Research Article

Bifurcation Analysis in a Two-Dimensional Neutral Differential Equation

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The dynamics of a 2-dimensional neural network model in neutral form are investigated. We prove that a sequence of Hopf bifurcations occurs at the origin as the delay increases. The direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions are determined by using normal form method and center manifold theory. Global existence of periodic solutions is established using a global Hopf bifurcation result of Krawcewicz et al. Finally, some numerical simulations are carried out to support the analytic results.

1. Introduction

The fundamental theories and applications of neutral functional differential equations have been largely developed and there are a great deal of important conclusions on the stability, bifurcation theory, and numerical solutions of neutral delay differential equations during the last few decades. It is well known that Wei and Ruan [1] have established a basic theory to analyze the distribution of the zeros of general transcendental equations and applied it to the neutral functional differential equations. One can use the theory to study the existence of Hopf bifurcation to a neutral delay differential equation. In [2], Wang and Wei extend the computation of the properties of Hopf bifurcation, such as the direction of bifurcation and stability of bifurcating periodic solutions, of DDE introduced by Kazarinoff et al. Based on combining the global Hopf bifurcation theory of neutral equations due to Krawcewicz et al. and the higher dimensional Bendixson's criterion for ordinary differential equations due to Li and Muldowney, the global existence of periodic solutions of neutral differential equations have been studied by Qu et al. [3]. But research findings of bifurcation about high-dimension neutral differential equations can rarely be found.

In order to determine the dynamics of artificial neural network, the two-dimensional delay neural network without self-feedback is studied, and some significant results are

obtained (see [4–8]). In this paper, we consider the following system:

$$
\begin{aligned}
\dot{x}_1(t) &= -\mu x_1(t) + af(x_1(t-\tau)) + b\dot{x}_2(t-\tau), \\
\dot{x}_2(t) &= -\mu x_2(t) + af(x_2(t-\tau)) + b\dot{x}_1(t-\tau),
\end{aligned}
\tag{1}
$$

where μ, a, and b are real numbers with $\mu > 0$ and $|b| < 1$. f is assumed adequately smooth, for example, $f \in C^3$, and satisfies the following condition:

(H1) $f(0) = 0$, $f'(0) = 1$, and there exists $L > 0$, such that $|f(x)| \leqslant L$ for all $x \in \mathbb{R}$,

(H2) $f'(x) > 0$ for all $x \in \mathbb{R}$, $xf''(x) < 0$ for all $x \neq 0$.

In the present paper, we provide a detailed analysis of this equation. By applying the local Hopf bifurcation theory (see [9] and also [1, 2, 10, 11]), we investigate the existence of stable periodic oscillations for (1). More specially, we prove that the equilibrium $(0,0)$ loses its stability as τ increases, and a sequence of Hopf bifurcations occurs at the origin. Whereafter, based on the normal form and center manifold theory due to [12, 13], by using the method introduced in Wang and Wei [2], we derive a sufficient condition for determining the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions. Furthermore,

the existence of periodic solutions for τ far away from the Hopf bifurcation values is also established, by using a global Hopf bifurcation result due to Krawcewicz et al. [14] (also see [15, 16]).

The rest of this paper is organized as follows: in Section 2, taking a and τ as parameters, we give the analysis of stability and bifurcations at equilibria. Section 3 is devoted to establishing the direction and stability of Hopf bifurcation. Finally, a global Hopf bifurcation is established, and some numerical simulations are carried out to illustrate the analytic results in Section 4.

2. Stability and Bifurcation Analysis

In this section, we will investigate the stability and bifurcations of the equilibria by taking a and τ as bifurcation parameters.

Under the given hypothesis (H1), it is easy to check that $(0, 0)$ is an equilibrium point of system (1). At this trivial equilibrium, the linearization of system (1) is

$$\dot{X}(t) = AX(t) + BX(t - \tau) + C\dot{X}(t - \tau), \quad (2)$$

where

$$A = \begin{pmatrix} -\mu & 0 \\ 0 & -\mu \end{pmatrix}, \qquad B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}. \quad (3)$$

And the corresponding characteristic equation is

$$\Delta(\lambda) = \left(\lambda + \mu - ae^{-\lambda\tau}\right)^2 - b^2\lambda^2 e^{-2\lambda\tau} = 0, \quad (4)$$

that is,

$$\left(\lambda + \mu - (a - b\lambda)e^{-\lambda\tau}\right)\left(\lambda + \mu - (a + b\lambda)e^{-\lambda\tau}\right)$$
$$\overset{\text{def}}{=} \Delta_1(\lambda) \cdot \Delta_2(\lambda) = 0. \quad (5)$$

Case 1. Choose a as parameter.

For convenience, we give two claims at first.

Claim 1. All the roots of (4) with $a = 0$ have negative real parts for any $\tau \geqslant 0$.

Proof. When $a = 0$ and $\tau = 0$, (4) becomes $(\lambda + \mu)^2 - b^2\lambda^2 = 0$, and the roots are $\lambda_{1,2} = -\mu/(1 \pm b) < 0$.

iv_0 ($v_0 > 0$) is a zero of $\Delta(\lambda)$ with $a = 0$ if and only if v_0 solves

$$(iv_0 + \mu)^2 + b^2 v_0^2 (\cos 2v_0\tau - i\sin 2v_0\tau) = 0. \quad (6)$$

Separating the real and imaginary parts gives that $(b^2 - 1)v_0^2 = \mu^2$, which contradicts with $|b| < 1$. Then the conclusion follows Wei and Ruan [1], and the proof is complete. □

Claim 2. Equation (4) has at least two positive roots when $a > \mu$.

In fact, the conclusion follows from $\Delta_{1,2}(0) = \mu - a < 0$ and $\lim_{\lambda \to +\infty}\Delta_{1,2}(\lambda) = +\infty$.

