)$. From Lemma 7, we have

$$\phi_i(t) = \frac{e^{(r_i - (\sigma_i^2/2))t + \sigma_i B_i(t)}}{\left(1/x_i(0)\right) + a_{ii}\int_0^t e^{(r_i - (\sigma_i^2/2))s + \sigma_i B_i(s)}ds}, \quad t \ge 0. \tag{29}$$

From the result in [13], we know

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \phi_i(s)\,ds = \frac{r_i - \sigma_i^2/2}{a_{ii}}, \qquad \lim_{t\to\infty}\frac{\log\phi_i(t)}{t} = 0, \quad \text{a.s. } i = 1,2,3 \tag{30}$$

provided Assumption 2.

Theorem 9. *Under Assumptions 1 and 2, system* (9) *is persistent in time average.*

Proof. From Lemma 7, we know

$$x_i(t) = \frac{e^{\int_0^t (r_i + \sum_{j=1,j\ne i}^3 a_{ij}x_j(s-\tau) - (\sigma_i^2/2))ds + \sigma_i B_i(t)}}{(1/x_i(0)) + a_{ii}\int_0^t e^{\int_0^s (r_i + \sum_{j=1,j\ne i}^3 a_{ij}x_j(u-\tau) - (\sigma_i^2/2))du + \sigma_i B_i(s)}ds}$$
$$= (1)\left(e^{-\sigma_i B_i(t)} \times \left[\frac{1}{x_i(0)}e^{-\int_0^t (r_i + \sum_{j=1,j\ne i}^3 a_{ij}x_j(s-\tau) - (\sigma_i^2/2))ds} + a_{ii} \times \int_0^t e^{-\int_s^t (r_i + \sum_{j=1,j\ne i}^3 a_{ij}x_j(u-\tau) - (\sigma_i^2/2))du + \sigma_i B_i(s)}ds\right]\right)^{-1}. \tag{31}$$

From Remark 8, yields

$$x_i(t) \ge \phi_i(t), \quad i = 1,2,3. \tag{32}$$

Together with Lemma 7, it is easy to obtain

$$\frac{r_i - \sigma_i^2/2}{a_{ii}} \le \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)\,ds, \qquad \liminf_{t\to\infty}\frac{\log x_i(t)}{t} \ge 0 \quad \text{a.s. } i = 1,2,3. \tag{33}$$

The inequality $\limsup_{t\to\infty}(1/t)\int_0^t x_i(s)ds < +\infty$ a.s. $i = 1,2,3$ will be shown in Theorem 11. □

Theorem 9 shows that system (9) is persistent in time average if the intensity of noise is small. Next we want to obtain the limit of the solution in time average of system (9). We begin from the lemma in [29].

Lemma 10. *Let $f \in C[[0,\infty)\times\Omega, (0,\infty)]$, $F(t) \in ((0,\infty)\times\Omega, R)$. If there exist positive constants λ_0 and λ such that*

$$\log f(t) \ge \lambda t - \lambda_0\int_0^t f(s)\,ds + F(t), \quad t \ge 0, \ \text{a.s.}, \tag{34}$$

and $\lim_{t\to\infty}(F(t)/t) = 0$ a.s., then

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t f(s)\,ds \ge \frac{\lambda}{\lambda_0}, \quad \text{a.s.} \tag{35}$$

Theorem 11. *Under Assumptions 1 and 2, for any initial data $\xi \in C([-\tau, 0]; R_+^3)$, the solution $x(t,\xi)$ has the property that*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)\,ds = \tilde{x}_i^* \quad \text{a.s. } i = 1,2,3, \tag{36}$$

where

$$\begin{aligned}
\tilde{x}_1^* &= \Bigg(\left(a_{22}a_{33} - a_{23}a_{32}\right)\left(r_1 - \frac{\sigma_1^2}{2}\right) + \left(a_{12}a_{33} + a_{13}a_{32}\right) \\
&\quad \times \left(r_2 - \frac{\sigma_2^2}{2}\right) + \left(a_{12}a_{23} + a_{13}a_{22}\right)\left(r_3 - \frac{\sigma_3^2}{2}\right)\Bigg) \\
&\quad \times (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} \\
&\quad - a_{33}a_{12}a_{21} - a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13})^{-1}, \\
\tilde{x}_2^* &= \Bigg(\left(a_{21}a_{33} + a_{23}a_{31}\right)\left(r_1 - \frac{\sigma_1^2}{2}\right) + \left(a_{11}a_{33} - a_{13}a_{31}\right) \\
&\quad \times \left(r_2 - \frac{\sigma_2^2}{2}\right) + \left(a_{11}a_{23} + a_{13}a_{21}\right)\left(r_3 - \frac{\sigma_3^2}{2}\right)\Bigg) \\
&\quad \times (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} \\
&\quad - a_{33}a_{12}a_{21} - a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13})^{-1}, \\
\tilde{x}_3^* &= \Bigg(\left(a_{21}a_{32} + a_{22}a_{31}\right)\left(r_1 - \frac{\sigma_1^2}{2}\right) + \left(a_{11}a_{32} + a_{12}a_{31}\right) \\
&\quad \times \left(r_2 - \frac{\sigma_2^2}{2}\right) + \left(a_{11}a_{22} - a_{12}a_{21}\right)\left(r_3 - \frac{\sigma_3^2}{2}\right)\Bigg) \\
&\quad \times (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} \\
&\quad - a_{33}a_{12}a_{21} - a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13})^{-1}.
\end{aligned} \tag{37}$$

The mathematical derivations are lengthy; we will give the proof in the appendix.

4. Nonpersistence

In this section, we will show that the system (9) is nonpersistent if the intensity of the noise is big enough; however, it does not occur to the deterministic system. First of all, we give the definition of nonpersistence.

Definition 12. System (9) is nonpersistent, if there are positive constants b_1, b_2, b_3 such that

$$\lim_{t\to\infty}\prod_{i=1}^{3} x_i^{b_i}(t) = 0 \quad \text{a.s.} \tag{38}$$

Theorem 13. *Under Assumption 1, if $C < 0$ holds, system* (9) *is nonpersistent, where $C = b_{11}(r_1 - (\sigma_1^2/2)) + b_{12}(r_2 - (\sigma_2^2/2)) + b_{13}(r_3 - (\sigma_3^2/2))$, $b_{11} > 0$; b_{12}, b_{13} is defined by* (B.24).

Proof. It follows from (B.25) that

$$\begin{aligned}
&b_{11}\log x_1(t) + b_{12}\log x_2(t) + b_{13}\log x_3(t) \\
&\quad \le \left(b_{11}\bar{r}_1 + b_{12}\bar{r}_2 + b_{13}\bar{r}_3\right)t \\
&\qquad - m_1\int_0^t x_1(s)\,ds + \sum_{i=1}^{3} b_{1i}\sigma_i B_i(t) + \sum_{i=1}^{3} b_{1i}U_i \\
&\quad \le \left(b_{11}\bar{r}_1 + b_{12}\bar{r}_2 + b_{13}\bar{r}_3\right)t + \sum_{i=1}^{3} b_{1i}\sigma_i B_i(t) + \sum_{i=1}^{3} b_{1i}U_i.
\end{aligned} \tag{39}$$

Together with $\lim_{t\to\infty}(B_i(t)/t) = 0$ and $\lim_{t\to\infty}(U_i/t) = 0$, $i = 1,2,3$, we have

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^{3} b_{1i}\log x_i(t) \le b_{11}\bar{r}_1 + b_{12}\bar{r}_2 + b_{13}\bar{r}_3. \tag{40}$$

If $b_{11}\bar{r}_1 + b_{12}\bar{r}_2 + b_{13}\bar{r}_3 < 0$ holds, it follows

$$\lim_{t\to\infty}\prod_{i=1}^{3} x_i^{b_{1i}}(t) = 0 \quad \text{a.s.} \tag{41}$$

Hence, system (9) is nonpersistent. The proof is completed. □

By a similar method, we can yield the following theorems.

Theorem 14. *Under Assumption 1, if $C < 0$ holds, system* (9) *is nonpersistent, where $C = b_{21}(r_1 - (\sigma_1^2/2)) + b_{22}(r_2 - (\sigma_2^2/2)) + b_{23}(r_3 - (\sigma_3^2/2))$, $b_{22} > 0$, $b_{21} = ((-a_{21}a_{33} - a_{23}a_{31})/(a_{13}a_{31} - a_{11}a_{33}))b_{22}$, $b_{23} = ((-a_{13}a_{21} - a_{11}a_{23})/(a_{13}a_{31} - a_{11}a_{33}))b_{22}$.*

Theorem 15. *Under Assumption 1, if $C < 0$ holds, system* (9) *is nonpersistent, where $C = b_{31}(r_1 - (\sigma_1^2/2)) + b_{32}(r_2 - (\sigma_2^2/2)) + b_{23}(r_3 - (\sigma_3^2/2))$, $b_{33} > 0$, $b_{31} = ((-a_{31}a_{22} - a_{32}a_{21})/(a_{12}a_{21} - a_{11}a_{22}))b_{33}$, $b_{32} = ((-a_{31}a_{12} - a_{32}a_{11})/(a_{12}a_{21} - a_{11}a_{22}))b_{33}$.*

5. Numerical Examinations

In this section, we give the numerical examinations to illustrate above results. By the method mentioned in [30], consider the discrete equation:

$$\begin{aligned}
x_{1,k+1} &= x_{1,k} + x_{1,k}\Big[\left(r_1 - a_{11}x_{1,k} + a_{12}x_{2,k-m} + a_{13}x_{3,k-m}\right)\Delta t \\
&\qquad + \sigma_1\varepsilon_{1,k}\sqrt{\Delta t} + \frac{1}{2}\sigma_1^2\left(\varepsilon_{1,k}^2\Delta t - \Delta t\right)\Big], \\
x_{2,k+1} &= x_{2,k} + x_{2,k}\Big[\left(r_2 + a_{21}x_{1,k-m} - a_{22}x_{2,k} + a_{23}x_{3,k-m}\right)\Delta t \\
&\qquad + \sigma_2\varepsilon_{2,k}\sqrt{\Delta t} + \frac{1}{2}\sigma_2^2\left(\varepsilon_{2,k}^2\Delta t - \Delta t\right)\Big], \\
x_{3,k+1} &= x_{3,k} + x_{3,k}\Big[\left(r_3 + a_{31}x_{1,k-m} + a_{32}x_{2,k-m} - a_{33}x_{3,k}\right)\Delta t \\
&\qquad + \sigma_3\varepsilon_{3,k}\sqrt{\Delta t} + \frac{1}{2}\sigma_3^2\left(\varepsilon_{3,k}^2\Delta t - \Delta t\right)\Big],
\end{aligned} \tag{42}$$

where m represents the integer part of $\tau/\Delta t - 1$. Choosing suitable parameters in the system, by Matlab we get the simulation figures with initial value $(x_1(t), x_2(t), x_3(t)) \equiv (0.7, 0.3, 0.9)$, $t \in [-\tau, 0]$. (For convenience we let the initial value be a constant function; otherwise we have to give $m+1$ values.) The time step $\Delta t = 0.005$; we always choose

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.2 & 0.4 \\ 0.1 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.8 \end{pmatrix}, \qquad \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.3 \end{pmatrix}. \tag{43}$$

Then $a_{11} - a_{12} - a_{13} = 0.1$, $a_{22} - a_{21} - a_{23} = 0.2$, $a_{33} - a_{31} - a_{32} = 0.2$, $(x_1^*, x_2^*, x_3^*) = (2.2679, 2.0268, 1.9554)$. By choosing different intensities of the noise and time delays, we obtain the following cases to illustrate our results.