When $b \neq 0$, iv ($v > 0$) is a root of (4) if and only if v satisfies

$$a\sin v\tau = -v(1 + b\cos v\tau),$$
$$a\cos v\tau = u + bv\sin v\tau \quad (7)$$

or

$$a\sin v\tau = -v(1 - b\cos v\tau),$$
$$a\cos v\tau = u - bv\sin v\tau. \quad (8)$$

Applying the method in [3], we have that there exists a unique value of v, denoted by v_{1m}^+ (v_{1m}^-), satisfying (7) and a unique value of v, denoted by v_{2m}^+ (v_{2m}^-), satisfying (8) on the interval $(2(m-1)\pi/\tau, 2m\pi/\tau)$, $m \in \mathbb{Z}^+$ when $a > 0(< 0)$. We also can show that $v_{km}^+ \in ((2m-1)\pi/\tau, 2m\pi/\tau)$, $k = 1, 2$, $m \in \mathbb{Z}^+$ with $a > 0$ and $v_{km}^- \in (2(m-1)\pi/\tau, (2m-1)\pi/\tau)$, $k = 1, 2$, $m \in \mathbb{Z}^+$ with $a < 0$.

Define

$$a_{km}^\pm = \pm\sqrt{v_{km}^{\pm\,2}(1 - b^2) + \mu^2}, \quad k = 1, 2, \ m \in \mathbb{Z}^+. \quad (9)$$

$a_{km}^\pm$ make sense since $|b| < 1$, and a_{km}^+ and a_{km}^- are all dependent on τ. Hence, $\pm v_{km}^+$ ($\pm v_{km}^-$) are a pair of purely imaginary roots of (4) with $a = a_{km}^+(a_{km}^-)$. And we have the transversal condition

$$\text{Sign Re}\left(\frac{d\lambda}{da}\right)\Bigg|_{a=a_{km}^\pm}$$
$$= \begin{cases} 1, & a = a_{km}^+, \\ -1, & a = a_{km}^-, \end{cases} \quad k = 1, 2, \ m \in \mathbb{Z}^+. \quad (10)$$

Moreover, it is fulfilled that $a_{k\times(m+1)}^- < a_{km}^- < 0 < a_{km}^+ < a_{k\times(m+1)}^+$, following the fact that $0 < v_{km}^- < v_{km}^+ < v_{k\times(m+1)}^- < v_{k\times(m+1)}^+$. Now we can distinguish two cases. First, $b > 0$. From (7) and (8), we can obtain that $a_{1m}^- \cos v_{1m}^-\tau > a_{2m}^- \cos v_{2m}^-\tau$, which implies $v_{1m}^- > v_{2m}^-$ and $a_{1m}^- < a_{2m}^-$. Second, $b < 0$. With the similar process as above, it is obtained that $v_{1m}^- < v_{2m}^-$ and $a_{1m}^- > a_{2m}^-$.

Summarizing the discussion above and applying Claims 1 and 2, we have the following.

Lemma 1. *There exist sequence values of a defined by* (9) *such that all the roots of* (4) *have negative real parts when $a \in (a_1^-(\tau), \mu)$, and* (4) *has at least two roots with positive real part when $a \in (-\infty, a_1^-(\tau)) \cup (\mu, +\infty)$, a pair of purely imaginary roots when $a = a_{km}^\pm(\tau)$ ($k = 1, 2, m \in \mathbb{Z}^+$), where $a_1^-(\tau) = a_{11}^-(a_{21}^-)$ when $b < 0(> 0)$.*

Lemma 2. *Equation* (1) *undergoes a pitchfork bifurcation at $(0, 0)$ when $a = \mu$.*

Proof. Let $h(x) = -\mu x + af(x)$. Then we have $h(0) = 0$, $h(\pm\infty) = \mp\infty$, and $h'(x) = -\mu + af'(x)$. From (H1) and (H2), there exists a unique zero $x = 0$ of $h(x)$, and (1) has only one steady-state $X = (0, 0)$ when $a \leq \mu$. On the other hand, when $a > \mu$, we have $h'(0) = a - \mu > 0$ and $xh''(x) < 0$ when $x \neq 0$. Thus, there exists a unique $c^+ \in \mathbb{R}^+(c^- \in \mathbb{R}^-)$ satisfying $h(x) = 0$, and (1) has exactly nonzero equilibria $(c^\pm, 0)$, $(0, c^\pm)$, and $(c^\pm, c^\pm)$. This completes the proof. □

In the following, we will investigate the stability of $(c^{\pm}, 0)$, $(0, c^{\pm})$, and $(c^{\pm}, c^{\pm})$.

Firstly, we consider the stability of $(c^+, 0)$ when $a > \mu$. (It has similar results with $(c^-, 0)$ and $(0, c^{\pm})$.)

The characteristic equation of linearization of (1) at $(c^+, 0)$ is

$$\Delta_{(c^+,0)}(\lambda) = \left(\lambda + \mu - ae^{-\lambda\tau}\right)\left(\lambda + \mu - af'(c^+)e^{-\lambda\tau}\right) - b^2\lambda^2 e^{-2\lambda\tau} = 0. \tag{11}$$

Equation (11) has at least one positive root for any $\tau \geq 0$. In fact, we have $h'(c^{\pm}) < 0$ and $0 < af'(c^{\pm}) < \mu$ from the proof of Lemma 2. Then the conclusion follows from $\Delta_{(c^+,0)}(0) = (\mu-a)(\mu-af'(c^+)) < 0$ and $\lim_{\lambda \to +\infty}\Delta_{(c^+,0)}(\lambda) = +\infty$, which implies that $(c^+, 0)$ is unstable.

Secondly, we consider the stability of (c^+, c^+) when $a > \mu$. (It has similar results with (c^+, c^-) and $(c^-, c^{\pm})$.)

The characteristic equation of linearization of (1) at (c^+, c^+) is

$$\Delta_{(c^+,c^+)}(\lambda, \tau) = \left(\lambda + \mu - af'(c^+)e^{-\lambda\tau}\right)^2 - b^2\lambda^2 e^{-2\lambda\tau} = 0. \tag{12}$$

In particular,

$$\Delta_{(c^+,c^+)}(\lambda, 0) = \left(\lambda + \mu - af'(c^+)\right)^2 - b^2\lambda^2 = 0, \tag{13}$$

which implies

$$\lambda_{1,2} = \frac{-\mu + af'(c^+)}{1 \pm b} < 0. \tag{14}$$

Moreover, (12) has no zero root from $af'(c^+) < \mu$.

In order to prove that (c^+, c^+) is asymptotically stable for all $\tau \in \mathbb{R}^+$, what we need is to verify that (12) has no purely imaginary roots. Let $i\eta$ $(\eta > 0)$ be the root of (12), then η solves

$$\begin{aligned} af'(c^+)\sin\eta\tau + b\eta\cos\eta\tau &= -\eta, \\ af'(c^+)\cos\eta\tau - b\eta\sin\eta\tau &= \mu \end{aligned} \tag{15}$$

or

$$\begin{aligned} af'(c^+)\sin\eta\tau - b\eta\cos\eta\tau &= -\eta, \\ af'(c^+)\cos\eta\tau + b\eta\sin\eta\tau &= \mu. \end{aligned} \tag{16}$$

This leads to

$$\eta^2 = \frac{\left(af'(c^+)\right)^2 - \mu^2}{1 - b^2} < 0. \tag{17}$$

The assertion follows.

Applying Lemmas 1 and 2, we have the following results.

Theorem 3. *(i) The zero solution of (1) is asymptotically stable when $a \in (a_1^-(\tau), \mu)$ and unstable when $a \in (-\infty, a_1^-(\tau)) \cup (\mu, +\infty)$.*

(ii) Equation (1) undergoes a pitchfork bifurcation at $(0,0)$ when $a = \mu$. More precisely, some new equilibria bifurcate from zero and $(c^{\pm}, 0)$, $(0, c^{\pm})$ are unstable, and $(c^{\pm}, c^{\pm})$ are asymptotically stable for $\tau \in \mathbb{R}^+$ when $a > \mu$.

(iii) Equation (1) undergoes a Hopf bifurcation at $(0,0)$ when $a = a_{km}^{\pm}(\tau)$ $(k = 1, 2, m \in \mathbb{Z}^+)$.

Case 2. Regard τ as parameter.

First of all, we know that the root of (4) with $\tau = 0$ satisfies that $\lambda_{1,2} = (a - \mu)/(1 \pm b) > 0$ when $a > \mu$, and $\lambda_{1,2} = (a - \mu)/(1 \pm b) < 0$ when $a < \mu$.

Let $i\omega$ $(\omega > 0)$ be a root of (4); then it follows that

$$\sin\omega\tau = -\frac{\omega(a \pm b\mu)}{\mu^2 + \omega^2}, \qquad \cos\omega\tau = \frac{a\mu \mp b\omega^2}{\mu^2 + \omega^2}. \tag{18}$$

This leads to $\omega^2 = (a^2 - \mu^2)/(1 - b^2)$, $\omega_0 \overset{\text{def}}{=} \sqrt{(a^2 - \mu^2)/(1 - b^2)}$ makes sense when $|a| > \mu$. Define

$$\tau_{1j}$$
$$= \begin{cases} \dfrac{1}{\omega_0}\left(\arccos\dfrac{a\mu - b\omega_0^2}{\mu^2 + \omega_0^2} + 2j\pi\right), & a < -\mu, \\[2mm] \dfrac{1}{\omega_0}\left(-\arccos\dfrac{a\mu - b\omega_0^2}{\mu^2 + \omega_0^2} + 2(j+1)\pi\right), & a > \mu, \end{cases} \quad j = 0, 1, 2, \ldots,$$

$$\tau_{2j}$$
$$= \begin{cases} \dfrac{1}{\omega_0}\left(\arccos\dfrac{a\mu + b\omega_0^2}{\mu^2 + \omega_0^2} + 2j\pi\right), & a < -\mu, \\[2mm] \dfrac{1}{\omega_0}\left(-\arccos\dfrac{a\mu + b\omega_0^2}{\mu^2 + \omega_0^2} + 2(j+1)\pi\right), & a > \mu, \end{cases} \quad j = 0, 1, 2, \ldots. \tag{19}$$

Then $i\omega_0$ is a purely imaginary root of (4) with $\tau = \tau_{kj}$, $k = 1, 2$ defined by (19).

Let $\lambda(\tau) = \alpha(\tau) + i\varrho(\tau)$ be the root of (4), satisfying

$$\alpha(\tau_{kj}) = 0, \qquad \varrho(\tau_{kj}) = \omega_0. \tag{20}$$

Differentiating both side of (4) gives that

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_{kj}} = \frac{1 - b^2}{a^2 + b^2\omega_0^2} > 0. \tag{21}$$

This implies that $\alpha'(\tau_{kj}) > 0$, $k = 1, 2$, $j = 0, 1, 2, \ldots$.

Summarizing the discussions above, one can obtain the following.

Lemma 4. *(i) If $|a| < \mu$, then all roots of (4) have negative real parts.*

(ii) If $|a| > \mu$, then there exist a sequence values of τ defined by (19) such that (4) has a pair of purely imaginary roots $\pm i\omega_0$ when $\tau = \tau_{kj}$, $k = 1, 2$. Additionally, if $a < -\mu$, then all roots of (4) have negative real parts when $\tau \in [0, \tau_0)$, all roots of (4), except $\pm i\omega_0$, have negative real parts when $\tau = \tau_0$, and (4) has at least a pair of root with positive real parts when $\tau > \tau_0$, where $\tau_0 = \tau_{10}(\tau_{20})$ when $b < 0 (> 0)$; if $a > \mu$, then (4) has at least two positive roots.

Spectral properties in Lemma 4 immediately lead to the dynamics near the origin described by the following theorem.

Theorem 5. *For* (1), *the following holds.*

(i) *If* $|a| < \mu$, *then* $(0, 0)$ *is asymptotically stable for all* $\tau > 0$.