Case 1 (persistence). Choosing $\tau = 20.2$ $\sigma_1 = 0.04$, $\sigma_2 = 0.08$, $\sigma_3 = 0.07$, then we have $r_i > \sigma_i^2/2$, $i = 1, 2, 3$. Hence all assumptions of Theorem 11 are satisfied. Figure 1 shows that the solution fluctuates in a small zone.

Case 2 (nonpersistence (we only illustrate the first situation)).

(1) $x_1(t)$ Is Disturbed by a Big Noise, Which Leads to Nonpersistence. Choosing $\tau = 20.2$, $\sigma_1 = 1.53$, $\sigma_2 = 0.08$, $\sigma_3 = 0.07$, $b_{11} = 0.10$, $b_{12} = 0.1143$, $b_{13} = 0.0929$, then we have $r_i > \sigma_i^2$, $i = 2, 3$ but $b_{11}(r_1 - (\sigma_1^2/2)) + b_{12}(r_2 - (\sigma_2^2/2)) + b_{13}(r_3 - (\sigma_3^2/2)) < 0$. For convenience we shorten b_{1i} to b_i. Hence the assumptions of Theorem 13 are satisfied, and system (9) is nonpersistent (see Figure 2(d)). In addition, since $r_i > \sigma_i^2$, $i = 2, 3$, $x_2(t), x_3(t)$ do not tend to zero in time average by Theorem 9. So $\lim_{t\to\infty} x_1(t) = 0$ a.s. (see Figures 2(a)–2(c)).

(2) The Second Specie $x_2(t)$ Is Disturbed by the Big White Noise, Which Leads to the Nonpersistence. Choosing $\tau = 20.2$, $\sigma_1 = 0.04$, $\sigma_2 = 1.17$, $\sigma_3 = 0.07$, $b_1 = 0.10$, $b_2 = 0.1143$, $b_3 = 0.0929$, then we have $r_i > \sigma_i^2$, $i = 1, 3$, but $b_1(r_1 - (\sigma_1^2/2)) + b_2(r_2 - (\sigma_2^2/2)) + b_3(r_3 - (\sigma_3^2/2)) < 0$. Hence the assumptions of Theorem 13 are satisfied, and system (9) is nonpersistent (see Figure 3(d)). In addition, since $r_i > \sigma_i^2$, $i = 1, 3$, $x_1(t), x_3(t)$ do not tend to zero in time average by Theorem 9. So $\lim_{t\to\infty} x_2(t) = 0$ a.s. (see Figures 3(a)–3(c)).

(3) The Third Specie $x_3(t)$ Is Disturbed by the Big White Noise, Which Leads to the Nonpersistence. Choosing $\tau = 20.2$, $\sigma_1 = 0.04$, $\sigma_2 = 0.08$, $\sigma_3 = 1.36$, $b_1 = 0.10$, $b_2 = 0.1143$, $b_3 = 0.0929$, then we have $r_i > \sigma_i^2$, $i = 1, 2$, but $b_1(r_1 - (\sigma_1^2/2)) + b_2(r_2 - (\sigma_2^2/2)) + b_3(r_3 - (\sigma_3^2/2)) < 0$. Hence the assumptions of Theorem 13 are satisfied, and system (9) is nonpersistent (see Figure 4(d)). In addition, since $r_i > \sigma_i^2$, $i = 1, 2$, $x_1(t), x_2(t)$ do not tend to zero in time average by Theorem 9. So we have $\lim_{t\to\infty} x_3(t) = 0$ a.s. (see Figures 4(a)–4(c)).

(4) The First Two Species $x_1(t), x_2(t)$ Are Disturbed by the Big White Noises, Which Leads to the Nonpersistence. Choosing $\tau = 20.2$, $\sigma_1 = 1.25$, $\sigma_2 = 1.01$, $\sigma_3 = 0.07$, $b_1 = 0.10$, $b_2 = 0.1143$, $b_3 = 0.0929$, then we have $r_3 > \sigma_3^2$, but $b_1(r_1 - (\sigma_1^2/2)) + b_2(r_2 - (\sigma_2^2/2)) + b_3(r_3 - (\sigma_3^2/2)) < 0$. Hence the assumptions of Theorem 13 are satisfied; then system (9) is nonpersistent (see Figure 5(d)). In addition, since $r_3 > \sigma_3^2$, $x_3(t)$ does not tend to zero in time average by Theorem 9. So we have $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, 2$ a.s. (see Figures 5(a)–5(c)).

(5) The First and the Third Species $x_1(t), x_3(t)$ Are Disturbed by the Big White Noises, Which Leads to the Nonpersistence. Choosing $\tau = 20.2$, $\sigma_1 = 1.25$, $\sigma_2 = 0.08$, $\sigma_3 = 1.12$, $b_1 = 0.10$, $b_2 = 0.1143$, $b_3 = 0.0929$, then we have $r_2 > \sigma_2^2$, but $b_1(r_1 - (\sigma_1^2/2)) + b_2(r_2 - (\sigma_2^2/2)) + b_3(r_3 - (\sigma_3^2/2)) < 0$. Hence the assumptions of Theorem 13 are satisfied, and system (9) is nonpersistent (see Figure 6(d)). In addition, since $r_2 > \sigma_2^2$, $x_2(t)$ does not tend to zero in time average by Theorem 9. So we have $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, 3$ a.s. (see Figures 6(a)–6(c)).

(6) The Last Two Species $x_2(t), x_3(t)$ Are Disturbed by the Big White Noises, Which Leads to Nonpersistence. Choosing $\tau = 20.2$, $\sigma_1 = 0.04$, $\sigma_2 = 1.17$, $\sigma_3 = 1.36$, $b_1 = 0.10$, $b_2 = 0.1143$, $b_3 = 0.0929$, then we have $r_1 > \sigma_1^2$, but $b_1(r_1 - (\sigma_1^2/2)) + b_2(r_2 - (\sigma_2^2/2)) + b_3(r_3 - (\sigma_3^2/2)) < 0$. Hence the assumptions of Theorem 13 are satisfied, and system (9) is nonpersistent (see Figure 7(d)). In addition, since $r_1 > \sigma_1^2$, $x_1(t)$ does not tend to zero in time average by Theorem 9. So we have $\lim_{t\to\infty} x_i(t) = 0$, $i = 2, 3$ a.s. (see Figures 7(a)–7(c)).

(7) All the Three Species $x_1(t), x_2(t), x_3(t)$ Are Disturbed by the Big White Noise, Which Leads to the Nonpersistence. Choosing $\tau = 20.2$, $\sigma_1 = 1.25$, $\sigma_1 = 1.01$, $\sigma_1 = 1.36$, $b_1 = 0.10$, $b_2 = 0.1143$, $b_3 = 0.0929$, then we have $b_1(r_1 - (\sigma_1^2/2)) + b_2(r_2 - (\sigma_2^2/2)) + b_3(r_3 - (\sigma_3^2/2)) < 0$. Hence the assumptions of Theorem 13 are satisfied, and system (9) is nonpersistent (see Figure 8(d)). So we have $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, 2, 3$ a.s. (see Figures 8(a)–8(c)).

Appendices

A. Proof of Theorem 5

Proof. Define a C^2 function $V_1 : R_+^3 \to R_+$ by

$$\begin{aligned} V_1(x_1, x_2, x_3) = {} & c_1 \left[x_1(t) - x_1^* - x_1^* \log \frac{x_1(t)}{x_1^*} \right] \\ & + c_2 \left[x_2(t) - x_2^* - x_2^* \log \frac{x_2(t)}{x_2^*} \right] \\ & + c_3 \left[x_3(t) - x_3^* - x_3^* \log \frac{x_3(t)}{x_3^*} \right]. \end{aligned} \tag{A.1}$$