(ii) *If* $a < -\mu$, *then* $(0, 0)$ *is asymptotically stable when* $\tau \in (0, \tau_0)$, *and unstable when* $\tau > \tau_0$.

(iii) *If* $a > \mu$, *then* $(0, 0)$ *is always unstable for all* $\tau > 0$.

(iv) *If* $|a| > \mu$, *then* (1) *undergoes a Hopf bifurcation at* $(0, 0)$ *when* $\tau = \tau_{kj}$, $k = 1, 2$, $j = 0, 1, 2, \ldots$.

3. Properties of Hopf BifurCation

Theorems 3(iii) and 5(iv) in the previous section give the sufficient conditions for (1) to undergo Hopf bifurcations with a and τ as bifurcation parameters. In this section, we will investigate the direction of Hopf bifurcations and stability of bifurcating periodic solutions, following the same algorithms as Wang and Wei's recent work and using a computation process similar to that in [2] (see also [17]). We should mention that we will choose τ as bifurcation parameter, and the similar results follow when choosing other coefficients as bifurcation parameters.

Let $y_i(t) = x_i(\tau t)$, $i = 1, 2$, and $Y(t) = (y_1(t), y_2(t))^T$, then (1) becomes

$$\frac{d}{dt}[Y(t) - CY(t-1)] = \tau AY(t) + \tau BF(Y(t-1)), \quad (22)$$

where A, B, and C are denoted by (2) and $F(Y(t-1)) = (f(y_1(t-1)), f(y_2(t-1)))^T$. And the corresponding characteristic equation around $Y = (0, 0)^T$ is

$$(z + \mu\tau - a\tau e^{-z})^2 - b^2 z^2 e^{-2z} = 0, \quad (23)$$

Comparing (23) with (4), if is found that $z = \lambda\tau$ for $\tau \neq 0$. Therefore, combining this fact with Lemma 4, one has the following.

Lemma 6. *Assume* $|a| > \mu$.

(i) *If* $\tau = \tau_j$, $j = 0, 1, 2, \ldots$, *then* (23) *has a pair of purely imaginary roots* $\pm i\omega_0\tau_j$, *where* $\tau_j = \tau_{kj}$ *and* ω_0 *are defined by* (19).

(ii) *Let* $z(\tau) = \gamma(\tau) + i\varsigma(\tau)$ *be the root of* (23), *satisfying*

$$\gamma(\tau_j) = 0, \qquad \varsigma(\tau_j) = \omega_0\tau_j, \quad (24)$$

then

$$\gamma'(\tau_j) = \tau_j\alpha'(\tau_j) > 0, \quad j = 0, 1, 2, \ldots. \quad (25)$$

(iii) *Equation* (23) *has at least a pair of roots with positive real parts when* $\tau = \tau_j$ *for* $j \geq 1$, *and it has two positive roots when* $a > \mu$ *and* $\tau = \tau_0$. *All roots of* (23) *with* $\tau = \tau_0$, *except* $\pm i\omega_0\tau_0$, *have negative real parts when* $a < -\mu$.

For convenience, denote $\tau = \tau_j + \nu$. Then we know that (22) undergoes a Hopf bifurcation at origin when $\nu = 0$. For $\phi \in C([-1, 0], \mathbb{R}^2)$, let

$$D(\phi) = \phi(0) - C\phi(-1),$$

$$L(\nu, \phi) = A(\tau_j + \nu)\phi(0) + B(\tau_j + \nu)\phi(-1), \quad (26)$$

$$F(\nu, \phi) = \frac{f'''(0)}{6}(\tau_j + \nu)B\phi^3(-1) + \cdots.$$

By the Riesz representation theorem, there exist functions $\eta(\theta)$ and $\mu(\theta)$ such that

$$D(\phi) = \phi(0) - \int_{-1}^{0} d\mu(\theta)\phi(\theta),$$

$$L(\nu, \phi) = \int_{-1}^{0} d\eta(\theta)\phi(\theta). \quad (27)$$

In fact, we can choose

$$\mu(\theta) = \begin{cases} -C, & \theta = -1, \\ 0, & \theta \in (-1, 0], \end{cases}$$

$$\eta(\theta) = \begin{cases} -B(\tau_j + \nu), & \theta = -1, \\ 0, & \theta \in (-1, 0), \\ A(\tau_j + \nu), & \theta = 0. \end{cases} \quad (28)$$

Define

$$A(\nu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \phi'(0) - D\phi'(0) + L\phi, & \theta = 0, \end{cases} \quad (29)$$

$$R(\nu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\nu, \phi), & \theta = 0. \end{cases}$$

Then (22) can be written as

$$\dot{Y}_t = A(\nu)Y_t + R(\nu)Y_t. \quad (30)$$

Clearly, (30) is an abstract ODE on the phase space BC' of (22), where $BC' = \{\phi : [-1, 0] \to \mathbb{C}^2, \phi$ is continuous on $[-1, 0)$, and $\lim_{\theta \to 0}\phi(\theta)$ exists$\}$.

The adjoint operator $A^*\psi = -d\psi/ds$ with domain

$$D(A^*)$$
$$= \left\{\psi \in C' \overset{\text{def}}{=} C\left([0, 1], \mathbb{R}^2\right) : \frac{d\psi}{ds} \in C'; D\frac{d\psi}{ds} = -L\psi\right\}. \quad (31)$$

For $(\psi, \phi) \in C' \times C$, define a bilinear form:

$$(\psi, \phi) = \overline{\psi}(0)\phi(0)$$
$$- \int_{-1}^{0} d\left[\int_{\alpha=0}^{\theta} \overline{\psi}(\theta - \alpha)d\mu(\alpha)\right]\phi(\theta) \quad (32)$$
$$- \int_{-1}^{0}\int_{\theta=0}^{s}\overline{\psi}(\theta - s)d\eta(s)\phi(\theta)d\theta.$$