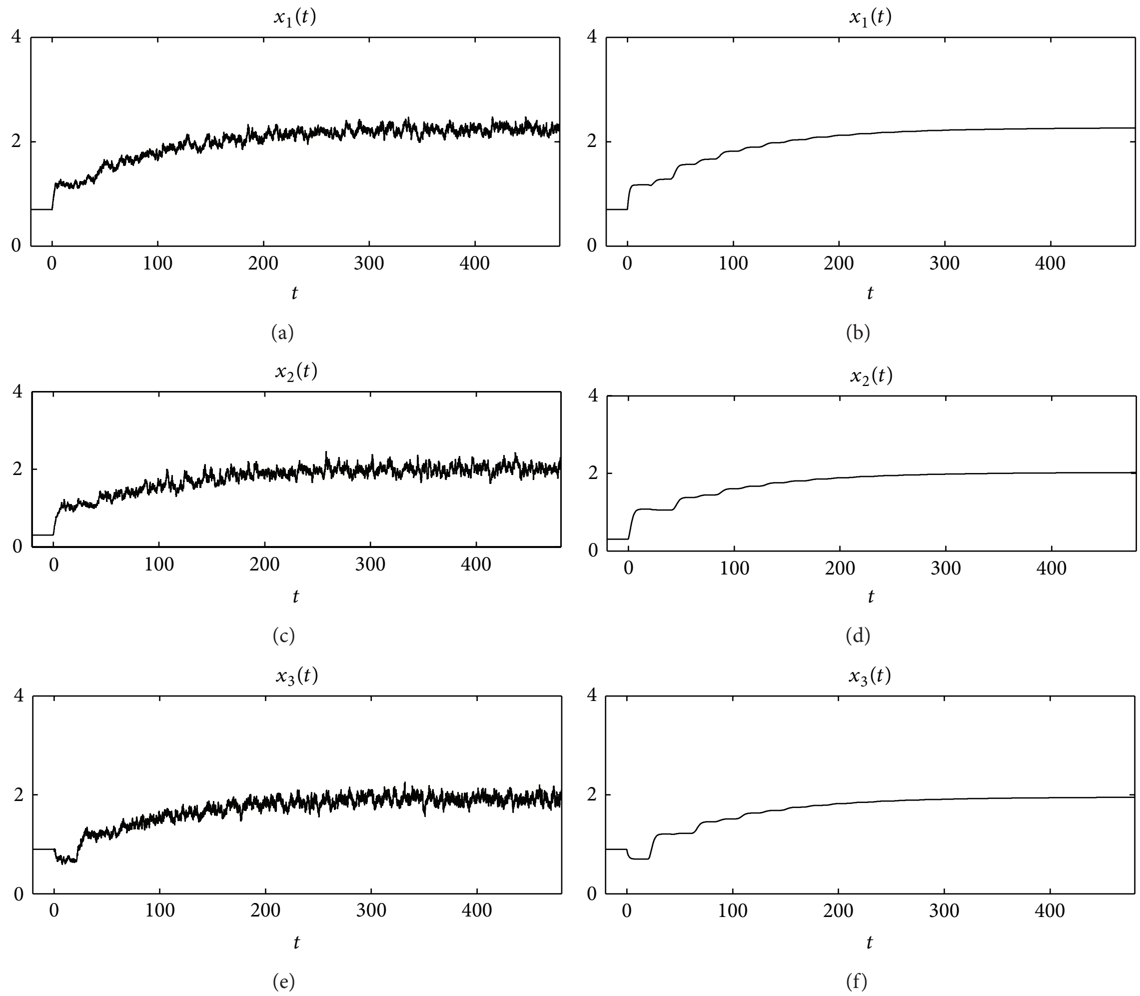

Figure 1: The solution (imaginary line) of system (8) and the solution (real line) of system (9) with $\sigma_1 = 0.04$, $\sigma_2 = 0.08$, $\sigma_3 = 0.07$, $\Delta t = 0.005$.

By Itô's formula, we have

$$\begin{aligned} dV_1(x_1, x_2, x_3) &= c_1\left[1 - \frac{x_1^*}{x_1(t)}\right]dx_1(t) + c_2\left[1 - \frac{x_2^*}{x_2(t)}\right]dx_2(t) \\ &\quad + c_3\left[1 - \frac{x_3^*}{x_3(t)}\right]dx_3(t) + \frac{c_1}{2}x_1^*\sigma_1^2 dt \\ &\quad + \frac{c_2}{2}x_2^*\sigma_2^2 dt + \frac{c_3}{2}x_3^*\sigma_3^2 dt \\ &= c_1\left[x_1(t) - x_1^*\right]\left[(r_1 - a_{11}x_1(t) + a_{12}x_2(t-\tau) \right. \\ &\qquad \left. + a_{13}x_3(t-\tau))\,dt + \sigma_1 dB_1(t)\right] \\ &\quad + c_2\left[x_2(t) - x_2^*\right]\left[(r_2 + a_{21}x_1(t-\tau) - a_{22}x_2(t) \right. \\ &\qquad \left. + a_{23}x_3(t-\tau))\,dt + \sigma_2 dB_2(t)\right] \\ &\quad + c_3\left[x_3(t) - x_3^*\right]\left[(r_3 + a_{31}x_1(t-\tau) + a_{32}x_2(t-\tau) \right. \\ &\qquad \left. - a_{33}x_3(t))\,dt + \sigma_3 dB_3(t)\right] \\ &\quad + \left(\frac{c_1}{2}x_1^*\sigma_1^2 + \frac{c_2}{2}x_2^*\sigma_2^2 + \frac{c_3}{2}x_3^*\sigma_3^2\right)dt, \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} LV_1(x_1, x_2, x_3) &= c_1\left[x_1(t) - x_1^*\right]\left[r_1 - a_{11}x_1(t) \right. \\ &\qquad \left. + a_{12}x_2(t-\tau) + a_{13}x_3(t-\tau)\right] \\ &\quad + c_2\left[x_2(t) - x_2^*\right]\left[r_2 + a_{21}x_1(t-\tau) \right. \\ &\qquad \left. - a_{22}x_2(t) + a_{23}x_3(t-\tau)\right] \\ &\quad + c_3\left[x_3(t) - x_3^*\right]\left[r_3 + a_{31}x_1(t-\tau) \right. \\ &\qquad \left. + a_{32}x_2(t-\tau) - a_{33}x_3(t)\right] \\ &\quad + \frac{c_1x_1^*\sigma_1^2 + c_2x_2^*\sigma_2^2 + c_3x_3^*\sigma_3^2}{2}. \end{aligned} \tag{A.3}$$

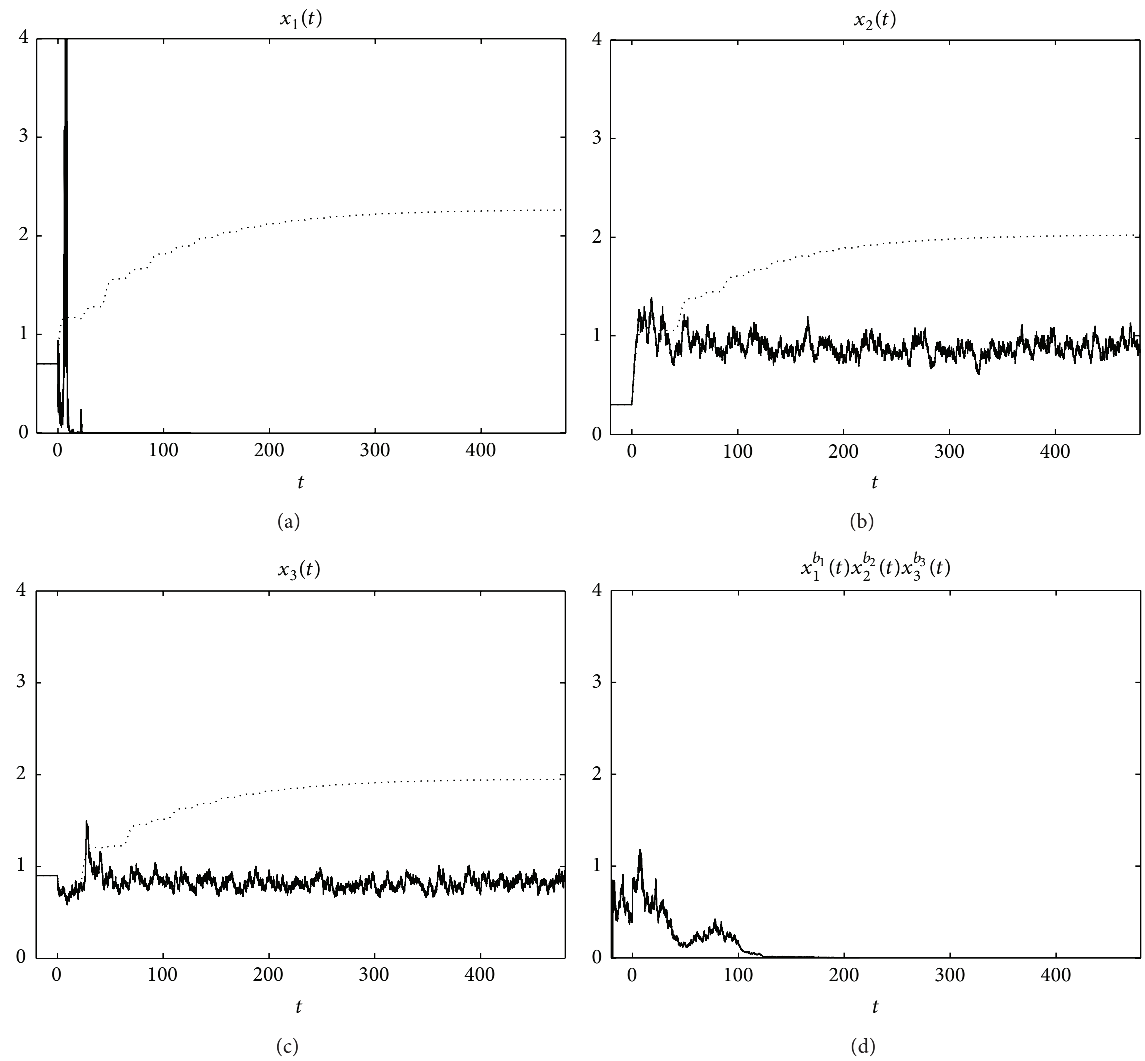

FIGURE 2: The solution (imaginary line) of system (8) and the solution (real line) of system (9) with $\sigma_1 = 1.53$, $\sigma_2 = 0.08$, $\sigma_3 = 0.07$, $\Delta t = 0.005$.

Since $x^* = (x_1^*, x_2^*, x_3^*)$ is the equilibrium point of system (8), we have

$$\begin{aligned}
LV_1(x_1, x_2, x_3) &= c_1[x_1(t) - x_1^*][-a_{11}(x_1(t) - x_1^*) + a_{12}(x_2(t-\tau) - x_2^*) \\
&\quad + a_{13}(x_3(t-\tau) - x_3^*)] \\
&\quad + c_2[x_2(t) - x_2^*][a_{21}(x_1(t-\tau) - x_1^*) - a_{22}(x_2(t) - x_2^*) \\
&\quad + a_{23}(x_3(t-\tau) - x_3^*)] \\
&\quad + c_3[x_3(t) - x_3^*][a_{31}(x_1(t-\tau) - x_1^*) \\
&\quad + a_{32}(x_2(t-\tau) - x_2^*) \\
&\quad - a_{33}(x_3(t) - x_3^*)] \\
&\quad + \frac{c_1x_1^*\sigma_1^2 + c_2x_2^*\sigma_2^2 + c_3x_3^*\sigma_3^2}{2} \\
&= -a_{11}c_1[x_1(t) - x_1^*]^2 + a_{12}c_1[x_1(t) - x_1^*][x_2(t-\tau) - x_2^*] \\
&\quad + a_{13}c_1[x_1(t) - x_1^*][x_3(t-\tau) - x_3^*] \\
&\quad + a_{21}c_2[x_2(t) - x_2^*][x_1(t-\tau) - x_1^*] \\
&\quad - a_{22}c_2[x_2(t) - x_2^*]^2 + a_{23}c_2[x_2(t) - x_2^*][x_3(t-\tau) - x_3^*] \\
&\quad + a_{31}c_3[x_3(t) - x_3^*][x_1(t-\tau) - x_1^*] \\
&\quad + a_{32}c_3[x_3(t) - x_3^*][x_2(t-\tau) - x_2^*] \\
&\quad - a_{33}c_3[x_3(t) - x_3^*]^2 \\
&\quad + \frac{c_1x_1^*\sigma_1^2 + c_2x_2^*\sigma_2^2 + c_3x_3^*\sigma_3^2}{2}.
\end{aligned} \tag{A.4}$$

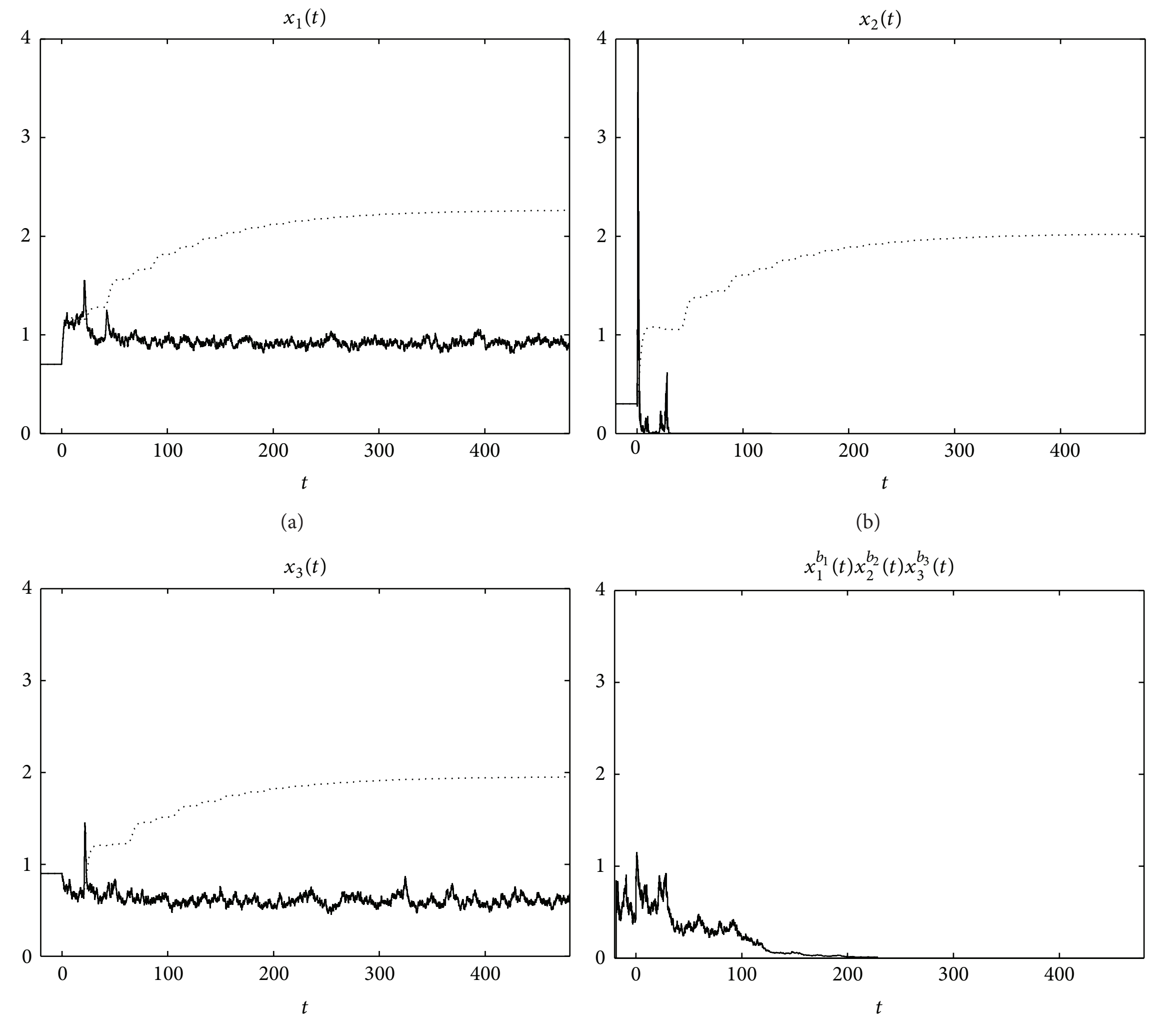

Figure 3: The solution (imaginary line) of system (8) and the solution (real line) of system (9) with $\sigma_1 = 0.04$, $\sigma_2 = 1.17$, $\sigma_3 = 0.07$, $\Delta t = 0.005$.

By Young inequality, we have

$$\begin{aligned}
&a_{12}c_1\left[x_1(t)-x_1^*\right]\left[x_2(t-\tau)-x_2^*\right] \\
&\quad\le \frac{a_{12}c_1}{2}\left\{\left(x_1(t)-x_1^*\right)^2+\left[x_2(t-\tau)-x_2^*\right]^2\right\}, \\
&a_{13}c_1\left[x_1(t)-x_1^*\right]\left[x_3(t-\tau)-x_3^*\right] \\
&\quad\le \frac{a_{13}c_1}{2}\left\{\left(x_1(t)-x_1^*\right)^2+\left[x_3(t-\tau)-x_3^*\right]^2\right\}, \\
&a_{21}c_2\left[x_2(t)-x_2^*\right]\left[x_1(t-\tau)-x_1^*\right] \\
&\quad\le \frac{a_{21}c_2}{2}\left\{\left(x_2(t)-x_2^*\right)^2+\left[x_1(t-\tau)-x_1^*\right]^2\right\}, \\
&a_{23}c_2\left[x_2(t)-x_2^*\right]\left[x_3(t-\tau)-x_3^*\right] \\
&\quad\le \frac{a_{23}c_2}{2}\left\{\left(x_2(t)-x_2^*\right)^2+\left[x_3(t-\tau)-x_3^*\right]^2\right\}, \\
&a_{31}c_3\left[x_3(t)-x_3^*\right]\left[x_1(t-\tau)-x_1^*\right] \\
&\quad\le \frac{a_{23}c_3}{2}\left\{\left(x_3(t)-x_3^*\right)^2+\left[x_1(t-\tau)-x_1^*\right]^2\right\}, \\
&a_{32}c_3\left[x_3(t)-x_3^*\right]\left[x_2(t-\tau)-x_2^*\right] \\
&\quad\le \frac{a_{32}c_3}{2}\left\{\left(x_3(t)-x_3^*\right)^2+\left[x_2(t-\tau)-x_2^*\right]^2\right\}.
\end{aligned} \tag{A.5}$$

Substituting (A.5) into (A.4), we yield

$$\begin{aligned}
&LV_1(x_1,x_2,x_3) \\
&\quad\le \left(-a_{11}c_1+\frac{a_{12}c_1}{2}+\frac{a_{13}c_1}{2}\right)\left[x_1(t)-x_1^*\right]^2 \\
&\qquad+\left(-a_{22}c_2+\frac{a_{21}c_2}{2}+\frac{a_{23}c_2}{2}\right)\left[x_2(t)-x_2^*\right]^2 \\
&\qquad+\left(-a_{33}c_3+\frac{a_{31}c_3}{2}+\frac{a_{32}c_3}{2}\right)\left[x_3(t)-x_3^*\right]^2
\end{aligned}$$