It is not difficult to verify that $q(\theta) = q_0 e^{i\omega_0\tau_j\theta}(\theta \in [-1,0])$ and $q^*(s) = \bar{l}q_0^* e^{i\omega_0\tau_j s}(s \in [0,1])$ are the eigenvectors of $A(0)$ and A^* corresponding to the eigenvalues $i\omega_0\tau_j$ and $-i\omega_0\tau_j$, respectively, where

$$q_0 = \begin{cases} (1,-1)^T & \tau_j = \tau_{1j}, \\ (1,1)^T & \tau_j = \tau_{2j}, \end{cases}$$
$$q_0^* = \begin{cases} (1,-1) & \tau_j = \tau_{1j}, \\ (1,1) & \tau_j = \tau_{2j}, \end{cases}$$
$$l = \frac{1}{q_0^* \left(I - Ce^{-i\omega_0\tau_j} + B\tau_j e^{-i\omega_0\tau_j}\right)\bar{q}_0}, \tag{33}$$

and $(q^*, q) = 1$.

$$Y_t(-1) = \begin{pmatrix} q_{01}e^{-i\omega_0\tau_j}z + \bar{q}_{01}e^{i\omega_0\tau_j}\bar{z} + W_{20}^{(1)}(-1)\dfrac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\dfrac{\bar{z}^2}{2} + \cdots \\ q_{02}e^{-i\omega_0\tau_j}z + \bar{q}_{02}e^{i\omega_0\tau_j}\bar{z} + W_{20}^{(2)}(-1)\dfrac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\dfrac{\bar{z}^2}{2} + \cdots \end{pmatrix}, \tag{38}$$

where $q_0 = (q_{01}, q_{02})^T$. Therefore, from (26), we have

$$F(0, Y_t) = \frac{f'''(0)}{6}\tau_j Y_t^3(-1) + \cdots$$
$$\stackrel{\text{def}}{=} F_{z^2}\frac{z^2}{2} + F_{z\bar{z}}z\bar{z} + F_{\bar{z}^2}\frac{\bar{z}^2}{2} + F_{z^2\bar{z}}\frac{z^2\bar{z}}{2} + \cdots. \tag{39}$$

Substituting the expression of $F(0, Y_t)$ into (35) and comparing its coefficients with that of (36) give that

$$g_{20} = g_{02} = g_{11} = 0,$$
$$g_{21} = \bar{l}\bar{q}_0^* \frac{f'''(0)}{6}\tau_j B \begin{pmatrix} q_{01}^2\bar{q}_{01} \\ q_{02}^2\bar{q}_{02} \end{pmatrix} e^{-i\omega_0\tau_j}. \tag{40}$$

It is well known that the coefficient $c_1(0)$ of third degree term of Poincaré normal form of (35) is given by (see [12])

$$c_1(0) = \frac{i}{2\omega_0\tau_j}\left(g_{20}g_{11} - 2\|g_{11}\|^2 - \frac{1}{3}\|g_{02}\|^2\right) + \frac{g_{21}}{2}. \tag{41}$$

Consequently, when $\tau = \tau_{1j}$ or $\tau = \tau_{2j}$,

$$c_1(0) = \frac{\tau_{kj}}{12\|1 + (\mp b + a\tau_{kj})e^{-i\omega_0\tau_{kj}}\|^2}$$
$$\times af'''(0)\left[1 + (\mp b + a\tau_{kj})e^{-i\omega_0\tau_{kj}}\right]e^{-i\omega_0\tau_{kj}}. \tag{42}$$

Now we compute the center manifold C_0 at $\nu = 0$. Define

$$z(t) = (q^*, Y_t),$$
$$W(t, \theta) = Y_t(\theta) - 2\operatorname{Re} z(t)q(\theta), \tag{34}$$

then we have

$$\dot{z}(t) = i\omega_0\tau_j z + \bar{q}^*(0)F(0, Y_t). \tag{35}$$

Equation (35) can be written in the abbreviated form as

$$\dot{z}(t) = i\omega_0\tau_j z + g(z, \bar{z}), \tag{36}$$

with

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{37}$$

Noting that $Y_t(\theta) = W(t, \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta)$, we have

Hence we obtain

$$\operatorname{Sign}\operatorname{Re} c_1(0)$$
$$= \operatorname{Sign} af'''(0)$$
$$\times \left[\frac{a\mu \mp b\omega_0^2}{\mu^2 + \omega_0^2} + (\mp b + a\tau_{kj})\left(2\left(\frac{a\mu \mp b\omega_0^2}{\mu^2 + \omega_0^2}\right)^2 - 1\right)\right]. \tag{43}$$

Summarizing the above analysis, we have the following theorem.

Theorem 7. *For* (22), *the direction of Hopf bifurcation at* $\tau = \tau_{kj}$, $k = 1, 2$, $j = 0, 1, 2, \ldots$, *is supercritical (subcritical), and the bifurcating periodic solutions are asymptotically stable (unstable) when* $\operatorname{Re} c_1(0) < 0$ $(\operatorname{Re} c_1(0) > 0)$.

4. Global Hopf Bifurcation Analysis

Our objective in this section is to obtain the global continuation of periodic solution bifurcating from the point $(0, \tau_{kj})$, $k = 1, 2$, $j = 0, 1, 2, \ldots$, for (1) by using a global Hopf bifurcation theorem given by Krawcewicz et al. [14]. For the reader's convenience, we copy (22), which is equivalent to (1), as follows

$$\dot{y}_1(t) - b\dot{y}_2(t-1) = -\mu\tau y_1(t) + a\tau f(y_1(t-1)),$$
$$\dot{y}_2(t) - b\dot{y}_1(t-1) = -\mu\tau y_2(t) + a\tau f(y_2(t-1)). \tag{44}$$

We have known that (44) undergoes a local Hopf bifurcation at the origin when $\tau = \tau_{kj}$, $k = 1, 2$, $j = 0, 1, 2, \ldots$. Now we

begin to show that each bifurcation branch can be continued with respect to the parameter τ under certain conditions. To bring out the ideas in the results of subsequent part, it is convenient to introduce the following notations:

$$X = C([-1,0], \mathbb{R}^2),$$

$$\Sigma = Cl\{(Y, \tau, T) : Y \text{ is a } T\text{-periodic solution of (44)}\} \subset X \times \mathbb{R}_+ \times \mathbb{R}_+,$$

$$N = \{(\widehat{Y}, \tau, T) : \mu\widehat{Y} = aF(\widehat{Y})\}.$$

Denote $C(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ the connected component of $(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ in Σ, where $\omega_0 = \sqrt{(a^2 - \mu^2)/(1 - b^2)}$ and τ_j $(j = 0, 1, 2, \ldots)$ are defined by Lemma 6.