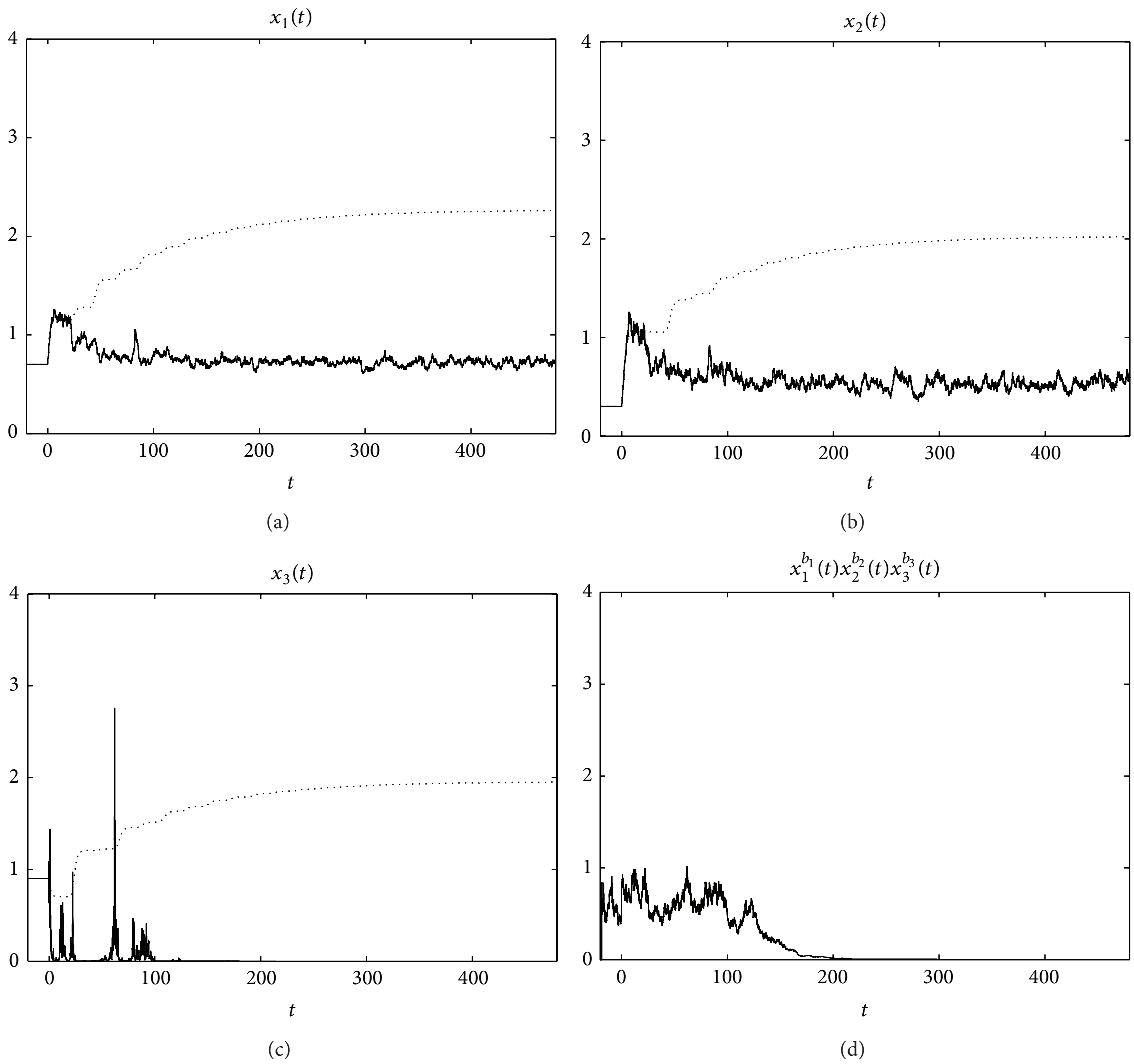

Figure 4: The solution (imaginary line) of system (8) and the solution (real line) of system (9) with $\sigma_1 = 0.04$, $\sigma_2 = 0.08$, $\sigma_3 = 1.36$, $\Delta t = 0.005$.

$$\begin{aligned}
&+\left(\frac{a_{21}c_2}{2}+\frac{a_{31}c_3}{2}\right)\left[x_1(t-\tau)-x_1^*\right]^2 \\
&+\left(\frac{a_{12}c_1}{2}+\frac{a_{32}c_3}{2}\right)\left[x_2(t-\tau)-x_2^*\right]^2 \\
&+\left(\frac{a_{13}c_1}{2}+\frac{a_{23}c_2}{2}\right)\left[x_3(t-\tau)-x_3^*\right]^2 \\
&+\frac{c_1x_1^*\sigma_1^2+c_2x_2^*\sigma_2^2+c_3x_3^*\sigma_3^2}{2}.
\end{aligned} \tag{A.6}$$

From (A.6), we have

$$\begin{aligned}
&dV_1(x_1,x_2,x_3) \\
&\quad\le\left\{\left(-a_{11}c_1+\frac{a_{12}c_1}{2}+\frac{a_{13}c_1}{2}\right)\left[x_1(t)-x_1^*\right]^2\right. \\
&\qquad+\left(-a_{22}c_2+\frac{a_{21}c_2}{2}+\frac{a_{23}c_2}{2}\right)\left[x_2(t)-x_2^*\right]^2 \\
&\qquad+\left(-a_{33}c_3+\frac{a_{31}c_3}{2}+\frac{a_{32}c_3}{2}\right)\left[x_3(t)-x_3^*\right]^2 \\
&\qquad+\left(\frac{a_{21}c_2}{2}+\frac{a_{31}c_3}{2}\right)\left[x_1(t-\tau)-x_1^*\right]^2 \\
&\qquad+\left(\frac{a_{12}c_1}{2}+\frac{a_{32}c_3}{2}\right)\left[x_2(t-\tau)-x_2^*\right]^2 \\
&\qquad+\left(\frac{a_{13}c_1}{2}+\frac{a_{23}c_2}{2}\right)\left[x_3(t-\tau)-x_3^*\right]^2 \\
&\qquad\left.+\frac{c_1x_1^*\sigma_1^2+c_2x_2^*\sigma_2^2+c_3x_3^*\sigma_3^2}{2}\right\}dt+\sum_{i=1}^{3}\sigma_i dB_i.
\end{aligned} \tag{A.7}$$

Define

$$\begin{aligned}
&V_2(x_1,x_2,x_3) \\
&\quad=\int_t^{t+\tau}\left\{\left(\frac{a_{12}c_1}{2}+\frac{a_{32}c_3}{2}\right)\left[x_2(s-\tau)-x_2^*\right]^2\right.
\end{aligned}$$

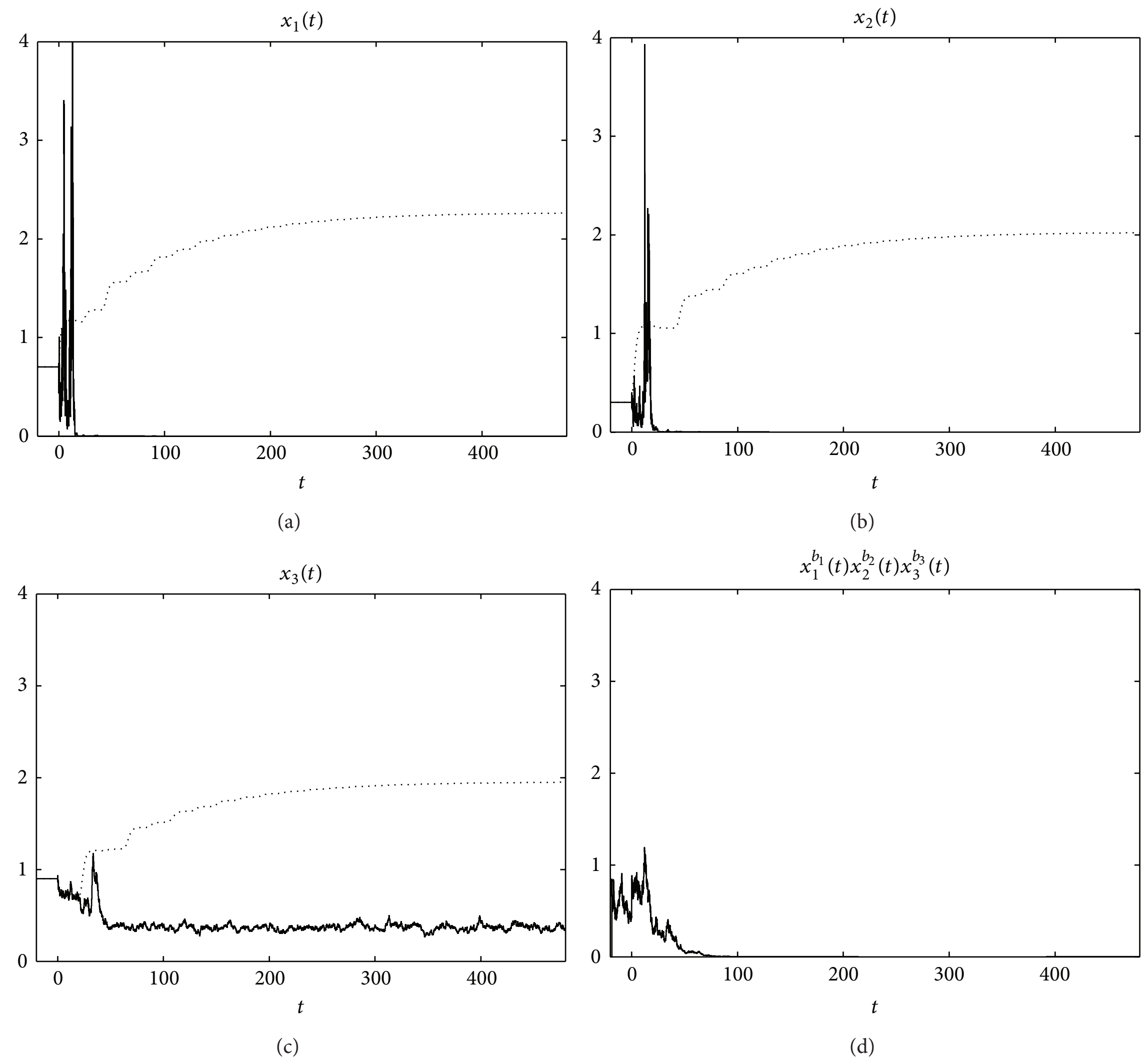

Figure 5: The solution (imaginary line) of system (8) and the solution (real line) of system (9) with $\sigma_1 = 1.25$, $\sigma_2 = 1.01$, $\sigma_3 = 0.07$, $\Delta t = 0.005$.

$$+\left(\frac{a_{13}c_1}{2}+\frac{a_{23}c_2}{2}\right)\left[x_3(s-\tau)-x_3^*\right]^2$$
$$+\left(\frac{a_{21}c_2}{2}+\frac{a_{31}c_3}{2}\right)\left[x_1(s-\tau)-x_1^*\right]^2\Bigg\}\,ds. \tag{A.8}$$

Then by Itô's formula, we have

$$\begin{aligned}
dV_2&(x_1,x_2,x_3)\\
&=\Bigg\{\left(\frac{a_{12}c_1}{2}+\frac{a_{32}c_3}{2}\right)\left[x_2(t)-x_2^*\right]^2\\
&\quad+\left(\frac{a_{13}c_1}{2}+\frac{a_{23}c_2}{2}\right)\left[x_3(t)-x_3^*\right]^2\\
&\quad+\left(\frac{a_{21}c_2}{2}+\frac{a_{31}c_3}{2}\right)+\left[x_1(t)-x_1^*\right]^2\\
&\quad-\left(\frac{a_{12}c_1}{2}+\frac{a_{32}c_3}{2}\right)\left[x_2(t-\tau)-x_2^*\right]^2\\
&\quad-\left(\frac{a_{13}c_1}{2}+\frac{a_{23}c_2}{2}\right)\left[x_3(t-\tau)-x_3^*\right]^2\\
&\quad-\left(\frac{a_{21}c_2}{2}+\frac{a_{31}c_3}{2}\right)\left[x_1(t-\tau)-x_1^*\right]^2\Bigg\}\,dt.
\end{aligned} \tag{A.9}$$