Lemma 8. *Equation (44) has no periodic nonconstant solution when $\tau = 0$.*

Proof. Let $u(t) = y_1(t) + y_2(t)$. When $\tau = 0$, we from (44) have

$$\frac{d}{dt}[u(t) - bu(t-1)] = 0. \tag{45}$$

Therefore, there exists $c_1 \in \mathbb{R}$ such that

$$u(t) = c_1 + bu(t-1). \tag{46}$$

Which implies that when $t \in [n, n+1]$, we have

$$u(t) = c_1 \sum_{i=0}^{n-1} b^i + b^n u(t-n). \tag{47}$$

Hence we obtain $\lim_{t \to +\infty} u(t) = c_1/(1-b)$.

Let $v(t) = y_1(t) - y_2(t)$. With the similar process as above, it is obtained that there exists $c_2 \in \mathbb{R}$ such that $\lim_{t \to +\infty} v(t) = c_2/(1+b)$.

Then the conclusion follows from $\lim_{t \to +\infty} y_1(t) = (1/2)[c_1/(1-b) + c_2/(1+b)]$, and the proof is complete. □

Lemma 9. *If $|b| < 1/2$, then all periodic solution of (44) are uniformly bounded.*

Proof. Let $Y(t) = (y_1(t), y_2(t))^T$ be a periodic solution of (44) and $u(t) = y_1(t) + y_2(t)$. Then there exist t_1 and t_2 such that

$$u(t_1) - bu(t_1 - 1) = \max_{t \in \mathbb{R}}[u(t) - bu(t-1)],$$
$$u(t_2) - bu(t_2 - 1) = \min_{t \in \mathbb{R}}[u(t) - bu(t-1)], \tag{48}$$

together with T^0 and T_0 such that

$$u(T^0) = \max_{t \in \mathbb{R}^+} u(t), \qquad u(T_0) = \min_{t \in \mathbb{R}^+} u(t), \tag{49}$$

Now we take the case $-1/2 < b < 0$ as an example, and it has the same result when $0 \leq b < 1/2$.

By (44), (48), and (H1), we have that

$$u(t_i) = \frac{a}{u} \sum_{j=1}^{2} f(y_j(t_i - 1)) \in \left[-\frac{2|a|L}{\mu}, \frac{2|a|L}{\mu}\right],$$
$$i = 1, 2,$$

$$u(t_2) - bu(t_2 - 1) \leq u(t) - bu(t-1)$$
$$\leq u(t_1) - bu(t_1 - 1). \tag{50}$$

On one hand, we have

$$u(t) \leq bu(t-1) + [u(t_1) - bu(t_1 - 1)]$$
$$\leq b^2 u(t-2) + b[u(t_2) - bu(t_2 - 1)]$$
$$\quad + [u(t_1) - bu(t_1 - 1)]$$
$$\leq b^{2m} u(t-2m) + \sum_{i=1}^{m} b^{2i-1}[u(t_2) - bu(t_2 - 1)]$$
$$\quad + \sum_{i=1}^{m} b^{2(i-1)}[u(t_1) - bu(t_1 - 1)] \tag{51}$$
$$\leq \frac{b}{1-b^2}[u(t_2) - bu(t_2 - 1)]$$
$$\quad + \frac{1}{1-b^2}[u(t_1) - bu(t_1 - 1)]$$
$$\leq \frac{2|a|L}{(1+b)\mu} - \frac{b^2}{1-b^2}u(T_0) - \frac{b}{1-b^2}u(T^0).$$

And on the other hand, we can obtain

$$u(t) \geq -\frac{2|a|L}{(1+b)\mu} - \frac{b}{1-b^2}u(T_0) - \frac{b^2}{1-b^2}u(T^0). \tag{52}$$

It follows that

$$u(T_0) \geq \frac{1-b^2}{1+b-b^2}\left[-\frac{2|a|L}{(1+b)\mu} - \frac{b^2}{1-b^2}u(T^0)\right],$$
$$u(T^0) \leq \frac{2|a|L}{(1+2b)\mu}. \tag{53}$$

Similarly, one can prove that

$$u(T_0) \geq -\frac{2|a|L}{(1+2b)\mu}. \tag{54}$$

Which implies that

$$-\frac{2|a|L}{(1+2b)\mu} \leq u(t) \leq \frac{2|a|L}{(1+2b)\mu}. \tag{55}$$

Let $v(t) = y_1(t) - y_2(t)$. In the same way, it can be verified that

$$-\frac{2|a|L}{(1+2b)\mu} \leq v(t) \leq \frac{2|a|L}{(1+2b)\mu}. \tag{56}$$

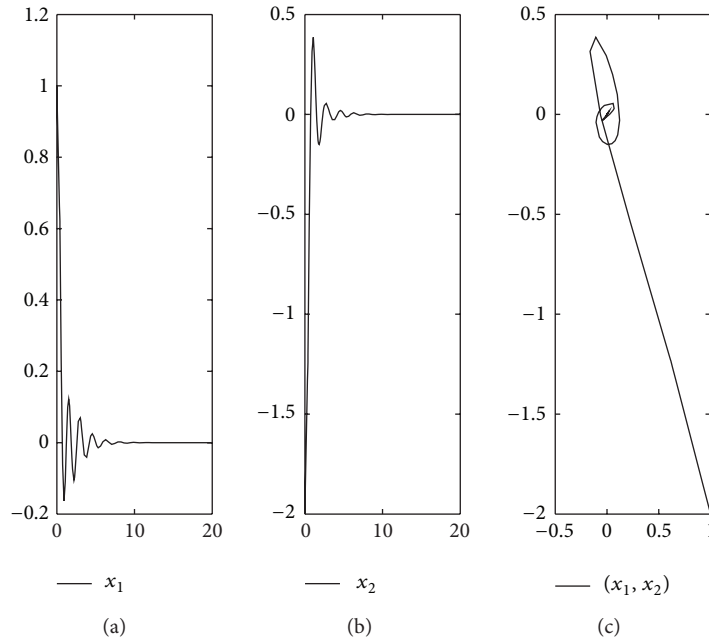Then $y_1(t)$ and $y_2(t)$ are uniformly bounded, and the proof is complete. □