Define

$$V(x_1,x_2,x_3)=V_1(x_1,x_2,x_3)+V_2(x_1,x_2,x_3). \tag{A.10}$$

Together with (A.7) and (A.9), it implies

$$\begin{aligned}
dV&(x_1,x_2,x_3)\\
&\le\Bigg\{\frac{(a_{12}+a_{13}-a_{11})c_1-1}{2}\left[x_1(t)-x_1^*\right]^2\\
&\quad+\frac{(a_{21}+a_{23}-a_{22})c_2-1}{2}\left[x_2(t)-x_2^*\right]^2
\end{aligned}$$

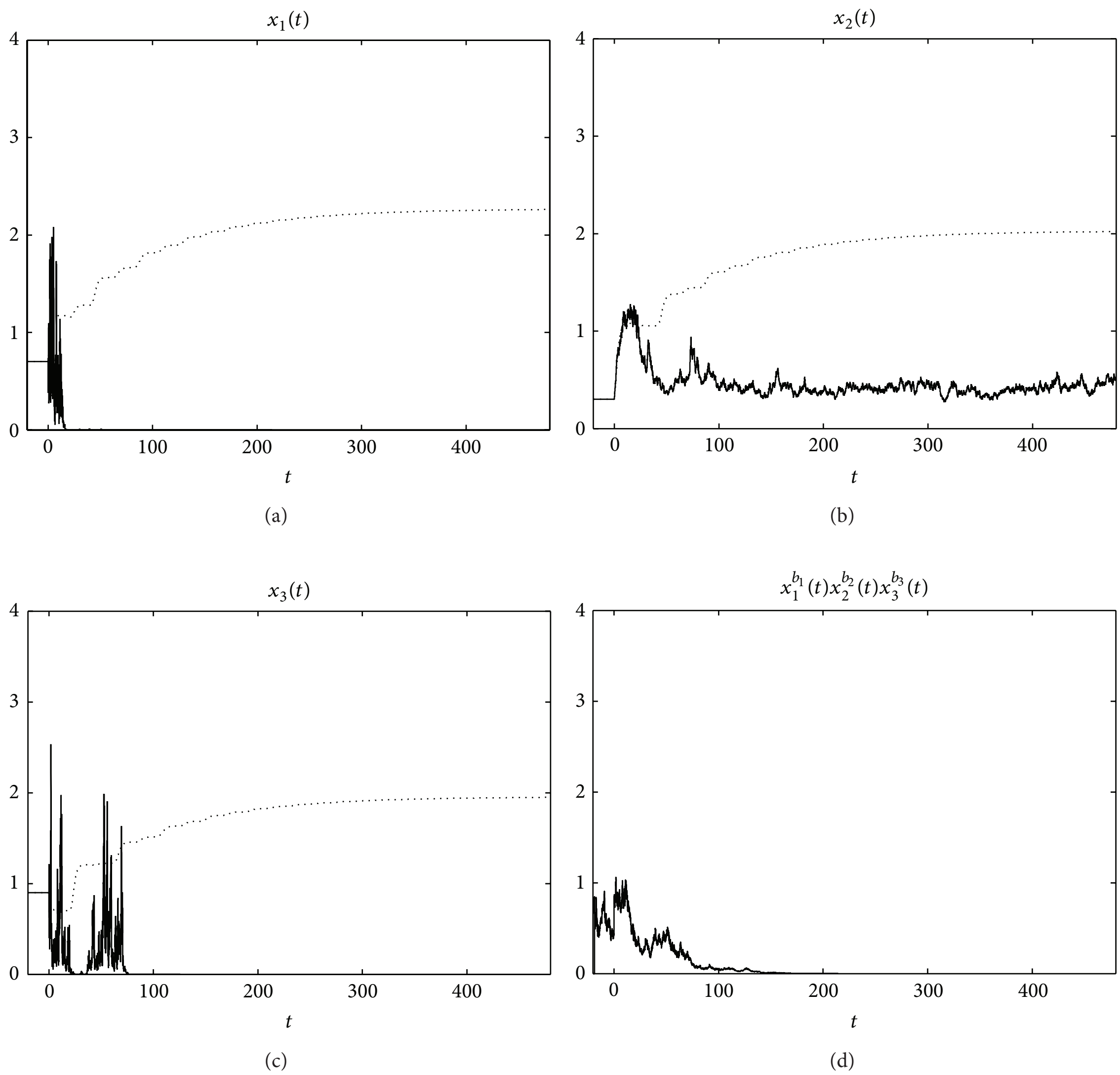

Figure 6: The solution (imaginary line) of system (8) and the solution (real line) of system (9) with $\sigma_1 = 1.25$, $\sigma_2 = 0.08$, $\sigma_3 = 1.12$, $\Delta t = 0.005$.

$$+\frac{(a_{31}+a_{32}-a_{33})c_3-1}{2}[x_3(t)-x_3^*]^2 + \frac{c_1x_1^*\sigma_1^2+c_2x_2^*\sigma_2^2+c_3x_3^*\sigma_3^2}{2}\Bigg\}dt+\sum_{i=1}^{3}\sigma_i dB_i. \tag{A.11}$$

Integrating from 0 to t, taking the expectation, we have

$$\begin{aligned} E[V(t)]-V(0) &\le -E\int_0^t\Bigg\{\frac{(a_{11}-a_{12}-a_{13})c_1+1}{2}[x_1(s)-x_1^*]^2 \\ &+\frac{(a_{22}-a_{21}-a_{23})c_2+1}{2}[x_2(s)-x_2^*]^2 \\ &+\frac{(a_{33}-a_{31}-a_{32})c_3+1}{2}[x_3(s)-x_3^*]^2\Bigg\}ds \\ &+\frac{c_1x_1^*\sigma_1^2+c_2x_2^*\sigma_2^2+c_3x_3^*\sigma_3^2}{2}t. \end{aligned} \tag{A.12}$$

Then we yield

$$\frac{E[V(t)]}{t} \le \frac{V(0)}{t} - \frac{(a_{11}-a_{12}-a_{13})c_1+1}{2}E\frac{1}{t}\int_0^t[x_1(s)-x_1^*]^2ds$$

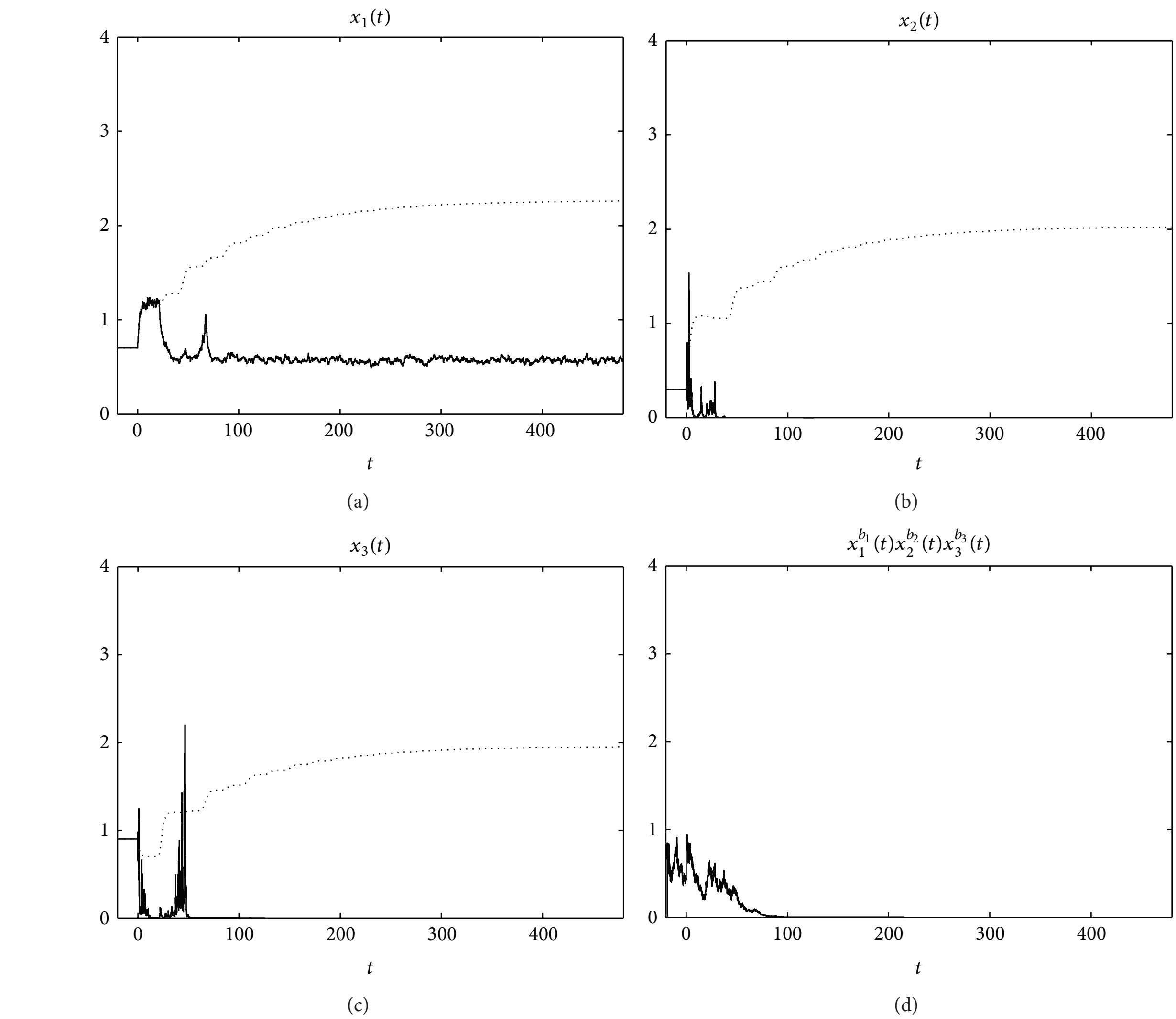

FIGURE 7: The solution (imaginary line) of system (8) and the solution (real line) of system (9) with $\sigma_1 = 0.04$, $\sigma_2 = 1.17$, $\sigma_3 = 1.36$, $\Delta t = 0.005$.

$$
\begin{aligned}
&- \frac{(a_{22} - a_{21} - a_{23}) c_2 + 1}{2} E \frac{1}{t} \int_0^t [x_2(s) - x_2^*]^2 ds \\
&- \frac{(a_{33} - a_{31} - a_{32}) c_3 + 1}{2} E \frac{1}{t} \int_0^t [x_3(s) - x_3^*]^2 ds \\
&+ \frac{c_1 x_1^* \sigma_1^2 + c_2 x_2^* \sigma_2^2 + c_3 x_3^* \sigma_3^2}{2}.
\end{aligned}
\tag{A.13}
$$