FIGURE 1: For system (1), the equilibrium $(0, 0)$ is asymptotically stable when $\tau = 0.55 < \tau_0 \approx 0.6378$.
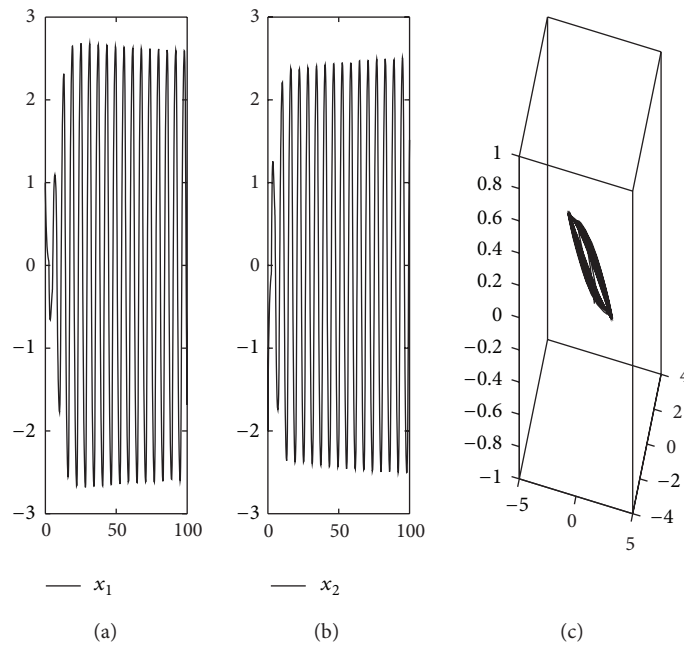

FIGURE 2: The system (1) undergoes a Hopf bifurcation at $(0, 0)$, and the bifurcating periodic solution is asymptotically stable when $\tau = 2.5 \in (\tau_0, \tau_1)$.

Lemma 10. *If* $a < -\mu$ *or* $a > \mu/M$, *then* (44) *has no periodic nonconstant solution of period 1, where* $M = \min\{f'(-2|a|L/(1 + 2b)\mu), f'(2|a|L/(1 + 2b)\mu)\}$.

Proof. Let $(y_1(t), y_2(t))$ be a periodic solution to (44) of period 1. Then it is a periodic solution to the following system of ordinary differential equations:

$$\dot{u}_1(t) - b\dot{u}_2(t) = -\mu\tau u_1(t) + a\tau f(u_1(t)),$$
$$\dot{u}_2(t) - b\dot{u}_1(t) = -\mu\tau u_2(t) + a\tau f(u_2(t)),$$

(57)

where $\cdot$ denotes d/dt. Denote

$$G = \left\{ u \in \mathbb{R}^2 : u_j \in \left[-\frac{2|a|L}{(1 + 2b)\mu}, \frac{2|a|L}{(1 + 2b)\mu} \right], \ j = 1, 2 \right\}.$$
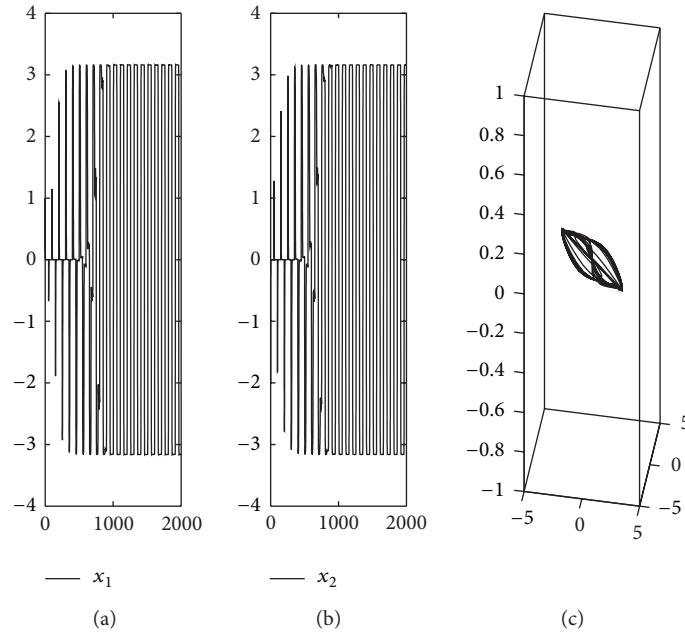
(58)

FIGURE 3: The system (1) undergoes a Hopf bifurcation at $(0, 0)$ with sufficiently large $\tau = 50$.

Lemma 9 shows that the periodic solution of (44) belong to the region G. Clearly, (57) is equivalent to

$$
\begin{aligned}
\dot{u}_1 &= \frac{\tau}{1 - b^2} \left[-\mu \left(u_1 + b u_2 \right) + a \left(f \left(u_1 \right) + b f \left(u_2 \right) \right) \right] \\
&\stackrel{\text{def}}{=} P \left(u_1, u_2 \right), \\
\dot{u}_2 &= \frac{\tau}{1 - b^2} \left[-\mu \left(u_2 + b u_1 \right) + a \left(f \left(u_2 \right) + b f \left(u_1 \right) \right) \right] \\
&\stackrel{\text{def}}{=} Q \left(u_1, u_2 \right),
\end{aligned}
\tag{59}
$$

Then

$$
\frac{\partial P}{\partial u_1} + \frac{\partial Q}{\partial u_2} = \frac{\tau}{1 - b^2} \left[-2\mu + a \left(f' \left(u_1 \right) + f' \left(u_2 \right) \right) \right]. \tag{60}
$$

From (H2), it is easy to see that $\partial P / \partial u_1 + \partial Q / \partial u_2 < 0$ when $a < -\mu$ and $\partial P / \partial u_1 + \partial Q / \partial u_2 > 0$ when $a > \mu/M$. Thus, the classical Bendixson's criterion implies that (57) has no nonconstant periodic solution in the region G, and the proof is complete. $\qquad\square$

Up to now, we have prepared sufficiently to state the following global Hopf bifurcation results.