Letting $t \to \infty$, therefore we have

$$
\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} E \int_0^t \Bigg\{ & \frac{(a_{11} - a_{12} - a_{13}) c_1 + 1}{2} [x_1(s) - x_1^*]^2 \\
& + \frac{(a_{22} - a_{21} - a_{23}) c_2 + 1}{2} [x_2(s) - x_2^*]^2 \\
& + \frac{(a_{33} - a_{31} - a_{32}) c_3 + 1}{2} [x_3(s) - x_3^*]^2 \Bigg\} ds \\
\le \frac{\sum_{i=1}^3 c_i x_i^* \sigma_i^2}{2}&,
\end{aligned}
\tag{A.14}
$$

which is the required assertion. The proof is completed. □

B. Proof of Theorem 11

Proof. It is sufficient to prove

$$
\begin{aligned}
\tilde{x}_i^* &\le \liminf_{t\to\infty} \frac{1}{t} \int_0^t x_i(s)\, ds \\
&\le \limsup_{t\to\infty} \frac{1}{t} \int_0^t x_i(s)\, ds \le \tilde{x}_i^* \quad \text{a.s. } i = 1, 2, 3.
\end{aligned}
\tag{B.1}
$$

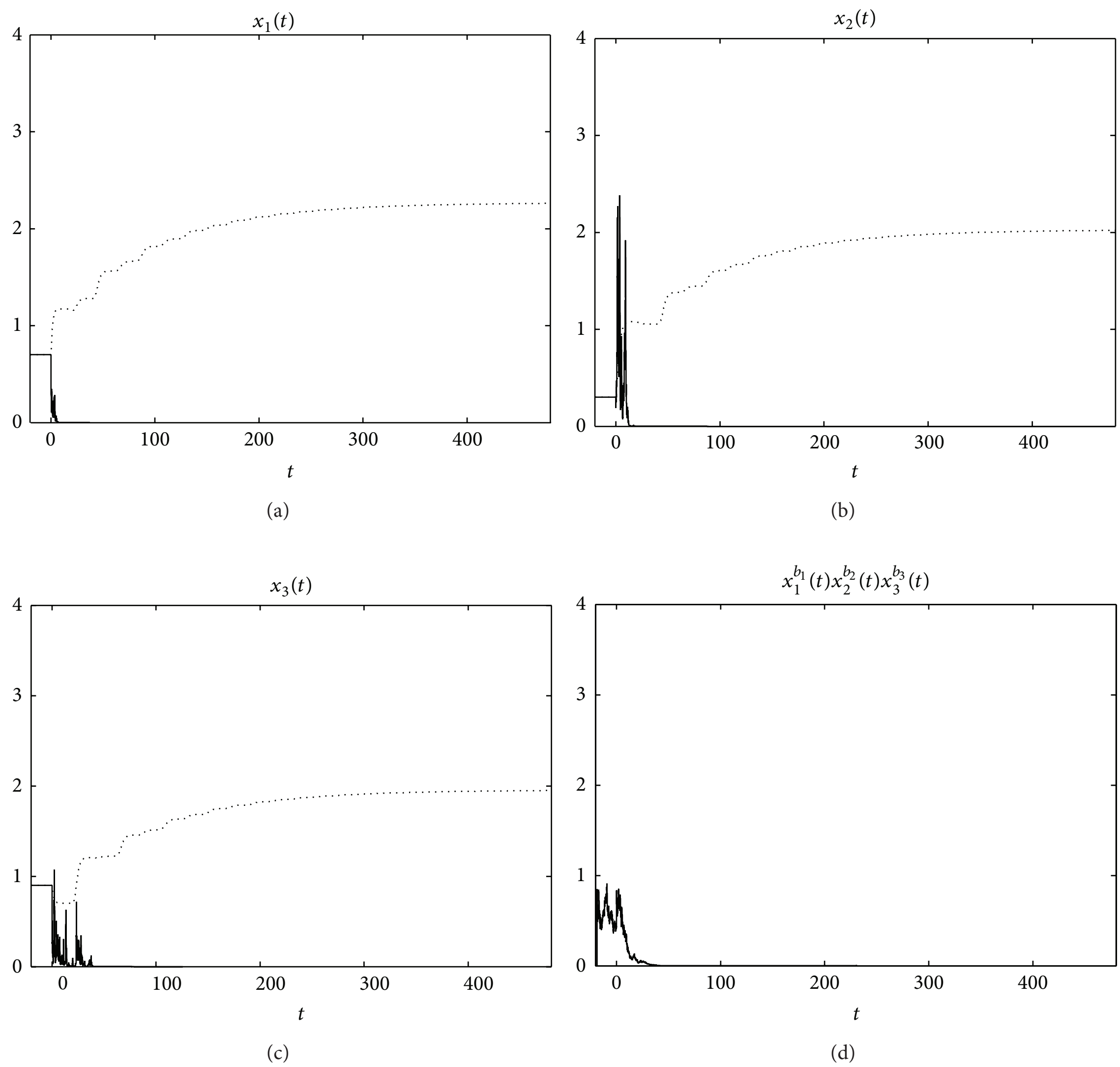

Figure 8: The solution (imaginary line) of system (8) and the solution (real line) of system (9) with $\sigma_1 = 1.25$, $\sigma_2 = 1.01$, $\sigma_3 = 1.36$, $\Delta t = 0.005$.

By Itô's formula, we have

$$\begin{aligned} &d\log x_1(t) \\ &\quad= \left(r_1 - \frac{\sigma_1^2}{2} - a_{11}x_1(t) + a_{12}x_2(t-\tau) + a_{13}x_3(t-\tau)\right)dt \\ &\qquad + \sigma_1 dB_1(t), \\ &d\log x_2(t) \\ &\quad= \left(r_2 - \frac{\sigma_2^2}{2} + a_{21}x_1(t-\tau) - a_{22}x_2(t) + a_{23}x_3(t-\tau)\right)dt \\ &\qquad + \sigma_2 dB_2(t), \\ &d\log x_3(t) \\ &\quad= \left(r_3 - \frac{\sigma_3^2}{2} + a_{31}x_1(t-\tau) + a_{32}x_2(t-\tau) - a_{33}x_3(t)\right)dt \\ &\qquad + \sigma_3 dB_3(t). \end{aligned} \tag{B.2}$$

Integrating both sides of (B.2) from 0 to t, then we have

$$\begin{aligned} &\log x_1(t) \\ &\quad= \log x_1(0) + \bar{r}_1 t + a_{12}\int_{-\tau}^{0}\xi_2(s)\,ds + a_{13}\int_{-\tau}^{0}\xi_3(s)\,ds \\ &\qquad - a_{11}\int_0^t x_1(s)\,ds + a_{12}\int_0^{t-\tau} x_2(s)\,ds \\ &\qquad + a_{13}\int_0^{t-\tau} x_3(s)\,ds + \sigma_1 B_1(t), \end{aligned}$$

$$\begin{aligned}
&\log x_2(t) \\
&= \log x_2(0) + \bar{r}_2 t + a_{21}\int_{-\tau}^{0}\xi_1(s)\,ds + a_{23}\int_{-\tau}^{0}\xi_3(s)\,ds \\
&\quad + a_{21}\int_{0}^{t-\tau}x_1(s)\,ds - a_{22}\int_{0}^{t}x_2(s)\,ds \\
&\quad + a_{23}\int_{0}^{t-\tau}x_3(s)\,ds + \sigma_2 B_2(t), \\
&\log x_3(t) \\
&= \log x_3(0) + \bar{r}_3 t + a_{31}\int_{-\tau}^{0}\xi_1(s)\,ds + a_{32}\int_{-\tau}^{0}\xi_2(s)\,ds \\
&\quad + a_{31}\int_{0}^{t-\tau}x_1(s)\,ds + a_{32}\int_{0}^{t-\tau}x_2(s)\,ds \\
&\quad - a_{33}\int_{0}^{t}x_3(s)\,ds + \sigma_3 B_3(t),
\end{aligned} \tag{B.3}$$

where $\bar{r}_i = r_i - (\sigma_i^2/2)$, $i = 1, 2, 3$. From (33), we know

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds \geq \frac{\bar{r}_1}{a_{11}} := M_1, \\
&\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds \geq \frac{\bar{r}_2}{a_{22}} := N_1, \\
&\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_3(s)\,ds \geq \frac{\bar{r}_3}{a_{33}} := L_1.
\end{aligned} \tag{B.4}$$

Hence, for any sufficiently small $0 < \varepsilon_1 < (1/2)\min\{M_1, N_1, L_1\}$, there is a $T_1(\omega) > 0$ such that if $t > T_1(\omega)$, $(t-\tau)/t \geq 1 - \varepsilon_1$, $(1/t)\int_0^t x_1(s)ds \geq M_1 - \varepsilon_1$, $(1/t)\int_0^t x_2(s)ds \geq N_1 - \varepsilon_1$, $(1/t)\int_0^t x_3(s)ds \geq L_1 - \varepsilon_1$. It follows from (B.3) that for $t > T_1(\omega) + \tau$,

$$\begin{aligned}
\log x_1(t) &= \log x_1(0) + \bar{r}_1 t + a_{12}\int_{-\tau}^{0}x_2(s)\,ds \\
&\quad + a_{13}\int_{-\tau}^{0}x_3(s)\,ds - a_{11}\int_0^t x_1(s)\,ds \\
&\quad + a_{12}\int_0^{t-\tau}x_2(s)\,ds + a_{13}\int_0^{t-\tau}x_3(s)\,ds + \sigma_1 B_1(t) \\
&= \log x_1(0) + \bar{r}_1 t + a_{12}\int_{-\tau}^{0}x_2(s)\,ds \\
&\quad + a_{13}\int_{-\tau}^{0}x_3(s)\,ds - a_{11}\int_0^t x_1(s)\,ds \\
&\quad + a_{12}(t-\tau)\frac{1}{t-\tau}\int_0^{t-\tau}x_2(s)\,ds \\
&\quad + a_{13}(t-\tau)\frac{1}{t-\tau}\int_0^{t-\tau}x_3(s)\,ds + \sigma_1 B_1(t) \\
&\geq \log x_1(0) + \bar{r}_1 t + a_{12}\int_{-\tau}^{0}x_2(s)\,ds \\
&\quad + a_{13}\int_{-\tau}^{0}x_3(s)\,ds - a_{11}\int_0^t x_1(s)\,ds + a_{12}(t-\tau) \\
&\quad \times(N_1-\varepsilon_1) + a_{13}(t-\tau)(L_1-\varepsilon_1) + \sigma_1 B_1(t) \\
&= [\bar{r}_1 + a_{12}(N_1-\varepsilon_1) + a_{13}(L_1-\varepsilon_1)]\,t \\
&\quad - a_{11}\int_0^t x_1(s)\,ds + \sigma_1 B_1(t) + S_1,
\end{aligned} \tag{B.5}$$

where

$$\begin{aligned}
S_1 = {} & \log x_1(0) - [a_{12}(N_1-\varepsilon_1) + a_{13}(L_1-\varepsilon_1)]\,\tau \\
& + a_{12}\int_{-\tau}^{0}x_2(s)\,ds + a_{13}\int_{-\tau}^{0}x_3(s)\,ds.
\end{aligned} \tag{B.6}$$

We know

$$\lim_{t\to\infty}\frac{\sigma_1 B_1(t) + S_1}{t} = 0. \tag{B.7}$$

From Lemma 10, we have

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds \\
&\quad \geq \frac{\bar{r}_1 + a_{12}(N_1-\varepsilon_1) + a_{13}(L_1-\varepsilon_1)}{a_{11}} := M_2.
\end{aligned} \tag{B.8}$$

Similarly,

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds \\
&\quad \geq \frac{\bar{r}_2 + a_{21}(M_1-\varepsilon_1) + a_{23}(L_1-\varepsilon_1)}{a_{22}} := N_2, \\
&\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_3(s)\,ds \\
&\quad \geq \frac{\bar{r}_3 + a_{31}(M_1-\varepsilon_1) + a_{32}(N_1-\varepsilon_1)}{a_{33}} := L_2.
\end{aligned} \tag{B.9}$$

Let $\varepsilon_2 = (1/2)\min\{M_2, N_2, L_2, \varepsilon_1\}$, continuing this process, we obtain sequences $\{M_n\}$, $\{N_n\}$, $\{L_n\}$, $\{\varepsilon_n\}$, where

$$\begin{aligned}
M_n &= \frac{\bar{r}_1 + a_{12}(N_{n-1}-\varepsilon_{n-1}) + a_{13}(L_{n-1}-\varepsilon_{n-1})}{a_{11}}, \\
N_n &= \frac{\bar{r}_2 + a_{21}(M_{n-1}-\varepsilon_{n-1}) + a_{23}(L_{n-1}-\varepsilon_{n-1})}{a_{22}}, \\
L_n &= \frac{\bar{r}_3 + a_{31}(M_{n-1}-\varepsilon_{n-1}) + a_{32}(N_{n-1}-\varepsilon_{n-1})}{a_{33}}, \\
\varepsilon_n &= \frac{1}{2}\min\{M_n, N_n, L_n, \varepsilon_{n-1}\}.
\end{aligned} \tag{B.10}$$

Sequence $\{M_n\}, \{N_n\}, \{L_n\}$ is nondecreasing and bounded; then we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)\,ds \ge M_n,$$
$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)\,ds \ge N_n, \tag{B.11}$$
$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_3(s)\,ds \ge L_n.$$

Let $n \to \infty$; we derive from (B.10) that

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad \lim_{n\to\infty} M_n = \tilde{x}_1^*,$$
$$\lim_{n\to\infty} N_n = \tilde{x}_2^*, \qquad \lim_{n\to\infty} L_n = \tilde{x}_3^*, \tag{B.12}$$

where $\tilde{x}_1^*, \tilde{x}_2^*, \tilde{x}_3^*$ is the unique solution of the following equation:

$$\bar{r}_1 - a_{11}\tilde{x}_1^* + a_{12}\tilde{x}_2^* + a_{13}\tilde{x}_3^* = 0,$$
$$\bar{r}_2 + a_{21}\tilde{x}_1^* - a_{22}\tilde{x}_2^* + a_{23}\tilde{x}_3^* = 0, \tag{B.13}$$
$$\bar{r}_3 + a_{31}\tilde{x}_1^* + a_{32}\tilde{x}_2^* - a_{33}\tilde{x}_3^* = 0.$$

Thus we obtain the assertion

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \ge \tilde{x}_i^* \quad i = 1,2,3 \text{ a.s.} \tag{B.14}$$

Next, we will prove

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \le \tilde{x}_i^* \quad i = 1,2,3 \text{ a.s.} \tag{B.15}$$

It follows from (B.2) that

$$d\left[\log x_1(t) + a_{12}\int_t^{t+\tau} x_2(s-\tau)\,ds + a_{13}\int_t^{t+\tau} x_3(s-\tau)\,ds\right]$$
$$= \left[\bar{r}_1 - a_{11}x_1(t) + a_{12}x_2(t) + a_{13}x_3(t)\right]dt + \sigma_1 dB_1(t). \tag{B.16}$$

Integrating both sides of (B.16) from 0 to t, we have

$$\log x_1(t) + a_{12}\int_t^{t+\tau} x_2(s-\tau)\,ds - a_{12}\int_0^{\tau} x_2(s-\tau)\,ds$$
$$+ a_{13}\int_t^{t+\tau} x_3(s-\tau)\,ds - a_{13}\int_0^{\tau} x_3(s-\tau)\,ds$$
$$= \log x_1(0) + \bar{r}_1 t - a_{11}\int_0^t x_1(s)\,ds + a_{12}\int_0^t x_2(s)\,ds$$
$$+ a_{13}\int_0^t x_3(s)\,ds + \sigma_1 B_1(t). \tag{B.17}$$

Since $x_i(t) > 0$, $i = 1,2,3$, we have

$$\log x_1(t) \le U_1 + \bar{r}_1 t - a_{11}\int_0^t x_1(s)\,ds$$
$$+ a_{12}\int_0^t x_2(s)\,ds + a_{13}\int_0^t x_3(s)\,ds + \sigma_1 B_1(t), \tag{B.18}$$

where

$$U_1 = a_{12}\int_0^{\tau} x_2(s-\tau)\,ds + a_{13}\int_0^{\tau} x_3(s-\tau)\,ds + \log x_1(0). \tag{B.19}$$

Similarly,

$$\log x_2(t) \le U_2 + \bar{r}_2 t - a_{22}\int_0^t x_2(s)\,ds$$
$$+ a_{21}\int_0^t x_1(s)\,ds + a_{23}\int_0^t x_3(s)\,ds + \sigma_2 B_2(t), \tag{B.20}$$

where

$$U_2 = a_{21}\int_0^{\tau} x_1(s-\tau)\,ds + a_{23}\int_0^{\tau} x_3(s-\tau)\,ds + \log x_2(0). \tag{B.21}$$

$$\log x_3(t) \le U_3 + \bar{r}_3 t - a_{33}\int_0^t x_3(s)\,ds + a_{31}\int_0^t x_1(s)\,ds$$
$$+ a_{32}\int_0^t x_2(s)\,ds + \sigma_3 B_3(t), \tag{B.22}$$

where

$$U_3 = a_{31}\int_0^{\tau} x_1(s-\tau)\,ds + a_{32}\int_0^{\tau} x_2(s-\tau)\,ds + \log x_3(0). \tag{B.23}$$

Let b_{11} be a positive constant

$$b_{12} = \frac{-a_{12}a_{33} - a_{13}a_{32}}{a_{23}a_{32} - a_{33}a_{22}} b_{11},$$
$$b_{13} = \frac{-a_{23}a_{12} - a_{13}a_{22}}{a_{23}a_{32} - a_{33}a_{22}} b_{11},$$
$$m_1$$
$$= a_{11}b_{11} - a_{21}\frac{-a_{12}a_{33} - a_{13}a_{32}}{a_{23}a_{32} - a_{33}a_{22}} b_{11} - a_{31}\frac{-a_{23}a_{12} - a_{13}a_{22}}{a_{23}a_{32} - a_{33}a_{22}} b_{11}$$
$$= \Big(\big(-a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{22}a_{13}a_{31} + a_{33}a_{12}a_{21}$$
$$+ a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}\big)\left(a_{23}a_{32} - a_{33}a_{22}\right)^{-1}\Big) b_{11}. \tag{B.24}$$

From Lemma 3, we know $b_{12} > 0$, $b_{13} > 0$, $m_1 > 0$. Since $x_i(t) > 0$, from (B.18)–(B.22) we have

$$b_{11}\log x_1(t) + b_{12}\log x_2(t) + b_{13}\log x_3(t)$$
$$= \left[b_{11}\bar{r}_1 + b_{12}\bar{r}_2 + b_{13}\bar{r}_3\right] t - m_1\int_0^t x_1(s)\,ds \tag{B.25}$$
$$+ \sum_{i=1}^{3} b_{1i}\sigma_i B_i(t) + \sum_{i=1}^{3} b_{1i}U_i.$$

From (33), we know

$$\liminf_{t\to\infty} \frac{\log x_i(t)}{t} \geq 0 \quad i = 1, 2, 3, \tag{B.26}$$

which implies

$$\lim_{t\to\infty} \frac{b_{11}\log x_1(t) + b_{12}\log x_2(t) + b_{13}\log x_3(t)}{t} \geq 0. \tag{B.27}$$

From

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{3} b_{1i}\sigma_i B_i(t) + \sum_{i=1}^{3} b_{1i}U_i}{t} = 0, \tag{B.28}$$

we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)\,ds \leq \frac{b_{11}\bar{r}_1 + b_{12}\bar{r}_2 + b_{13}\bar{r}_3}{m_1} = \tilde{x}_1^*. \tag{B.29}$$

Similarly,

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)\,ds &\leq \tilde{x}_2^*, \\ \limsup_{t\to\infty} \frac{1}{t}\int_0^t x_3(s)\,ds &\leq \tilde{x}_3^*. \end{aligned} \tag{B.30}$$

Therefore, $\limsup_{t\to\infty}(1/t)\int_0^t x(s)ds \leq \tilde{x}^*$. The proof is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The work was supported by the NSFC of China (no. 11371085) and the Ph.D. Programs Foundation of Ministry of China (no. 200918).

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