Theorem 11. *If $|b| < 1/2$ and either $a < -\mu$ or $a > \mu/M$ holds, then* (44) *has at least one periodic solution for $\tau > \tau_j$ $(j \geqslant 1)$.*

Proof. First note that the stationary points $(\mathbf{0}, \tau_j, 2\pi/\omega_0 \tau_j)$ of (44) are nonsingular and isolate centers (see [14]) under the assumption $|a| > \mu$ for each $\tau > \tau_j$ $(j \geqslant 1)$. By (25), there exist

$\epsilon > 0$, $\delta > 0$, and a smooth curve $\lambda : (\tau_j - \delta, \tau_j + \delta) \to \mathbb{C}$, such that

$$
\begin{aligned}
\Delta \left(\lambda \left(\tau \right) \right) &= \Delta_{(\mathbf{0}, \tau, T)} \left(\lambda \left(\tau \right) \right) = 0, \\
\left| \lambda \left(\tau \right) - i \omega_0 \tau_j \right| &< \epsilon,
\end{aligned}
\tag{61}
$$

for all $\tau \in [\tau_j - \delta, \tau_j + \delta]$, where Δ is defined as (23), and

$$
\lambda \left(\tau_j \right) = i \omega_0 \tau_j, \qquad \left. \frac{d \operatorname{Re}(\lambda(\tau))}{d\tau} \right|_{\tau = \tau_j} > 0. \tag{62}
$$

Let

$$
\Omega_\epsilon = \left\{ (u, p) : 0 < u < \epsilon, \left| p - \frac{2\pi}{\omega_0 \tau_j} \right| < \epsilon \right\}. \tag{63}
$$

Clearly, if $\tau - \tau_j \leqslant \delta$ and $(u, p) \in \Omega_\epsilon$ such that $\Delta_{(\mathbf{0}, \tau, T)}(u + i2\pi/p) = 0$, then $\tau = \tau_j$, $u = 0$, and $p = 2\pi/\omega_0 \tau_j$. Moreover, if we put

$$
H^\pm \left(0, \tau_j, \frac{2\pi}{\omega_0 \tau_j} \right) (u, p) = \Delta_{(\mathbf{0}, \tau \pm \delta, T)} \left(u + \frac{i2\pi}{p} \right), \tag{64}
$$

at $m = 1$, we have, from $\operatorname{Re} \lambda'(\tau_j) > 0$, that the crossing number is

$$
\begin{aligned}
\gamma_1 \left(0, \tau_j, \frac{2\pi}{\omega_0 \tau_j} \right) &= \deg_B \left(H^- \left(0, \tau_j, \frac{2\pi}{\omega_0 \tau_j} \right), \Omega_\epsilon \right) \\
&\quad - \deg_B \left(H^+ \left(0, \tau_j, \frac{2\pi}{\omega_0 \tau_j} \right), \Omega_\epsilon \right) = -1.
\end{aligned}
\tag{65}
$$

Using the local Hopf bifurcation theorem for NFDE [14, Theorem 5.6], we conclude that the connected component

$C(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ through $(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ in Σ is nonempty, and thus $C(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ is unbounded by the global Hopf bifurcation theorem given by Krawcewicz et al. [14, Theorem 5.14].

Lemma 9 implies that the projection of $C(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ onto the Y-space is bounded. Meanwhile, the projection of $C(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ onto τ-space is bounded below from Lemma 8.

By the definition of τ_j, we know that

$$2\pi < \omega_0\tau_j < 2(j+1)\pi, \quad j \geqslant 1, \tag{66}$$

under the assumptions that $|a| > \mu$ and $|b| < 1$, which implies

$$\frac{1}{j+1} < \frac{2\pi}{\omega_0\tau_j} < 1. \tag{67}$$

Applying Lemma 10, one has $1/(j+1) < T < 1$ if $(Y, \tau, T) \in C(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ for $j \geqslant 1$, when $|b| < 1/2$ and either $a < -\mu$ or $a > \mu/M$ holds. Thus in order for $C(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ to be unbounded, its projection onto the τ-space must be unbounded. Consequently, the projection of $C(\mathbf{0}, \tau_j, 2\pi/\omega_0\tau_j)$ onto the τ-space include $[\tau_j, \infty)$ for $j \geqslant 1$ when $|b| < 1/2$ and either $a < -\mu$ or $a > \mu/M$ holds. The proof is complete. □

Next we carry out some numerical simulations for (1).

Let $f(x) = \arctan(x)$, and assume that $\mu = 1.2$, $a = -3$, and $b = 0.2$. From Lemma 4, it is obtained that $\omega_0 \approx 2.8062$ and $\tau_0 \approx 0.6378$. Accordingly, it is known that $(0,0)$ is asymptotically stable for $\tau \in (0, \tau_0)$ and unstable for $\tau > \tau_0$, and Hopf bifurcation at the origin occurs when $\tau = \tau_0$ by Theorem 5. From Theorem 7, the direction of the Hopf bifurcation at $\tau = \tau_0$ is supercritical, and the bifurcating periodic solutions are asymptotically stable. Furthermore, according to Theorem 11, (1) with this set of parameters has at least one periodic solution when $\tau > \tau_j$ ($j \geqslant 1$). The corresponding numerical simulation results are shown in Figures 1, 2, and 3.

Acknowledgment

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